PARTIAL DIFFERENTIAL EQUATIONS 2, MATS340

MIKKO PARVIAINEN
UNIVERSITY OF JYVÄSKYLÄ

Contents

Date: January 23, 2019, PDE2 lecture note in 2013, 2017 and 2019 at JyU.

1. Introduction

This lecture note contains a sketch of the lectures. More illustrations and examples are presented during the lectures.

Partial differential equations (PDEs) have a great variety of applications to mechanics, electrostatics, quantum mechanics and many other fields of physics as well as to finance.

In addition, PDEs have a rich mathematical theory. In the ICM at 1900, a German mathematician David Hilbert published part of a nowadays legendary list of 23 mathematical problems that have been very influential for 20th century mathematics. We are interested in particular with the problems:

(1) 20th problem: Has not every regular variational problem a solution provided certain assumptions regarding the given boundary conditions, and provided that, if needed, the notion of solutions shall be suitably extended?
(2) 19 th problem: Are the solutions of regular problems in the calculus of variations always necessarily analytic?

Comments:

- Variational problems and PDEs have a tight connection. We will return to this later.
- As Hilbert suggested, in most of the cases we will have to relax the definition of the solution to PDEs to obtain **existence** of a solution. Still we would like to preserve the **uniqueness** and to some extend **regularity** and **stability**. These are the question we will deal with in this course.

2. Sobolev spaces

2.1. Notations.

DOM = Lebesgue's dominated convergence theorem,

$\Omega \subset \mathbb{R}^n$ open set, bounded unless otherwise stated

$|x| = \sqrt{x_1^2 + \ldots + x_n^2}$ for $x \in \mathbb{R}^n$,

$m(E) = |E| =$ a Lebesgue measure of a set E

$$\fint_{B(0,\varepsilon)} \ldots \, dy = \frac{1}{|B(0,\varepsilon)|} \int_{B(0,\varepsilon)} \ldots \, dy$$

$f : \Omega \to \mathbb{R}$ a function

$\operatorname{spt} f = \overline{\{x \in \Omega \, : \, f(x) \neq 0\}} =$ the support of f

$C(\Omega) = \{f \, : \, f \text{ continuous in } \Omega\}$

$$\begin{aligned} C_0(\Omega) &= \{f \in C(\Omega) \,:\, \operatorname{spt} f \text{ is compact subset of } \Omega\} \\ C^k(\Omega) &= \{f \in C(\Omega) \,:\, f \text{ is } k \text{ times continuously differentiable}\} \\ C_0^k(\Omega) &= C^k(\Omega) \cap C_0(\Omega) \\ C^\infty(\Omega) &= \cap_{k=1}^\infty C^k(\Omega) = \text{smooth functions} \end{aligned}$$

$$C_0^\infty(\Omega) = C^\infty(\Omega) \cap C_0(\Omega) = \text{compactly supported smooth functions}$$

Remark 2.1. *Recall that*

$$u \in C^k(\Omega) \iff D^\alpha u \in C(\Omega)$$

for multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ *and* $|\alpha| := \alpha_1 + \ldots + \alpha_n \leq k$, *where*

$$D^\alpha u := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}.$$

Example 2.2 (Warning)**.** *It is not always the case that* $\operatorname{spt} f \subset \Omega$.

Example 2.3. *(1)*

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$$

$$f \in C^1(\Omega) \setminus C^2(\Omega)$$

(2)

$$\varphi : \mathbb{R}^n \to \mathbb{R}, \quad \varphi(x) = \begin{cases} e^{1/(|x|^2-1)}, & |x| < 1 \\ 0, & |x| \geq 1. \end{cases}$$

$$\varphi \in C_0^\infty(\Omega), \operatorname{spt} \varphi \subset \overline{B}(0,1)$$

Exercise.

2.2. **Reminders (from the Measure and Integration).** Let E be Lebesgue measurable, $1 \leq p \leq \infty$, and $f : E \to [-\infty, \infty]$ a Lebesgue measurable function. Then we define

$$||f||_{L^p(E)} = \begin{cases} \left(\int_E |f|^p \, dx \right)^{1/p}, & p < \infty \\ \operatorname{ess\,sup}_E |f|, & p = \infty. \end{cases}$$

where

$$\operatorname*{ess\,sup}_E |f| := \inf\{M \,:\, |f| \leq M \text{ a.e. in } E\}.$$

Then we define $L^p(E)$ to be a a linear space of all Lebesgue measurable functions $f : E \to [-\infty, \infty]$ for which

$$||f||_{L^p(E)} < \infty.$$

If we identify functions that coincide a.e., then this will be a Banach space with the norm defined above.

We also recall

$$L^p_{\text{loc}}(E) := \{f : E \to [-\infty, \infty] \, : \, f \in L^p(F) \text{ for each } F \Subset E\},$$

where $\Subset$ means that $\overline{F}$ is a compact subset of E.

Remark 2.4. *There is usually no inclusions between L^p spaces:*

$$L^p \nsubseteq L^q \qquad L^q \nsubseteq L^p.$$

This can be seen by recalling that

$$x^\alpha \in L^1((0,1)) \iff \alpha > -1$$
$$x^\alpha \in L^1((1,\infty)) \iff \alpha < -1.$$

Thus if we let $1 \le p < q \le \infty$ and choose $\beta > 0$ such that

$$-\frac{1}{q} > -\beta > -\frac{1}{p}$$

we have

$$x^{-\beta} \in L^p((0,1)), \text{ but } x^{-\beta} \notin L^q((0,1))$$
$$x^{-\beta} \notin L^p((1,\infty)), \text{ but } x^{-\beta} \in L^q((1,\infty)).$$

Nonetheless, Hölder's inequality is often a useful tool:

$$||fg||_{L^1} \le ||f||_{L^p} \, ||g||_{L^q} \,,$$

that is

$$\int |fg| \, dx \le \Big(\int |f|^p \, dx \Big)^{1/p} \Big(\int |g|^q \, dx \Big)^{1/q},$$

where $1 \le p, q \le \infty$ are Hölder-conjugates that is

$$\frac{1}{q} + \frac{1}{p} = 1.$$

This implies, in particular, for $1 \le p' < q' \le \infty$ and for a set $|E| < \infty$ that

$$f \in L^{q'}(E) \Rightarrow f \in L^{p'}(E)$$

because $(1 - p'/q' = (q' - p')/q')$

$$\begin{aligned}\int_E |f|^{p'}\,dx &\leq \Big(\int_E 1^{q'/(q'-p')}\,dx\Big)^{(q'-p')/q'}\Big(\int_E |f|^{q'}\,dx\Big)^{p'/q'}\\ &\leq |E|^{(q'-p')/q'}\,||f||^{p'}_{L^{q'}(E)}.\end{aligned}$$

Also the following inequalities are worth recalling. Young's inequality: for each $\varepsilon > 0$*,* $1 < p, q < \infty, 1/p + 1/q = 1$ *and* $a, b \geq 0$ *it holds*

$$ab \leq \varepsilon a^p + Cb^q,$$

where $C = C(\varepsilon, p, q)$ *(meaning that* C *depends on the quantities in the parenthesis). Minkowski's inequality: for* $1 \leq p \leq \infty$ *and* $f, g \in L^p(E)$ *it holds that*

$$||f + g||_{L^p(E)} \leq ||f||_{L^p(E)} + ||g||_{L^p(E)}.$$

2.3. **Weak derivatives.** Let $u \in C^1(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$. Then by integrating by parts

$$\int_\Omega u\frac{\partial\varphi}{\partial x_i}\,dx = -\int_\Omega \frac{\partial u}{\partial x_i}\varphi\,dx, \quad \text{for } i = 1, \ldots, n.$$

Observe that φ vanishes at the boundary and thus there is no boundary term above.

More generally for multi-index α, $|\alpha| \leq k$, and $u \in C^k(\Omega)$, we have

$$\int_\Omega uD^\alpha\varphi\,dx = (-1)^{|\alpha|}\int_\Omega D^\alpha u\varphi\,dx.$$

Remark 2.5. *Observe that the left hand side does not require* u *to be continuously differentiable. This will be our starting point for defining weak derivatives for functions that are not continuous differentiable.*

Definition 2.6. *Let* $u, v \in L^1_{loc}(\Omega)$ *and* α *a multi-index. Then* v *is* α*th weak partial derivative of* u *if*

$$\int_\Omega uD^\alpha\varphi\,dx = (-1)^{|\alpha|}\int_\Omega v\varphi\,dx,$$

for every test function $\varphi \in C_0^\infty(\Omega)$*. We denote*

$$D^\alpha u := v.$$

We denote weak partial derivatives with the familiar notation

$$\frac{\partial u}{\partial x_i}.$$

We also use

$$Du = (\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n})$$

for the weak gradient.

Example 2.7.

$$u : (0,2) \to \mathbb{R}, \quad u(x) = \begin{cases} x, & 0 < x \le 1 \\ 1, & 1 < x < 2. \end{cases}$$

We claim that a weak derivative is

$$u'(x) = v(x) = \begin{cases} 1, & 0 < x \le 1 \\ 0, & 1 < x < 2. \end{cases}$$

This is in L^1_{loc} *so by definition, the task is to show that*

$$\int_{(0,2)} v\varphi \, dx = -1 \int_{(0,2)} u\varphi' \, dx.$$

To see this, we calculate using the integration by parts

$$\begin{aligned}
\int_{(0,2)} u\varphi' \, dx &\overset{DOM}{=} \lim_{\varepsilon \to 0} \Big(\int_{(0,1-\varepsilon)} u\varphi' \, dx + \int_{(1+\varepsilon,2)} u\varphi' \, dx \Big) \\
&= u(1)\varphi(1) - \underbrace{u(0)\varphi(0)}_{0} + \underbrace{u(2)\varphi(2)}_{0} - u(1)\varphi(1) \\
&\quad - \int_{(0,1)} \underbrace{u'}_{1} \varphi \, dx - \int_{(1,2)} \underbrace{u'}_{0} \varphi \, dx \\
&= - \int_{(0,1)} \varphi \, dx \\
&= - \int_{(0,2)} v\varphi \, dx.
\end{aligned}$$

Note that above $u \notin C^1((0,2))$ *and* $u' \notin C((0,2))$. *Also observe that weak derivatives are only defined a.e. and thus it is irrelevant what is the point value for example at 1.*

We found one weak derivative but could there be several? Answer: **No**, weak derivatives are unique up to a set of measure zero.

