EXERCISE SET 7
PARTIAL DIFFERENTIAL EQUATIONS 2, 2019
EXERCISES ON WEDNESDAY 12.15-14, MAD381

1. Show that the definition of a weak solution to the heat equation $u_t = \Delta u$ is equivalent to

$$\int_\Omega u(x,t_2)\varphi(x,t_2)\ dx - \int_\Omega u(x,t_1)\varphi(x,t_1)\ dx - \int_{\Omega\times(t_1,t_2)} u\varphi_t\ dx\,dt + \int_{\Omega\times(t_1,t_2)} Du\cdot D\varphi\ dx\,dt = 0$$

for $0 < t_1 < t_2 < T$.

2. Provide the definition of weak sub- and supersolution to $u_t + Lu = f$ in Ω_T.

3. Show that the function

$$u(x,t) = \begin{cases} 0 & t \le \frac{1}{2} \\ 1 & t > \frac{1}{2} \end{cases}$$

is a weak supersolution to $u_t - \Delta u \ge 0$.

4. Let $u, g \in C(0,T;L^2(\Omega))$. Show that the following three initial conditions are equivalent:

$$\frac{1}{h}\int_{\Omega_h} |u(x,t) - g(x,t)|^2\ dx\,dt \to 0 \quad \text{as} \quad h \to 0,$$

$$\int_\Omega |u(x,t) - g(x,0)|^2\ dx\,dt \to 0 \quad \text{as} \quad t \to 0,$$

and

$$\int_\Omega u(x,0)\varphi(x)\ dx = \int_\Omega g(x,0)\varphi(x)\ dx \qquad \text{for every } \varphi \in C_0^\infty(\Omega).$$

5. Let u be a weak solution with zero boundary condition and an initial data g of the equation $u_t = \Delta u + f$, where $f \in L^2(\Omega_T)$ and $g \in W_0^{1,2}(\Omega)$. Show that the following energy estimate holds for u:

$$\sup_{t\in[0,T]} \int_\Omega u(x,t)^2\ dx + \int_{\Omega_t} |Du(x,t)|^2\ dx\,dt \le C\|g\|^2_{L^2(\Omega)} + C\|f\|^2_{L^2(\Omega_T)}.$$

6. Under the assumptions of the previous exercise, show the estimate

$$\|u_t\|_{L^2(\Omega)} \leq C\|Dg\|_{L^2(\Omega)} + C\|f\|_{L^2(\Omega_T)}.$$