LECTURE EXERCISES
PARTIAL DIFFERENTIAL EQUATIONS 2, 2019

1. [1] Recall the notations $m(E) = |E|$, $\fint_{B(0,\varepsilon)} u(y)\,dy$, $\operatorname{spt} f$. Let Ω be an open bounded set. Recall $C(\Omega), C_0(\Omega), C^k(\Omega), C^\infty(\Omega), C_0^\infty(\Omega)$. Also recall $D^\alpha u$ where α is a multi-index.
2. Give an example of $f : (0,1) \to \mathbb{R}$ such that $f \in L^1_{\text{loc}}((0,1))$ but $f \notin L^1((0,1))$
3. Recall Hölder's inequality. Show for $1 \le p < q < \infty$ that $f \in L^q((0,1))$ implies $f \in L^p((0,1))$.
4. Show that for $1 \le p < q < \infty$

$$
\begin{aligned}
&f \in L^p((0,1)) \text{ does not imply } f \in L^q((0,1)) \\
&f \in L^q((1,\infty)) \text{ does not imply } f \in L^p((1,\infty)).
\end{aligned}
$$

5. Recall Young's inequality. Let $1 < p, q < \infty$ where p, q are Hölder conjugates. Show that if $f \in L^p(\Omega)$, $g \in L^q(\Omega)$, and

$$\int_\Omega |f|^p \, dx \le \int_\Omega |fg| \, dx,$$

then there exists $C > 0$ such that

$$\int_\Omega |f|^p \, dx \le C \int_\Omega |g|^q \, dx.$$

6. Recall the definition of a weak derivative. Show that for

$$u : (0,2) \to \mathbb{R}, \quad u(x) = \begin{cases} x, & 0 < x \le 1 \\ 1, & 1 < x < 2. \end{cases}$$

a weak derivative is

$$u'(x) = \begin{cases} 1, & 0 < x \le 1 \\ 0, & 1 < x < 2. \end{cases}$$

7. Show that u does not have a weak derivative, when

$$u : (0,2) \to \mathbb{R}, \quad u(x) = \begin{cases} x, & 0 < x \le 1 \\ 2, & 1 < x < 2. \end{cases}$$

8. Recall the definition of the Sobolev space and the norm.

[1]These exercises help to review background material and are different from the actual homework exercises.

9. Describe, how to study for which values of p

$$u : B(0,1) \to [0,\infty], \quad u(x) = |x|^{-\beta}, x \in \mathbb{R}^n, \beta > 0, n \geq 2$$

belongs to $W^{1,p}(B(0,1))$.