Theorem 2.8. *A weak* α*th derivate of* u *is uniquely defined up to a set of measure zero.*

Proof. Suppose that $v, \overline{v} \in L^1_{\text{loc}}(\Omega)$ satisfy

$$\begin{aligned}
\int_\Omega u D^\alpha \varphi \, dx &= (-1)^{|\alpha|} \int_\Omega v\varphi \, dx \\
&= (-1)^{|\alpha|} \int_\Omega \overline{v}\varphi \, dx
\end{aligned}$$

for all $\varphi \in C_0^\infty(\Omega)$. It follows that

$$\int_\Omega (v - \overline{v})\varphi \, dx = 0$$

for every $\varphi \in C_0^\infty(\Omega)$. This implies that $v = \overline{v}$ a.e. by the following reason:
Let $\Omega' \Subset \Omega$ and observe that $C_0^\infty(\Omega')$ is dense in $L^1(\Omega')$. Indeed, then there exists

$$\varphi_i \in C_0^\infty(\Omega'), \quad |\varphi_i| \leq 2$$

such that

$$\varphi_i \to \operatorname{sign}(v - \overline{v}) \quad \text{a.e. in } \Omega',$$

(more about approximations later) where

$$\operatorname{sign}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0. \end{cases}$$

Then

$$\begin{aligned} 0 &= \lim_i \int_{\Omega'} (v - \overline{v})\varphi_i \, dx \\ &\overset{\text{DOM, below}}{=} \int_{\Omega'} \lim_i ((v - \overline{v})\varphi_i) \, dx \\ &= \int_{\Omega'} (v - \overline{v}) \operatorname{sign}(v - \overline{v}) \, dx \\ &= \int_{\Omega'} |v - \overline{v}| \, dx, \end{aligned}$$

where the use of DOM is based on $|(v - \overline{v})\varphi_i| \leq 2(|v| + |\overline{v}|) \in L^1(\Omega')$. This implies that $v = \overline{v}$ a.e. in Ω', for any $\Omega' \Subset \Omega$, and thus a.e. in Ω. □

The above proof also yields a useful result.

Lemma 2.9 (Fundamental lemma in calc var). *If $f \in L^1_{loc}(\Omega)$, and*

$$\int_\Omega f\varphi \, dx = 0$$

for every $\varphi \in C_0^\infty(\Omega)$, then $f = 0$ a.e.

Example 2.10.

$$u : (0, 2) \to \mathbb{R}, \quad u(x) = \begin{cases} x & 0 < x \leq 1 \\ 2 & 1 < x < 2. \end{cases}$$

This time u' does not exist even in the weak sense.

Counterproposition: Suppose that there is $v \in L^1_{loc}(\Omega)$ such that

$$\int_{(0,2)} u\varphi' \, dx = -1 \int_{(0,2)} v\varphi \, dx,$$

for every test function $\varphi \in C_0^\infty(\Omega)$. Then

$$\begin{aligned}
\int_{(0,2)} v\varphi \, dx &= -\int_{(0,2)} u\varphi' \, dx \\
&= -\int_{(0,1)} u\varphi' \, dx - \int_{(1,2)} u\varphi' \, dx \\
&= -\int_{(0,1)} x\varphi' \, dx - \int_{(1,2)} 2\varphi' \, dx \\
&= -\varphi(1) + 2\varphi(1) + \int_{(0,1)} \varphi \, dx \\
&= \varphi(1) + \int_{(0,1)} \varphi \, dx.
\end{aligned}$$

Then we can choose a sequence $\varphi_i \in C_0^\infty(\Omega), |\varphi_i| \leq 2$ such that $\varphi_i(1) = 1$ and $\varphi_i(x) \to 0$ if $x \neq 1$. We obtain the desired contradiction by calculating

$$\begin{aligned}
0 &= \lim_i \Big(\int_{(0,2)} v\varphi_i \, dx - \int_{(0,1)} \varphi_i \, dx - \varphi_i(1) \Big) \\
&\overset{DOM}{=} \Big(\int_{(0,2)} v \lim_i \varphi_i \, dx - \int_{(0,1)} \lim_i \varphi_i \, dx - 1 \Big) \\
&= 0 - 0 - 1 = -1.
\end{aligned}$$

The Sobolev spaces are named after a Soviet mathematician S.L. Sobolev for his significant contributions to the theory starting 1930's.

Definition 2.11 (Sobolev space)**.** *Let $1 \leq p \leq \infty$ and $k \in \mathbb{N}$. A function $u : \Omega \to [-\infty, \infty]$ belongs to a Sobolev space $W^{k,p}(\Omega)$ if $u \in L^p(\Omega)$ and its weak derivatives $D^\alpha u$, $|\alpha| \leq k$ exist and belong to $L^p(\Omega)$.*

The function u belongs to the local Sobolev space $W^{k,p}_{loc}$, if $u \in W^{k,p}(\Omega')$ for each $\Omega' \Subset \Omega$.

Remark 2.12.

1. *(1) Sobolev functions are only defined up to a measure zero similarly as L^p functions.*
2. *(2) Notation $H^k := W^{k,2}$ as well as some further variants are encountered in the literature*

Example 2.13. *For the function in Example 2.7, it holds*

$$u \in W^{1,p}((0,2)) \quad \textit{for every } p \geq 1$$

and

$$u \notin W^{k,p}((0,2)) \quad \textit{for any } k \geq 2.$$

Example 2.14.

$$u : B(0,1) \to [0,\infty], \quad u(x) = |x|^{-\beta}, x \in \mathbb{R}^n, \beta > 0, n \geq 2$$

will be in a Sobolev space for a suitable β. When $x \neq 0$

$$\frac{\partial u}{\partial x_i}(x) = -\beta |x|^{-\beta-1} \frac{x_i}{|x|} = -\beta \frac{x_i}{|x|^{\beta+2}}$$

as well as

$$Du(x) = -\beta \frac{x}{|x|^{\beta+2}}.$$

We aim at showing that this function satisfies the definition of the weak derivative but we will have to be careful with the singularity. Therefore let $\varphi \in C_0^\infty(B(0,1))$ and use Gauss' theorem

$$\int_{B(0,1)\setminus\overline{B}(0,\varepsilon)} \frac{\partial(u\varphi)}{\partial x_i}\, dx = \int_{\partial\left((B(0,1)\setminus\overline{B}(0,\varepsilon)\right)} u\varphi\nu_i\, dS$$

where $\nu = (\nu_1, \ldots, \nu_n)$ is the outer unit normal vector of the boundary. Recalling that $\varphi = 0$ on $\partial B(0,1)$ we get

$$\int_{B(0,1)\setminus\overline{B}(0,\varepsilon)} \frac{\partial u}{\partial x_i}\varphi\, dx = -\int_{B(0,1)\setminus\overline{B}(0,\varepsilon)} u \frac{\partial \varphi}{\partial x_i}\, dx + \int_{\partial B(0,\varepsilon)} u\varphi\nu_i\, dS \tag{2.1}$$

If we can pass to the limit $\varepsilon \to 0$ and to show that $\int_{\partial B(0,\varepsilon)} u\varphi\nu_i\, dS \to 0$, we are done. To establish this we estimate

$$\begin{aligned} \left|\int_{\partial B(0,\varepsilon)} u\varphi\nu_i\, dS\right| &\leq ||\varphi||_{L^\infty(B(0,1))} \int_{\partial B(0,\varepsilon)} \varepsilon^{-\beta}\, dS \\ &\leq ||\varphi||_{L^\infty(B(0,1))}\, \omega_{n-1}\varepsilon^{n-1-\beta} \to 0 \end{aligned}$$

as $\varepsilon \to 0$, *if* $n-1-\beta > 0$. *Next we calculate*

$$
\begin{aligned}
\int_{B(0,1)} \left|\frac{\partial u}{\partial x_i}\right| dx &= \int_{B(0,1)} \beta \frac{|x_i|}{|x|^{\beta+2}}\, dx \\
&\le \beta \int_{B(0,1)} \frac{|x|}{|x|^{\beta+2}}\, dx \\
&= \beta \int_{B(0,1)} \frac{1}{|x|^{\beta+1}}\, dx \\
&= \beta \int_0^1 \int_{\partial B(0,\rho)} \frac{1}{\rho^{\beta+1}}\, dS\, d\rho \\
&= \beta \int_0^1 \omega_{n-1}\rho^{n-2-\beta}\, d\rho \\
&= \beta\omega_{n-1} \Big/_0^1 \frac{\rho^{n-1-\beta}}{n-1-\beta} < \infty,
\end{aligned}
\tag{2.2}
$$

whenever $n-1-\beta > 0$. *Thus, we have integrable upper bound for* $\chi_{B(0,1)\setminus \overline{B}(0,\varepsilon)} \frac{\partial u}{\partial x_i}$ *and we have*

$$
\begin{aligned}
\lim_{\varepsilon\to 0} \int_{B(0,1)\setminus \overline{B}(0,\varepsilon)} \frac{\partial u}{\partial x_i}\varphi\, dx &\overset{DOM}{=} \int_{B(0,1)} \lim_{\varepsilon\to 0} \chi_{B(0,1)\setminus \overline{B}(0,\varepsilon)} \frac{\partial u}{\partial x_i}\varphi\, dx \\
&= \int_{B(0,1)} \frac{\partial u}{\partial x_i}\varphi\, dx
\end{aligned}
$$

Similarly as in (2.2), *we see that*

$$
\begin{aligned}
\int_{B(0,1)} |u|\, dx &= \int_0^1 \omega_{n-1}\rho^{n-1-\beta}\, d\rho \\
&= \omega_{n-1} \Big/_0^1 \frac{\rho^{n-\beta}}{n-\beta} < \infty,
\end{aligned}
$$

whenever $n-\beta > 0$. *Thus we can again pass to the limit*

$$
\lim_{\varepsilon\to 0} \int_{B(0,1)\setminus \overline{B}(0,\varepsilon)} u\frac{\partial \varphi}{\partial x_i}\, dx \overset{DOM}{=} \int_{B(0,1)} u\frac{\partial \varphi}{\partial x_i}\, dx.
$$

Recalling (2.1), *passing to the limit* $\varepsilon \to 0$ *and combining the above estimates, we deduce*

$$
\int_{B(0,1)} \frac{\partial u}{\partial x_i}\varphi\, dx = -\int_{B(0,1)} u\frac{\partial \varphi}{\partial x_i}\, dx + 0
$$

for all $\varphi \in C_0^\infty(\Omega)$.

By modifying calculation (2.2), we have

$$\frac{\partial u}{\partial x_i} \in L^p \iff n - p(\beta+1) > 0 \iff \beta < \frac{n-p}{p}$$

and

$$u \in L^p(\Omega) \iff n - p\beta > 0 \iff \frac{n}{p} > \beta.$$

As a conclusion

$$u \in W^{1,p}(B(0,1)) \iff \beta < \frac{n-p}{p}$$

Observe: *If $p \geq n$, then $u \notin W^{1,p}(B(0,1))$ for all $\beta > 0$. Actually, we will later see that when $p > n$, Sobolev functions have a Hölder continuous representative.*

Example 2.15. *A Sobolev function can be rather singular! Indeed, let q_i be a set of points with rational coordinates in $B(0,1) \subset \mathbb{R}^n$. Then for*

$$u : B(0,1) \to [0,\infty], \quad u(x) = \sum_{i=1}^{\infty} \frac{1}{2^i} |x - q_i|^{-\beta}$$

holds

$$u \in W^{1,p}(B(0,1)) \iff \beta < \frac{n-p}{p}.$$

Observe: *u explodes at every rational point!*

Example 2.16. *Without a proof, we state that Cantor function is* not *in $W^{1,1}(0,1)$.*

Theorem 2.17 (Calculation rules)**.** *Let $u, v \in W^{k,p}(\Omega)$ and $|\alpha| \leq k$. Then*

(1) $D^\alpha u \in W^{k-|\alpha|,p}(\Omega)$.
(2) $D^\alpha(D^\beta u) = D^\beta(D^\alpha u)$ for all multi-indexes with $|\alpha| + |\beta| \leq k$.
(3) Let $\lambda, \mu \in \mathbb{R}$. Then $\lambda u + \mu v \in W^{k,p}(\Omega)$ and

$$D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v.$$

(4) If $\xi \in C_0^\infty(\Omega)$, then $\xi u \in W^{k,p}(\Omega)$ and

$$D^\alpha(\xi u) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \xi \, D^{\alpha-\beta} u$$

where

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}, \quad \alpha! = \alpha_1! \cdot \ldots \cdot \alpha_n!$$

and $\beta \leq \alpha$ means $\beta_i \leq \alpha_i$ for each $i = 1, \ldots, n$.

Proof. (1) Clear.

(2) Let $\varphi \in C_0^\infty(\Omega)$. By the first statement, the weak derivatives exist and

$$\begin{aligned}(-1)^{|\beta|}\int_\Omega D^\beta D^\alpha u\varphi\,dx &\overset{\varphi \text{ smooth}}{=} (-1)^{|\alpha|}\int_\Omega uD^\beta D^\alpha \varphi\,dx \\ &\overset{\text{def}}{=} (-1)^{|\alpha|}(-1)^{|\alpha|+|\beta|}\int_\Omega \varphi D^\alpha D^\beta u\,dx \\ &= (-1)^{|\beta|}\int_\Omega \varphi D^\alpha D^\beta u\,dx.\end{aligned}$$

(3) Clear.

(4) When $|\alpha| = 1$, then (4) says

$$D^\alpha(\xi u) = uD^\alpha \xi + \xi D^\alpha u$$

which follows from the definition by observing

$$\begin{aligned}\int_\Omega \xi u D^\alpha \varphi\,dx &= \int_\Omega uD^\alpha(\xi\varphi) - u\varphi D^\alpha \xi\,dx \\ &= -\int_\Omega \xi D^\alpha u\varphi\,dx - \int_\Omega u(D^\alpha \xi)\varphi\,dx \\ &= -\int_\Omega (\xi D^\alpha u + uD^\alpha \xi)\varphi\,dx.\end{aligned}$$

The rest follows by induction, but details are omitted.

□

Remark 2.18 (Reminder). *Vector space with the norm satisfying*

(1) $0 \leq ||u|| < \infty$
(2) $||u|| = 0 \iff u = 0$
(3) $||cu|| = |c|\,||u||$ *for each* $c \in \mathbb{R}$
(4) $||u + v|| \leq ||u|| + ||v||$

is a normed vector space. If, in addition, the space is complete, it is called Banach *space. Completeness means that all of its Cauchy sequences converge.*

Definition 2.19 (Sobo norm). *If $u \in W^{k,p}(\Omega)$, we define its norm to be*

$$||u||_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha|\leq k}\int_\Omega |D^\alpha u|^p\,dx\right)^{1/p} & 1 \leq p < \infty \\ \sum_{|\alpha|\leq k} \operatorname{ess\,sup}_\Omega |D^\alpha u| & p = \infty. \end{cases}$$

Remark 2.20. *The norm* $||u||_{W^{k,p}(\Omega)}$ *is equivalent with the norm*

$$\sum_{|\alpha|\le k}\Big(\int_\Omega |D^\alpha u|^p\,dx\Big)^{1/p} \quad \textit{if } 1\le p\le\infty.$$

Further in the case $p=\infty$ *the norm* $||u||_{W^{k,\infty}(\Omega)}$ *is equivalent with*

$$\max_{|\alpha|\le k}\operatorname*{ess\,sup}_{\Omega}|D^\alpha u| = \max_{|\alpha|\le k}||D^\alpha u||_{L^\infty(\Omega)}\,.$$

Definition 2.21. *Let* $u_i, u\in W^{k,p}(\Omega)$*. We say that* u_i *converges to* u *in* $W^{k,p}(\Omega)$ *denoted by*

$$u_i\to u \quad \textit{in } W^{k,p}(\Omega),$$

if

$$\lim_{i\to\infty}||u-u_i||_{W^{k,p}(\Omega)}=0.$$

Let $u_i, u\in W^{k,p}_{loc}(\Omega)$*. We say that* u_i *converges to* u *locally in* $W^{k,p}(\Omega)$ *denoted by*

$$u_i\to u \quad \textit{in } W^{k,p}_{loc}(\Omega),$$

if

$$\lim_{i\to\infty}||u-u_i||_{W^{k,p}(\Omega')}=0$$

for every $\Omega'\Subset\Omega$*.*

The space $C^1(\Omega)$ is not complete with respect to the Sobolev norm: to see this approximate in Example 2.7 the weak derivative by a smooth function v_i in L^p. Then by integrating v_i, we obtain $u_i\in C^1((0,2))$ so that

$$u_i\to u \quad \text{in } W^{1,p}((0,2)),$$

but clearly $u\notin C^1((0,2))$. However, the Sobolev space 'fixes' this issue.

Theorem 2.22. *The Sobolev space* $W^{k,p}(\Omega)$ *is a Banach space.*

Proof. First we check that $||u||_{W^{k,p}(\Omega)}$ is a norm.

(1) $||u||_{W^{k,p}(\Omega)}=0 \iff u=0$ a.e. in Ω
"$\Rightarrow$"
$||u||_{W^{k,p}(\Omega)}=0$ implies that $||u||_{L^p(\Omega)}=0$ and this implies by Chebysev's inequality (see Measure and integration 1) that $u=0$ a.e. in Ω.

"$\Leftarrow$"
Suppose that $u = 0$ a.e. in Ω. Then

$$0 = \int_\Omega u D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_\Omega 0 \varphi \, dx$$

for every $\varphi \in C_0^\infty(\Omega)$, i.e. $D^\alpha u = 0$.

(2) $||\lambda u||_{W^{k,p}(\Omega)} = |\lambda| \, ||u||_{W^{k,p}(\Omega)}$ is clear.

(3) Let ($1 \le p < \infty$, if $p = \infty$ a similar proof applies). Then

$$\begin{aligned}
||u+v||_{W^{k,p}(\Omega)} &\le \Big(\sum_{|\alpha| \le k} ||D^\alpha u + D^\alpha v||^p_{L^p(\Omega)} \Big)^{1/p} \\
&\overset{\text{Minkowski}}{\le} \Big(\sum_{|\alpha| \le k} \big(||D^\alpha u||_{L^p(\Omega)} + ||D^\alpha v|| \big)^p_{L^p(\Omega)} \Big)^{1/p} \\
&\overset{\text{Minkowski for } |\cdot|_p}{\le} \big(\sum_{|\alpha| \le k} ||D^\alpha u||^p_{L^p(\Omega)} \big)^{1/p} + \big(\sum_{|\alpha| \le k} ||D^\alpha v||^p_{L^p(\Omega)} \big)^{1/p}.
\end{aligned}$$

Next we show that if u_i is a Cauchy sequence in $W^{k,p}(\Omega)$, then it converges in $W^{k,p}(\Omega)$ i.e. $W^{k,p}(\Omega)$ is complete. To this end, let u_i be a Cauchy sequence in $W^{k,p}(\Omega)$.
Claim: $D^\alpha u_i$ is a Cauchy sequence in $L^p(\Omega)$ for each $\alpha, |\alpha| \le k$.

Proof: This follows by fixing $\varepsilon > 0$ and observing that

$$||D^\alpha u_i - D^\alpha u_j||_{L^p(\Omega)} \le ||u_i - u_j||_{W^{k,p}(\Omega)} < \varepsilon$$

whenever i, j are large enough, since u_i is a Cauchy sequence in $W^{k,p}(\Omega)$.///
The space L^p is complete and thus there exists $u_\alpha \in L^p(\Omega)$ such that

$$D^\alpha u_i \to g_\alpha \quad \text{in } L^p(\Omega).$$

In particular for $\alpha = 0$

$$u_i \to u \quad \text{in } L^p(\Omega).$$

Claim: g_α is the weak derivative $D^\alpha u$
Proof: Let $\varphi \in C_0^\infty(\Omega)$

$$\frac{1}{p} + \frac{1}{q} = 1, \quad p, q \ge 1$$

and observe that

$$\left| \int_\Omega (u - u_i) D^\alpha \varphi \, dx \right| \overset{\text{Hölder}}{\le} \big(\int_\Omega |u - u_i|^p \, dx \big)^{1/p} \big(\int_\Omega |D^\alpha \varphi|^q \, dx \big)^{1/q} \to 0 \tag{2.3}$$

by L^p convergence. Thus

$$\begin{aligned}\int_\Omega u D^\alpha \varphi\, dx &\overset{(2.3)}{=} \lim_i \int_\Omega u_i D^\alpha \varphi\, dx \\ &= \lim_i (-1)^{|\alpha|} \int_\Omega D^\alpha u_i \varphi\, dx \\ &\overset{\text{sim to } (2.3)}{=} (-1)^{|\alpha|} \int_\Omega g_\alpha \varphi\, dx.\end{aligned}$$

This completes the proof of the auxiliary claim.///

We have shown that $D^\alpha u := g_\alpha \in L^p(\Omega)$ exists and

$$D^\alpha u_i \to g_\alpha = D^\alpha u \quad \text{in } L^p(\Omega)$$

as desired. □

Remark 2.23 (Warning). *The Sobolev space $W^{k,p}(\Omega)$ is not compact in the sense that from*

$$||u_i||_{W^{k,p}(\Omega)} \leq C < \infty \tag{2.4}$$

it does not follow that there would be $u \in W^{k,p}(\Omega)$ and a subsequence such that

$$u_i \to u \quad \textit{in } W^{k,p}(\Omega).$$

If this were true some existence results would be much easier. For example, the functions

$$u_i : (0,2) \to \mathbb{R}, \quad u_i(x) = \begin{cases} 0 & 0 < x \leq 1 \\ (x-1)i & 1 \leq x \leq 1 + 1/i \\ 1 & 1 + 1/i < x < 2 \end{cases} \tag{2.5}$$

are in $W^{1,1}((0,2))$ and furthermore

$$||u_i||_{W^{1,1}((0,2))} \leq 2.$$

However, there is no in $W^{1,1}((0,2))$ convergent subsequence. If there was a limit, it should be (to have even L^1 convergence)

$$u(x) = \begin{cases} 0 & 0 < x \leq 1 \\ 1 & 1 < x < 2 \end{cases}$$

but this is not in $W^{1,1}((0,2))$.

When $p > 1$, $W^{k,p}(\Omega)$ is a reflexive Banach space and thus from (2.4) it follows that there is weakly convergent subsequence u_i (consequence *of Banach-Alaoglu's theorem). Especially, there is the weak limit* $u \in W^{k,p}(\Omega)$ *such that*

$$||u||_{W^{k,p}} \leq \liminf_i ||u_i||_{W^{k,p}}.$$

We omit the details here but observe that (2.5) *shows that this fails in the case* $p = 1$*. By modifying the example to be*

$$u_i(x) = \begin{cases} 0 & 0 < x \leq 1 \\ (x-1)\sqrt{i} & 1 \leq x \leq 1 + 1/i \\ 1/\sqrt{i} & 1 + 1/i < x < 2, \end{cases}$$

we have $u_i \in W^{1,2}((0,2))$*,* $||u_i||_{W^{1,2}((0,2))} \leq C$ *and*

$$u_i \to u \quad \text{weakly in } W^{1,2}(\Omega),$$

where $u = 0$*. It clearly holds that*

$$0 = ||u||_{W^{1,2}((0,2))} \leq \liminf_i ||u_i||_{W^{1,2}((0,2))}.$$

Observe carefully that strong convergence does not hold

$$u_i \nrightarrow u \quad \text{in } W^{1,2}((0,2)).$$

2.4. **Approximations.**

Below we denote

$$\Omega_\varepsilon = \{x \in \Omega \,:\, \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$$

which is an open set by continuity of $\operatorname{dist}(x, \partial\Omega)$.

Definition 2.24 (Standard mollifier)**.** *Let*

$$\eta : \mathbb{R}^n \to \mathbb{R}, \quad \eta(x) = \begin{cases} Ce^{1/(|x|^2-1)} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

where C *is chosen so that*

$$\int_{\mathbb{R}^n} \eta \, dx = 1.$$

Then we set for $\varepsilon > 0$

$$\eta_\varepsilon(x) := \frac{1}{\varepsilon^n} \eta\left(\frac{x}{\varepsilon}\right)$$

which is called a standard mollifier.

Remark 2.25. *Observe that*

$$\eta_\varepsilon \in C_0^\infty(\mathbb{R}^n), \quad \operatorname{spt} \eta_\varepsilon \subset \overline{B}(0, \varepsilon)$$

and

$$\int_{\mathbb{R}^n} \eta_\varepsilon(x)\,dx = \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n} \eta(\frac{x}{\varepsilon})\,dx \overset{y=x/\varepsilon,\varepsilon^n\,dy=dx}{=} \int_{\mathbb{R}^n} \eta(y)\,dy = 1.$$

Definition 2.26 (Standard mollification)**.** *Let*

$$f:\Omega\to[-\infty,\infty],\quad f\in L^1_{loc}(\Omega).$$

Then we define the standard mollification for f *by*

$$f_\varepsilon:\Omega_\varepsilon\to\mathbb{R},\quad f_\varepsilon := \eta_\varepsilon * f,$$

where $\eta_\varepsilon * f = \int_\Omega \eta_\varepsilon(x-y)f(y)\,dy$ *denotes the convolution for* $x\in\Omega_\varepsilon$.

Theorem 2.27. *The standard mollification has the following properties (*$f\in L^1_{loc}(\Omega)$ *unless otherwise specified)*

(1)

$$D^\alpha f_\varepsilon = f * D^\alpha \eta_\varepsilon \quad \text{in } \Omega_\varepsilon$$

and

$$f_\varepsilon \in C^\infty(\Omega_\varepsilon).$$

(2) Let $f\in L^p(\Omega)$. *Then*

$$f_\varepsilon \to f \quad \text{a.e. in } \Omega.$$

(3) If $f\in C(\Omega)$, *then*

$$f_\varepsilon\to f,\quad \text{uniformly in compact subsets of } \Omega.$$

(4) If $f\in L^p_{loc}(\Omega)$ *for* $1\le p\le\infty$, *then for* $\Omega'\Subset\Omega''\Subset\Omega$

$$||f_\varepsilon||_{L^p(\Omega')}\le ||f||_{L^p(\Omega'')}$$

for small enough $\varepsilon>0$, *and for* $1\le p<\infty$

$$f_\varepsilon\to f \quad \text{in } L^p_{loc}(\Omega).$$

Warning: *The convergence does not hold for* $p=\infty$.

(5) If $f\in W^{k,p}_{loc}(\Omega)$ *for* $1\le p\le\infty$, $k\in\mathbb{N}$, *then*

$$D^\alpha f_\varepsilon = \eta_\varepsilon * D^\alpha f \quad \text{in } \Omega_\varepsilon.$$

(6) If $f\in W^{k,p}_{loc}(\Omega)$, *for* $1\le p<\infty$, $k\in\mathbb{N}$, *then*

$$f_\varepsilon\to f \quad \text{in } W^{k,p}_{loc}(\Omega).$$

Proof. (1) Let

$$x \in \Omega_\varepsilon, \quad e_i = (0, \dots, \underbrace{1}_{ith}, \dots, 0)$$

and $h > 0$ such that $x + he_i \in \Omega_\varepsilon$. Intuitive idea is

$$\frac{\partial f_\varepsilon}{\partial x_i}(x) = \int_\Omega \frac{\partial \eta_\varepsilon(x-y)}{\partial x_i} f(y)\, dy.$$

To make this rigorous we would like to deduce

$$\begin{aligned}
\frac{\partial f_\varepsilon}{\partial x_i}(x) &= \lim_{h\to 0} \frac{f_\varepsilon(x+he_i) - f_\varepsilon(x)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\int_{\Omega'} \eta_\varepsilon(x+he_i-y) f(y)\, dy - \int_{\Omega'} \eta_\varepsilon(x-y) f(y)\, dy \right) \\
&\overset{\text{DOM,below}}{=} \frac{1}{\varepsilon^n} \int_{\Omega'} \lim_{h\to 0} \frac{1}{h}\Big(\eta(\frac{x+he_i-y}{\varepsilon}) - \eta(\frac{x-y}{\varepsilon}) \Big) f(y)\, dy \\
&= \int_{\Omega'} \frac{\partial \eta_\varepsilon(x-y)}{\partial x_i} f(y)\, dy \\
&= \frac{\partial \eta_\varepsilon}{\partial x_i} * f
\end{aligned} \tag{2.6}$$

where $B(x+he_i, \varepsilon) \cup B(x, \varepsilon) \subset \Omega' \Subset \Omega$. For this we need to calculate the limit inside the integral and to look for an integrable upper bound to be able to use DOM:

Claim 1:

$$\frac{1}{h}\Big(\eta(\frac{x+he_i-y}{\varepsilon}) - \eta(\frac{x-y}{\varepsilon}) \Big) \to \frac{1}{\varepsilon} \frac{\partial \eta}{\partial x_i}(\frac{x-y}{\varepsilon}).$$

Proof: This can be seen to hold by setting

$$\psi(x) = \eta(\frac{x-y}{\varepsilon})$$

and the limit is

$$\frac{\partial \psi}{\partial x_i}(x) = \frac{1}{\varepsilon}\frac{\partial \eta}{\partial x_i}(\frac{x-y}{\varepsilon}). \qquad ///$$

Claim 2: $\frac{1}{h}\big(\eta(\frac{x+he_i-y}{\varepsilon}) - \eta(\frac{x-y}{\varepsilon})\big) f(y)$ has an integrable upper bound in Ω'.

Proof:

$$\begin{aligned}\psi(x+he_i)-\psi(x) &= \int_0^h \frac{\partial}{\partial t}\psi(x+te_i)\,dt \\ &= \int_0^h D\psi(x+te_i)\cdot e_i\,dt\end{aligned}$$

Thus

$$|\psi(x+he_i)-\psi(x)| \le h\,||D\psi||_{L^\infty(\Omega)}$$

and

$$\left|\frac{1}{h}\big(\eta(\frac{x+he_i-y}{\varepsilon})-\eta(\frac{x-y}{\varepsilon})\big)f(y)\right| \le ||D\psi||_{L^\infty(\Omega)}\,|f(y)| \in L^1(\Omega'). \qquad ///$$

Thus the use of DOM in (2.6) was correct and the proof is complete. A similar argument shows that for every multi-index α, $D^\alpha f_\varepsilon$ exists and

$$D^\alpha f_\varepsilon = D^\alpha \eta_\varepsilon * f.$$

Moreover, the convolution on the RHS is continuous (ex). Now, repeating the argument for the higher derivatives of $f_\varepsilon \in C^\infty(\Omega_\varepsilon)$ implies the result.

(2) Let $x \in \Omega' \Subset \Omega$ so that the convolution below is well defined for a small enough ε, recall $\int_{\mathbb{R}^n} \eta_\varepsilon\,dy = 1$, and estimate

$$\begin{aligned}|f_\varepsilon(x)-f(x)| &= \left|\int_\Omega \eta_\varepsilon(x-y)f(y)\,dy - f(x)\right| \\ &= \left|\int_\Omega \eta_\varepsilon(x-y)(f(y)-f(x))\,dy\right| \\ &\le ||\eta||_{L^\infty(\mathbb{R}^n)}\,\frac{1}{\varepsilon^n}\int_{B(x,\varepsilon)}|f(y)-f(x)|\,dy \\ &\le C||\eta||_{L^\infty(\mathbb{R}^n)} \fint_{B(0,\varepsilon)}|f(y)-f(x)|\,dy \overset{*}{\to} 0\end{aligned} \tag{2.7}$$

a.e. in Ω, where at $*$ we used Lebesgue's differentiation theorem. Above $\fint_{B(0,\varepsilon)} \dots\,dy := \frac{1}{|B(0,\varepsilon)|}\int_{B(0,\varepsilon)} \dots\,dy$.

(3) Let $\Omega' \Subset \Omega'' \Subset \Omega$. Then f is uniformly continuous on a compact subset $\overline{\Omega}''$. Let $\varepsilon > 0$ be small enough so that for $x \in \Omega'$ we have $B(x,\varepsilon) \subset \Omega''$. By uniform continuity, for any $\delta > 0$, there exists $\varepsilon > 0$ such that

$$|x-y| < \varepsilon \Rightarrow |f(x)-f(y)| < \delta$$

for any $x, y \in \overline{\Omega}''$. Then by this and (2.7), we have

$$\begin{aligned}|f_\varepsilon(x) - f(x)| &\leq C||\eta||_{L^\infty(\mathbb{R}^n)} \fint_{B(x,\varepsilon)} |f(y) - f(x)|\, dy\\ &\leq C||\eta||_{L^\infty(\mathbb{R}^n)} \fint_{B(x,\varepsilon)} \delta\, dy\\ &\leq C\,||\eta||_{L^\infty(\mathbb{R}^n)}\,\delta\end{aligned}$$

independent of $x \in \Omega'$ for all small enough ε.

(4) Let $1 \leq p < \infty$ (bound in the case $p = \infty$ is straightforward) and $x \in \Omega' \Subset \Omega'' \Subset \Omega$. Then

$$\begin{aligned}|f_\varepsilon(x)| &= \left|\int_{B(x,\varepsilon)} \eta_\varepsilon(x-y) f(y)\, dy\right|\\ &\leq \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)^{1-1/p} \eta_\varepsilon(x-y)^{1/p} |f(y)|\, dy\\ &\overset{\text{Hölder}}{\leq} \underbrace{\Big(\int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)\, dy\Big)^{(p-1)/p}}_{1} \Big(\int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)|f(y)|^p\, dy\Big)^{1/p}.\end{aligned}$$

We apply this estimate together with Fubini's/Tonelli's theorem ($\mathbb{R}^{2n}$ measurability ok). Thus, whenever $\varepsilon > 0$ is small enough,

$$\begin{aligned}\int_{\Omega'} |f_\varepsilon(x)|^p\, dx &\leq \int_{\Omega'} \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)|f(y)|^p\, dy\, dx\\ &= \int_{\Omega'} \int_{\Omega''} \eta_\varepsilon(x-y)|f(y)|^p\, dy\, dx\\ &\overset{\text{Fubini}}{=} \int_{\Omega''} \int_{\Omega'} \eta_\varepsilon(x-y)|f(y)|^p\, dx\, dy\\ &= \int_{\Omega''} |f(y)|^p \int_{\Omega'} \eta_\varepsilon(x-y)\, dx\, dy\\ &\leq \int_{\Omega''} |f(y)|^p \underbrace{\int_{\mathbb{R}^n} \eta_\varepsilon(x-y)\, dx}_{1}\, dy\\ &= \int_{\Omega''} |f(y)|^p\, dy.\end{aligned}$$

It remains to show that $f_\varepsilon \to f$ in $L^p_{\text{loc}}(\Omega)$. Recall (not proven here) that $C(\Omega'')$ is dense in $L^p(\Omega'')$ ie. for any $f \in$

$L^p(\Omega'')$ and $\delta > 0$, there exists $g \in C(\Omega'')$ such that

$$\Big(\int_{\Omega''} |f-g|^p \, dy\Big)^{1/p} < \delta/3.$$

From this and the beginning of the proof, we deduce

$$\begin{aligned}
&\Big(\int_{\Omega'} |f-f_\varepsilon|^p \, dx\Big)^{1/p} \\
&\overset{\text{Minkowski}}{\leq} \Big(\int_{\Omega'} |f-g|^p \, dx\Big)^{1/p} + \Big(\int_{\Omega'} |g-g_\varepsilon|^p \, dx\Big)^{1/p} + \Big(\int_{\Omega'} |g_\varepsilon-f_\varepsilon|^p \, dx\Big)^{1/p} \\
&\leq \delta/3 + \Big(\int_{\Omega'} |g-g_\varepsilon|^p \, dx\Big)^{1/p} + \Big(\int_{\Omega''} |g-f|^p \, dx\Big)^{1/p} \\
&\leq \delta/3 + \Big(\int_{\Omega'} |g-g_\varepsilon|^p \, dx\Big)^{1/p} + \delta/3 \\
&\leq \delta/3 + \delta/3 + \delta/3,
\end{aligned}$$

where the last inequality follows from fact we proved earlier: for continuous functions the convergence is uniform and thus

$$\Big(\int_{\Omega'} |g-g_\varepsilon|^p \, dx\Big)^{1/p} \leq \sup_{x\in\Omega'} |g-g_\varepsilon| |\Omega'|^{1/p} \leq \delta/3$$

for small enough ε.

(5) Exercise.
(6) Exercise.

□

2.5. **Global approximation in Sobolev space.** We already stated in Theorem 2.27 (6) that Sobolev functions can be estimated locally by mollifying. At the vicinity of the boundary this does not hold as such since we need some space to mollify. To establish a global approximation the idea is to take smaller and smaller ε when approaching the boundary so that $B(x,\varepsilon(x)) \subset \Omega$ always holds.

Theorem 2.28. *Let $u \in W^{k,p}(\Omega)$ for some $1 \leq p < \infty$. Then there is a sequence $u_i \in C^\infty(\Omega) \cap W^{k,p}(\Omega)$ of functions such that*

$$u_i \to u \quad \textit{in } W^{k,p}(\Omega).$$

Proof. We define

$$\begin{aligned}
\Omega_0 &= \emptyset \\
\Omega_i &= \{x \in \Omega \,:\, \operatorname{dist}(x, \partial\Omega) > 1/i\} \cap B(0,i)
\end{aligned}$$

and observe that Ω_i are bounded sets such that $\Omega_0 \Subset \Omega_1 \Subset \ldots \Subset \Omega$ and

$$\Omega = \bigcup_{i=1}^{\infty} \Omega_i.$$

Claim: There are $\xi_i \in C_0^\infty(\Omega_{i+2} \setminus \overline{\Omega}_{i-1})$ such that

$$0 \le \xi_i \le 1, \quad \sum_{i=1}^{\infty} \xi_i = 1 \text{ in } \Omega.$$

This is called *partition of unity.*
Proof: Clearly we can choose functions $\tilde{\xi}_i \in C_0^\infty(\Omega_{i+2} \setminus \overline{\Omega}_{i-1})$ such that

$$0 \le \tilde{\xi}_i \le 1, \quad \text{and} \quad \tilde{\xi}_i = 1 \text{ in } \overline{\Omega}_{i+1} \setminus \Omega_i.$$

We set

$$\xi_i(x) = \frac{\tilde{\xi}_i(x)}{\sum_{j=1}^{\infty} \tilde{\xi}_j(x)}, \quad i = 1, \ldots$$

Observe that for any fixed $x \in \Omega$, only three terms in the sum will be nonzero. Similarly ξ_i is nonzero at the most for three indices. Then by $\sum_{i=1}^{\infty} \xi_i(x) = \sum_{i=1}^{\infty} \frac{\tilde{\xi}_i(x)}{\sum_{j=1}^{\infty} \tilde{\xi}_j(x)} = 1$ the claim follows.///

We continue with the original proof. By Theorem 2.17 (4) $\xi_i u \in W^{k,p}(\Omega)$ and

$$\operatorname{spt}(\xi_i u) \subset \Omega_{i+2} \setminus \overline{\Omega}_{i-1}.$$

Hence for small enough ε_i

$$\eta_{\varepsilon_i} * (\xi_i u) \in C_0^\infty(\Omega_{i+2} \setminus \overline{\Omega}_{i-1})$$

and

$$||\eta_{\varepsilon_i} * (\xi_i u) - \xi_i u||_{W^{k,p}(\Omega)} \le \frac{\delta}{2^i}.$$

We define

$$v = \sum_{i=1}^{\infty} \eta_{\varepsilon_i} * (\xi_i u).$$

Then it holds that $v \in C^\infty(\Omega)$ because at each point $x \in \Omega$ there are at the most three smooth functions that are nonzero in the sum. Then

$$\begin{aligned} ||v-u||_{W^{k,p}(\Omega)} &\overset{\sum \xi_i = 1}{=} \left|\left| \sum_{i=1}^{\infty} \eta_{\varepsilon_i} * (\xi_i u) - \sum_{i=1}^{\infty} \xi_i u \right|\right|_{W^{k,p}(\Omega)} \\ &\leq \sum_{i=1}^{\infty} ||\eta_{\varepsilon_i} * (\xi_i u) - \xi_i u||_{W^{k,p}(\Omega)} \\ &\leq \sum_{i=1}^{\infty} \frac{\delta}{2^i} \leq \delta. \end{aligned}$$

□

Corollary 2.29 (Approximation characterization of the Sobolev space)**.**

$$u \in W^{k,p}(\Omega)$$

if and only if there exists a sequence $u_i \in C^\infty(\Omega)$ such that

$$u_i \to u \text{ in } W^{k,p}(\Omega).$$

Proof. "$\Rightarrow$": This follows from the previous theorem.
"$\Leftarrow$": u_i is a Cauchy sequence, and since $W^{k,p}(\Omega)$ is a Banach space by Theorem 2.22, it follows that $u \in W^{k,p}(\Omega)$. □

In other words: $W^{k,p}(\Omega)$ can be characterized as a completion of $C^\infty(\Omega)$ (or $\left(C^\infty(\Omega), ||\cdot||_{W^{k,p}(\Omega)}\right)$ to be more precise).

2.6. **Sobolev spaces with zero boundary values: $W_0^{k,p}(\Omega)$.** Above, we showed that $W^{k,p}(\Omega)$ can be characterized as a completion of $C^\infty(\Omega)$. By following this idea, we define Sobolev spaces with zero boundary values as a completion of $C_0^\infty(\Omega)$.

Definition 2.30. *$u \in W_0^{k,p}(\Omega)$ if there exists a sequence $u_i \in C_0^\infty(\Omega)$ such that*

$$u_i \to u \quad \text{in } W^{k,p}(\Omega).$$

Remark 2.31 (Purpose)**.** *$u \in W_0^{k,p}(\Omega)$ has "zero boundary values in the Sobolev sense". Later, we want to set boundary values for weak solutions of PDEs: given $v \in W^{1,p}(\Omega)$, we say that u takes boundary values v in the "Sobolev sense" if*

$$u - v \in W_0^{1,p}(\Omega).$$

Remark 2.32 (Warning)**.** *The regularity of Ω affect the outcome, and $W_0^{1,p}(\Omega)$ functions do not always look what one might intuitively* *expect by thinking smooth functions with zero boundary values. Set* $\Omega = B(0,1) \setminus \{0\}$. *Then for*

$$u : \Omega \to \mathbb{R}, \quad u(x) = \operatorname{dist}(x, \partial B(0,1)) = 1 - |x|$$

it holds that $u \in W_0^{1,p}(\Omega)$ *whenever* $p < n$.
Reason (with omitting some details): Choose a cut-off function $\xi_\varepsilon \in C_0^\infty(B(0,1))$, $0 \le \xi_i \le 1$ *such that* $\xi_\varepsilon(x) = 1$ *in* $B(0,\varepsilon)$ *and in* $B(0,1) \setminus \overline{B}(0, 1-\varepsilon)$, $\xi_\varepsilon = 0$ *in* $B(0,1-2\varepsilon) \setminus \overline{B}(0,2\varepsilon)$ *and* $|D\xi_\varepsilon| \le C/\varepsilon$. *Then* $(1-\xi_\varepsilon)u \in C_0^\infty(\Omega)$ *and*

$$(1-\xi_\varepsilon)u \to u \quad \text{in } W^{1,p}(\Omega)$$

as $\varepsilon \to 0$, *whenever* $p < n$. *Indeed, by MON (=Lebesgue's monotone convergence thm)* $(1-\xi_\varepsilon)u \to u$ *in* $L^p(\Omega)$ *and we may concentrate on showing that* $\frac{\partial}{\partial x_i}((1-\xi_\varepsilon)u) \to \frac{\partial u}{\partial x_i}$ *in* $L^p(\Omega)$. *To see this, we calculate using Theorem 2.17*

$$\begin{aligned}
&\int_\Omega \left| \frac{\partial}{\partial x_i}((1-\xi_\varepsilon)u) - \frac{\partial u}{\partial x_i} \right|^p dx \\
&\quad = \int_\Omega \left| -\frac{\partial \xi_\varepsilon}{\partial x_i} u + (1-\xi_\varepsilon)\frac{\partial u}{\partial x_i} - \frac{\partial u}{\partial x_i} \right|^p dx \\
&\quad \le C \int_{B(0,2\varepsilon)} \left| \frac{\partial \xi_\varepsilon}{\partial x_i} \right|^p dx + C \int_{B(0,1)\setminus \overline{B}(0,1-2\varepsilon)} \left| \frac{\partial \xi_\varepsilon}{\partial x_i} \right|^p (1-|x|)^p \, dx \\
&\qquad + C \int_\Omega \left| \xi_\varepsilon \frac{\partial u}{\partial x_i} \right|^p dx \\
&\quad \le C \int_{B(0,2\varepsilon)} |D\xi_\varepsilon|^p \, dx + C\varepsilon^{1-p+p} + C \underbrace{||Du||_{L^\infty(\Omega)}}_{=1} \int_\Omega |\xi_\varepsilon|^p \, dx \\
&\quad \le C\varepsilon^n/\varepsilon^p + C\varepsilon + C(2\varepsilon + (2\varepsilon)^n) \to 0,
\end{aligned}$$

when $\varepsilon \to 0$ *and* $p < n$.

The problem in this example is that $\{0\}$ *is too small to be "seen" by* $W^{1,p}(\Omega)$ *function when* $p < n$. *Let us also remark that Lebesgue measure is not the most accurate gauge to measure smallness of sets in the Sobolev theory. In a sense right gauge is so called p-capacity.*

The following lemma shows that when considering Sobolev spaces over the whole $\mathbb{R}^n$, $W_0^{1,p}(\mathbb{R}^n)$ coincides with $W^{1,p}(\mathbb{R}^n)$.

Lemma 2.33. $W_0^{1,p}(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n)$.

Proof. Exercise. □

2.7. **Properties of $W^{1,p}(\Omega)$, $1 \le p < \infty$.**

Lemma 2.34 (Chain rule). *Let $f \in C^1(\mathbb{R})$, $||f'||_{L^\infty(\mathbb{R})} < \infty$, and $u \in W^{1,p}(\Omega)$. Then*

$$\frac{\partial f(u)}{\partial x_j} = f'(u)\frac{\partial u}{\partial x_j}, \quad j = 1, \ldots, n$$

a.e. in Ω, and where $\frac{\partial u}{\partial x_j}$, $\frac{\partial f(u)}{\partial x_j}$ denotes the weak derivative.

Proof. We have proven that we can choose $u_i \in C^\infty(\Omega) \cap W^{1,p}(\Omega)$ such that

$$u_i \to u \quad \text{in } W^{1,p}(\Omega).$$

Claim: For any $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega f(u)\frac{\partial \varphi}{\partial x_j}\,dx = \lim_{i\to\infty}\int_\Omega f(u_i)\frac{\partial \varphi}{\partial x_j}\,dx.$$

Proof: Let $1 < p < \infty$ (the case $p = 1$ is similar). Then since $1/p + (p-1)/p = 1$, we have

$$\begin{aligned}
&\left|\int_\Omega f(u)\frac{\partial \varphi}{\partial x_j}\,dx - \int_\Omega f(u_i)\frac{\partial \varphi}{\partial x_j}\,dx\right| \\
&\le \int_\Omega |f(u) - f(u_i)||D\varphi|\,dx \\
&\overset{\text{Hölder}}{\le} \Big(\int_\Omega |f(u) - f(u_i)|^p\,dx\Big)^{1/p}\Big(\int_\Omega |D\varphi|^{p/(p-1)}\,dx\Big)^{(p-1)/p} \\
&\overset{*}{\le} ||f'||_{L^\infty(\mathbb{R})}\Big(\int_\Omega |u - u_i|^p\,dx\Big)^{1/p}\Big(\int_\Omega |D\varphi|^{p/(p-1)}\,dx\Big)^{(p-1)/p} \to 0,
\end{aligned}$$

where * follows from $|f(u) - f(u_i)| = \left|\int_{u_i}^u f'(t)\,dt\right| \le ||f'||_{L^\infty(\mathbb{R})}\,|u_i - u|$.///

$$\begin{aligned}
\int_\Omega f(u)\frac{\partial \varphi}{\partial x_j}\,dx &= \lim_{i\to\infty}\int_\Omega f(u_i)\frac{\partial \varphi}{\partial x_j}\,dx \\
&\overset{\text{calc for smooth functions}}{=} -\lim_{i\to\infty}\int_\Omega f'(u_i)\frac{\partial u_i}{\partial x_j}\varphi\,dx \\
&\overset{*}{=} -\int_\Omega \lim_{i\to\infty} f'(u_i)\frac{\partial u_i}{\partial x_j}\varphi\,dx \\
&= -\int_\Omega f'(u)\frac{\partial u}{\partial x_j}\varphi\,dx.
\end{aligned}$$

Since the LHS above is as in the definition of the weak derivative of $\frac{\partial f(u)}{\partial x_j}$, the proof is complete. At * we used

$$\begin{aligned}&\left|\int_\Omega (f'(u_i)\frac{\partial u_i}{\partial x_j} - f'(u)\frac{\partial u}{\partial x_j})\varphi\,dx\right|\\ &= \left|\int_\Omega (f'(u_i)\frac{\partial u_i}{\partial x_j} - f'(u_i)\frac{\partial u}{\partial x_j} + f'(u_i)\frac{\partial u}{\partial x_j} - f'(u)\frac{\partial u}{\partial x_j})\varphi\,dx\right|\\ &= \left|\int_\Omega f'(u_i)(\frac{\partial u_i}{\partial x_j} - \frac{\partial u}{\partial x_j})\varphi + (f'(u_i) - f'(u))\frac{\partial u}{\partial x_j}\varphi\,dx\right| \to 0.\end{aligned}$$

The first term converges because of Hölder's inequality and the second by the fact that since $u_i \to u$ in L^p we can choose a.e. converging subsequence to u. Moreover, as f' is continuous, also $f'(u_i) \to f'(u)$ a.e., and the conditions of DOM are satisfied. □

Theorem 2.35. *If $u \in W^{1,p}(\Omega)$, then recalling $u_+ = \max(u,0)$ and $u_- = -\min(u,0)$, we have $u_+, u_-, |u| \in W^{1,p}(\Omega)$ and*

$$Du_+ = \begin{cases} Du & \textit{a.e. in } \{x \in \Omega : u(x) > 0\} \\ 0 & \textit{a.e. in } \{x \in \Omega : u(x) \le 0\} \end{cases}$$

$$Du_- = \begin{cases} -Du & \textit{a.e. in } \{x \in \Omega : u(x) < 0\} \\ 0 & \textit{a.e. in } \{x \in \Omega : u(x) \ge 0\} \end{cases}$$

and

$$D|u| = \begin{cases} Du & \textit{a.e. in } \{x \in \Omega : u(x) > 0\} \\ 0 & \textit{a.e. in } \{x \in \Omega : u(x) = 0\} \\ -Du & \textit{a.e. in } \{x \in \Omega : u(x) < 0\}. \end{cases}$$

Proof. We aim at using the previous theorem for a suitable f. Let

$$f_\varepsilon(s) = \begin{cases} \sqrt{s^2+\varepsilon^2} - \varepsilon & s \ge 0 \\ 0 & s < 0. \end{cases}$$

It holds that $f_\varepsilon \in C^1(\mathbb{R})$ and $\lim_{\varepsilon\to 0} f_\varepsilon(s) = f(s)$, where

$$f(s) = \begin{cases} s & s \ge 0 \\ 0 & s < 0. \end{cases}$$

Also observe that

$$||f_\varepsilon'||_{L^\infty(\mathbb{R})} < \infty.$$

Thus by Lemma 2.34

$$\int_\Omega f_\varepsilon(u)\frac{\partial \varphi}{\partial x_j}\,dx = -\int_\Omega f_\varepsilon'(u)\frac{\partial u}{\partial x_j}\varphi\,dx$$

for every $\varphi \in C_0^\infty(\Omega)$. Observe that

$$\lim_{\varepsilon \to 0} f_\varepsilon(u) = u_+ \quad \text{in } \Omega$$

and

$$\lim_{\varepsilon \to 0} f'_\varepsilon(u(x)) = \begin{cases} 1 & \text{in } \{x \in \Omega \,:\, u(x) > 0\} \\ 0 & \text{in } \{x \in \Omega \,:\, u(x) \leq 0\}. \end{cases}$$

By DOM

$$\begin{aligned} \int_\Omega u_+ \frac{\partial \varphi}{\partial x_j}\, dx &= \int_\Omega \lim_{\varepsilon \to 0} f_\varepsilon(u) \frac{\partial \varphi}{\partial x_j}\, dx \\ &= \lim_{\varepsilon \to 0} \int_\Omega f_\varepsilon(u) \frac{\partial \varphi}{\partial x_j}\, dx \\ &\overset{\text{prev. lemma}}{=} \lim_{\varepsilon \to 0} - \int_\Omega f'_\varepsilon(u) \frac{\partial u}{\partial x_j} \varphi\, dx \\ &\overset{\text{DOM}, ||f'_\varepsilon||_{L^\infty} < C}{=} - \int_\Omega \lim_{\varepsilon \to 0} f'_\varepsilon(u) \frac{\partial u}{\partial x_j} \varphi\, dx \\ &= - \int_{\{x \in \Omega \,:\, u(x) > 0\}} \frac{\partial u}{\partial x_j} \varphi\, dx. \end{aligned}$$

This proves the first part of the claim. The second and the third follow by observing

$$u_- = (-u)_+ \quad \text{and} \quad |u| = u_+ + u_-. \qquad \square$$

Corollary 2.36. *Let $u, v \in W^{1,p}(\Omega)$ and $\lambda \in \mathbb{R}$. Then*

$$\min(u, v), \max(u, v) \in W^{1,p}(\Omega),$$

and if Ω bounded

$$\min(u, \lambda) \in W^{1,p}(\Omega)$$

and

$$D \min(u, \lambda) = \begin{cases} Du & a.e. \ in\ \{x \in \Omega \,:\, u(x) < \lambda\} \\ 0 & a.e. \ in\ \{x \in \Omega \,:\, u(x) \geq \lambda\}. \end{cases}$$

Proof.

$$\max(u,v) = \begin{cases} u, & \{x \in \Omega : u(x) \geq v(x)\} \\ v, & \{x \in \Omega : u(x) < v(x)\} \end{cases}$$
$$= \begin{cases} \frac{1}{2}(u+v+(u-v)), & \{x \in \Omega : u(x) \geq v(x)\} \\ \frac{1}{2}(u+v-(u-v)), & \{x \in \Omega : u(x) < v(x)\} \end{cases}$$
$$= \frac{1}{2}(u+v+|u-v|)$$

and

$$\min(u,v) = \frac{1}{2}(u+v-|u-v|).$$ □

Corollary 2.37. *Let $u \in W^{1,p}(\Omega)$ and $\lambda > 0$. Then for*

$$u_\lambda := \min(\max(u,-\lambda),\lambda)) = \begin{cases} \lambda & \{x \in \Omega : u(x) \geq \lambda\} \\ u & \{x \in \Omega : \lambda < u(x) < \lambda\} \\ -\lambda & \{x \in \Omega : u(x) \leq \lambda\} \end{cases}$$

we have

$$u_\lambda \to u \quad \text{in } W^{1,p}(\Omega)$$

when $\lambda \to \infty$.

Proof. Exercise. □

Theorem 2.38. *If $u, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, then $uv \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, and*

$$\frac{\partial(uv)}{\partial x_j} = \frac{\partial u}{\partial x_j} v + u \frac{\partial v}{\partial x_j}$$

almost everywhere in Ω.

Proof. Exercise: The derivatives in the statement denote weak derivatives, so start from the integral definition and use similar techniques as in Lemma 2.34. □

2.8. Difference quotient characterization of Sobolev spaces.

Definition 2.39. *Let $u \in L^1_{loc}(\Omega)$ and $\Omega' \subset \Omega$ and $e_i = (0, \ldots, \underbrace{1}_{ith}, 0, \ldots, 0)$.*

Then difference quotient of u to direction e_i is

$$D_i^h u(x) := \frac{u(x+he_i) - u(x)}{h}$$

for $x \in \Omega'$ and $|h| < \operatorname{dist}(\Omega', \partial\Omega)$. Further, we denote

$$D^h u = (D_1^h u, \dots, D_n^h u).$$

Theorem 2.40. *Let $u \in W^{1,p}(\Omega)$ for $1 \le p < \infty$. Then there exists $C = C(n,p) > 0$ such that*

$$\left|\left|D^h u\right|\right|_{L^p(\Omega')} \le C \left|\left|Du\right|\right|_{L^p(\Omega)}$$

for every $\Omega' \Subset \Omega$ and $|h| < \operatorname{dist}(\Omega', \partial\Omega)$. Here $\left|\left|D^h u\right|\right|_{L^p(\Omega')} := \left|\left|\, \left|D^h u\right| \,\right|\right|_{L^p(\Omega')}$.

Proof. Let first $u \in C^\infty(\Omega) \cap W^{1,p}(\Omega)$. Then

$$\begin{aligned}
|u(x + he_i) - u(x)| &= \left| \int_0^h \frac{\partial}{\partial t} u(x + te_i)\, dt \right| \\
&= \left| \int_0^h Du(x + te_i) \cdot e_i \, dt \right| \\
&= \left| \int_0^h \frac{\partial u(x + te_i)}{\partial x_i}\, dt \right| \\
&\le \int_0^h \left| \frac{\partial u(x + te_i)}{\partial x_i} \right| dt.
\end{aligned}$$

Thus

$$\begin{aligned}
\left|D_i^h u(x)\right| &= \left| \frac{u(x + he_i) - u(x)}{h} \right| \\
&\le \frac{1}{|h|} \int_0^{|h|} \left| \frac{\partial u(x + te_i)}{\partial x_i} \right| dt \\
&\overset{\text{Hölder}}{\le} \Big(\frac{1}{|h|} \int_0^{|h|} \left| \frac{\partial u(x + te_i)}{\partial x_i} \right|^p dt \Big)^{1/p}
\end{aligned}$$

i.e.

$$\left|D_i^h u(x)\right|^p \le \frac{1}{|h|} \int_0^{|h|} \left| \frac{\partial u(x + te_i)}{\partial x_i} \right|^p dt.$$

Using this

$$
\begin{aligned}
\int_{\Omega'} \left|D_i^h u(x)\right|^p dx &\le \frac{1}{|h|}\int_{\Omega'}\int_0^{|h|}\left|\frac{\partial u(x+te_i)}{\partial x_i}\right|^p dt\,dx \\
&\overset{t=s|h|}{=} \int_{\Omega'}\int_0^1 \left|\frac{\partial u(x+s|h|e_i)}{\partial x_i}\right|^p ds\,dx \\
&\overset{\text{Fubini}}{=} \int_0^1\int_{\Omega'} \left|\frac{\partial u(x+s|h|e_i)}{\partial x_i}\right|^p dx\,ds \\
&\le \sup_{s\in[0,1]}\int_{\Omega'} \left|\frac{\partial u(x+s|h|e_i)}{\partial x_i}\right|^p dx \\
&\le \int_{\Omega} \left|\frac{\partial u(x)}{\partial x_i}\right|^p dx.
\end{aligned}
$$

Then we deduce the result for the full gradient

$$
\begin{aligned}
\int_{\Omega'} \left|D^h u(x)\right|^p dx &= \int_{\Omega'} \Big(\sum_{i=1}^n \left|D_i^h u(x)\right|^2\Big)^{p/2} dx \\
&\le C\int_{\Omega'}\sum_{i=1}^n \left|D_i^h u(x)\right|^p dx \\
&= C\sum_{i=1}^n \int_{\Omega'} \left|D_i^h u(x)\right|^p dx \\
&\overset{\text{previous}}{\le} C\sum_{i=1}^n\int_{\Omega}\left|\frac{\partial u(x)}{\partial x_i}\right|^p dx \\
&\le C\int_{\Omega}\Big(\sum_{i=1}^n\left|\frac{\partial u(x)}{\partial x_i}\right|^2\Big)^{p/2} dx \\
&= C\int_{\Omega}|Du(x)|^p\,dx.
\end{aligned}
$$

We assumed $u \in W^{1,p}(\Omega)\cap C^\infty(\Omega)$, but we can extend the result for $W^{1,p}(\Omega)$ by approximation. $\square$

Theorem 2.41. *Let $\Omega \Subset \Omega$. If $u \in L^p(\Omega)$, $1 < p < \infty$ and if there exists a uniform constant*

$$
\left|\left|D^h u\right|\right|_{L^p(\Omega')} \le C \tag{2.8}
$$

for all $|h| < \operatorname{dist}(\Omega', \partial\Omega)$, then $u \in W^{1,p}(\Omega')$ and

$$
||Du||_{L^p(\Omega')} \le C
$$

for the same constant C.

Proof. Let $\varphi \in C_0^\infty(\Omega')$. Then

$$\begin{aligned}
&\int_{\Omega'} u(x)\frac{\varphi(x+he_i)-\varphi(x)}{h}\,dx \\
&\quad = \frac{1}{h}\int_{\Omega'} u(x)\varphi(x+he_i)\,dx - \frac{1}{h}\int_{\Omega'} u(x)\varphi(x)\,dx \\
&\quad \overset{y\,=\,x+he_i}{=} \frac{1}{h}\int_{\Omega'} u(y-he_i)\varphi(y)\,dy - \frac{1}{h}\int_{\Omega'} u(x)\varphi(x)\,dx \\
&\quad = -\int_{\Omega'} \frac{u(x)-u(x-he_i)}{h}\varphi(x)\,dx \\
&\quad = -\int_{\Omega'} \frac{u(x-he_i)-u(x)}{-h}\varphi(x)\,dx
\end{aligned}$$

for $|h|$ so small that $\operatorname{spt}\varphi(\cdot+he_i)\subset\Omega'$. Then

$$\int_{\Omega'} uD_i^h\varphi\,dx = -\int_{\Omega'}(D_i^{-h}u)\varphi\,dx, \tag{2.9}$$

"integration by parts for difference quotients". From the assumption (2.8) it follows that

$$\sup_{0<|h|<\operatorname{dist}(\Omega',\partial\Omega)} \left|\left|D_i^{-h}u\right|\right|_{L^p(\Omega')} < \infty,$$

and because $L^p(\Omega')$, $p>1$ is reflexive, there exist $v_i \in L^p(\Omega')$ and a subsequence $h_j \to 0$ such that (see Remark 2.42)

$$D_i^{-h_j}u \to v_i \quad \text{weakly in } L^p(\Omega').$$

Next we check that this weak limit is a weak derivative. Recalling (2.9), it follows that

$$\begin{aligned}
\int_{\Omega'} u\frac{\partial\varphi}{\partial x_i}\,dx &= \int_{\Omega} u\frac{\partial\varphi}{\partial x_i}\,dx \\
&= \int_{\Omega}\lim_{h_j\to 0} D_i^{h_j}\varphi u\,dx \\
&\overset{\text{DOM}}{=} \lim_{h_j\to 0}\int_{\Omega} D_i^{h_j}\varphi u\,dx \\
&\overset{(2.9)}{=} -\lim_{h_j\to 0}\int_{\Omega}\varphi D_i^{-h_j}u\,dx \\
&\overset{\text{weak convergence}}{=} -\int_{\Omega}\varphi v_i\,dx.
\end{aligned}$$

As a conclusion $v_i = \frac{\partial u}{\partial x_i}$ in a weak sense, and thus $u \in W^{1,p}(\Omega')$. Moreover, for weakly convergent sequence, we have

$$||v_i||_{L^p(\Omega')} = \left|\left|\frac{\partial u}{\partial x_i}\right|\right|_{L^p(\Omega')} \leq \liminf_{h_j \to 0} \left|\left|D_i^{-h_j} u\right|\right|_{L^p(\Omega')} \leq C.$$

$\square$

Remark 2.42 (Reminder)**.**

$$f_j \to f \quad \textit{weakly in } L^p(\Omega')$$

if

$$\int_{\Omega'} f_j g\, dx = \int_{\Omega'} f g\, dx$$

for every $g \in L^{p'}(\Omega')$, *where* $1/p + 1/p' = 1$, $1 < p < \infty$. *If space is reflexive, it is weakly sequentially compact: every bounded (in the norm of the space) sequence has a weakly convergent subsequence. Moreover for this sequence*

$$||f||_{L^p(\Omega')} \leq \liminf_{j \to \infty} ||f_j||_{L^p(\Omega')}.$$

2.9. **Sobolev type inequalities.** Study of Sobolev type inequalities is divided in three intervals of exponents:

(1) $1 \leq p < n$, Gagliardo-Nirenberg-Sobolev inequality
(2) $p = n$
(3) $n < p \leq \infty$, Morrey's inequality

Also recall the notation $\frac{1}{|B(x,r)|}\int_{B(x,r)} \dots\, dy = \fint_{B(x,r)} \dots\, dy$.

2.9.1. *Gagliardo-Nirenberg-Sobolev inequality,* $1 \leq p < n$. We define a *Sobolev conjugate*

$$p^* = \frac{pn}{n-p} > p,$$

or in other words

$$\frac{1}{p} - \frac{1}{n} = \frac{1}{p^*}.$$

Motivation for this form of the Sobolev conjugate is as follows: We want to prove that an inequality of the form

$$\Big(\int_{\mathbb{R}^n} |u|^q\, dx\Big)^{1/q} \leq C\Big(\int_{\mathbb{R}^n} |Du|^p\, dx\Big)^{1/p},$$

for every $u \in C_0^\infty(\mathbb{R}^n)$ and constant *independent* of u. Then it should also hold for

$$u_\lambda(x) = u(\lambda x) \in C_0^\infty(\mathbb{R}^n), \lambda > 0.$$

For this function

$$\int_{\mathbb{R}^n} |u(\lambda x)|^q \, dx \overset{y=\lambda x}{=} \frac{1}{\lambda^n} \int_{\mathbb{R}^n} |u(y)|^q \, dy$$

and

$$\begin{aligned}\int_{\mathbb{R}^n} |Du_\lambda(x)|^p \, dx &= \int_{\mathbb{R}^n} |\lambda Du(\lambda x)|^p \, dx \\ &\overset{y=\lambda x}{=} \frac{1}{\lambda^{n-p}} \int_{\mathbb{R}^n} |Du(y)|^p \, dy.\end{aligned}$$

Thus we would have

$$\Big(\frac{1}{\lambda^n} \int_{\mathbb{R}^n} |u(y)|^q \, dy\Big)^{1/q} \leq \Big(\frac{1}{\lambda^{n-p}} \int_{\mathbb{R}^n} |Du(y)|^p \, dy\Big)^{1/p}$$

and constant would be independent of λ only if

$$\lambda^{n/q+1-n/p} = \lambda^0$$

that is

$$\frac{1}{p} - \frac{1}{n} = \frac{1}{q},$$

i.e. $q = p^*$.

Next theorem shows that any function in $W^{1,p}(\mathbb{R}^n)$ can be controlled by its gradient. Later we will see that this holds in general for $W_0^{1,p}(\Omega)$-functions (recall that $W_0^{1,p}(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n)$). Also observe that the constant below does not depend on the function u itself.

Theorem 2.43 (Sobolev's inequality, $1 \leq p < n$, $\mathbb{R}^n$)**.** *Let $1 \leq p < n$. Then there exists $C = C(n,p)$ such that*

$$\Big(\int_{\mathbb{R}^n} |u|^{p^*} \, dx\Big)^{1/p^*} \leq C\Big(\int_{\mathbb{R}^n} |Du|^p \, dx\Big)^{1/p}.$$

for any $u \in W^{1,p}(\mathbb{R}^n)$.

Proof. By approximation argument, as shown at the end of the proof, we may again assume that $u \in C_0^\infty(\mathbb{R}^n)$. Then

$$u(x_1, \dots, x_j, \dots, x_n) = \int_{-\infty}^{x_j} \frac{\partial u}{\partial x_j}(x_1, \dots, t_j, \dots, x_n) \, dt_j$$

implying

$$|u(x)| \leq \int_{\mathbb{R}} |Du(x_1, \ldots, x_j, \ldots, x_n)| \, dx_j.$$

Multiplying we obtain

$$|u(x)|^{n/(n-1)} \leq \prod_{j=1}^{n} \Big(\int_{\mathbb{R}} |Du(x_1, \ldots, x_j, \ldots, x_n)| \, dx_j \Big)^{1/(n-1)}$$

and further

$$\begin{aligned}\int_{\mathbb{R}} |u(x)|^{n/(n-1)} \, dx_1 &\leq \Big(\int_{\mathbb{R}} |Du| \, dx_1 \Big)^{1/(n-1)} \int_{\mathbb{R}} \prod_{j=2}^{n} \Big(\int_{\mathbb{R}} |Du| \, dx_j \Big)^{1/(n-1)} dx_1 \\ &\overset{*}{\leq} \Big(\int_{\mathbb{R}} |Du| \, dx_1 \Big)^{1/(n-1)} \prod_{j=2}^{n} \Big(\int_{\mathbb{R}} \int_{\mathbb{R}} |Du| \, dx_j \, dx_1 \Big)^{1/(n-1)},\end{aligned}$$

in * we used generalized Hölder's inequality, Lemma 2.45, with powers $\sum_{i=1}^{n-1} \frac{1}{n-1} = 1$. We repeat the argument for x_2:

$$\begin{aligned}&\int_{\mathbb{R}} \int_{\mathbb{R}} |u(x)|^{n/(n-1)} \, dx_1 \, dx_2 \\ &\leq \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |Du| \, dx_1 \Big)^{1/(n-1)} \prod_{j=2}^{n} \Big(\int_{\mathbb{R}} \int_{\mathbb{R}} |Du| \, dx_j \, dx_1 \Big)^{1/(n-1)} dx_2 \\ &\leq \Big(\int_{\mathbb{R}} \int_{\mathbb{R}} |Du| \, dx_2 \, dx_1 \Big)^{1/(n-1)} \\ &\qquad \cdot \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |Du| \, dx_1 \Big)^{1/(n-1)} \prod_{j=3}^{n} \Big(\int_{\mathbb{R}} \int_{\mathbb{R}} |Du| \, dx_j \, dx_1 \Big)^{1/(n-1)} dx_2 \\ &\overset{\text{gen Hölder}}{\leq} \Big(\int_{\mathbb{R}} \int_{\mathbb{R}} |Du| \, dx_1 \, dx_2 \Big)^{1/(n-1)} \Big(\int_{\mathbb{R}} \int_{\mathbb{R}} |Du| \, dx_1 \, dx_2 \Big)^{1/(n-1)} \\ &\qquad \cdot \prod_{j=3}^{n} \Big(\int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} |Du| \, dx_j \, dx_1 \, dx_2 \Big)^{1/(n-1)}\end{aligned}$$

Repeating the argument n times, we finally obtain

$$\int_{\mathbb{R}} \ldots \int_{\mathbb{R}} |u(x)|^{n/(n-1)} \, dx_1 \, dx_2 \ldots \, dx_n \leq \Big(\int_{\mathbb{R}} \ldots \int_{\mathbb{R}} |Du| \, dx_1 \, dx_2 \ldots \, dx_n \Big)^{n/(n-1)}.$$

This is the claim for $p = 1$.

When $1 < p < n$, we apply the estimate for

$$v = |u|^{\gamma}$$

where γ is to be selected. The above result yields

$$\begin{aligned}
&\Big(\int_{\mathbb{R}^n} |u|^{n\gamma/(n-1)}\,dx\Big)^{(n-1)/n} \\
&\le \int_{\mathbb{R}^n} |D|u|^\gamma|\,dx \\
&= \int_{\mathbb{R}^n} \gamma|u|^{\gamma-1}|Du|\,dx \\
&\overset{\text{Hölder}}{\le} \gamma\Big(\int_{\mathbb{R}^n} |u|^{(\gamma-1)p/(p-1)}\,dx\Big)^{(p-1)/p}\Big(\int_{\mathbb{R}^n} |Du|^p\,dx\Big)^{1/p}
\end{aligned}$$

Solving for γ so that on both sided u has a same power i.e.

$$\begin{aligned}
&n\gamma/(n-1) = (\gamma-1)p/(p-1) \\
&\iff n\gamma(p-1) = (\gamma-1)(n-1)p \\
&\iff \gamma(pn-n-np+p) = -(n-1)p \\
&\iff \gamma = \frac{p(n-1)}{n-p}.
\end{aligned}$$

Using this γ we have

$$\Big(\int_{\mathbb{R}^n} |u|^{p^*}\,dx\Big)^{(n-1)/n} \le \frac{p(n-1)}{n-p}\Big(\int_{\mathbb{R}^n} |u|^{p^*}\,dx\Big)^{(p-1)/p}\Big(\int_{\mathbb{R}^n} |Du|^p\,dx\Big)^{1/p}$$

and since

$$\frac{n-1}{n} - \frac{p-1}{p} = \frac{n-p}{np}$$

we are done for $C_0^\infty(\mathbb{R}^n)$.

We complete the proof by justifying the smoothness assumption. Let $u \in W^{1,p}(\mathbb{R}^n)$ and u_i a smooth sequence such that

$$u_i \to u \quad \text{in } W^{1,p}(\mathbb{R}^n).$$

We can also (not proven here) take a further subsequence so that

$$u_i \to u \text{ a.e.}$$

This u_i is a Cauchy sequence in $L^{p^*}(\mathbb{R}^n)$, since for any $\varepsilon > 0$

$$||u_i - u_j||_{L^{p^*}(\mathbb{R}^n)} \overset{u_i - u_j \in C_0^\infty(\mathbb{R}^n)}{\le} ||D(u_i - u_j)||_{L^p(\mathbb{R}^n)} \le \varepsilon,$$

for all large enough i, j. $L^{p^*}(\mathbb{R}^n)$ is complete and thus there exists $u \in L^{p^*}(\mathbb{R}^n)$ (more details at the end of the proof) such that

$$u_i \to u \quad \text{in } L^{p^*}(\mathbb{R}^n). \tag{2.10}$$

Thus

$$
\begin{aligned}
&||u||_{L^{p^*}(\mathbb{R}^n)} \\
&\overset{\text{Minkowski}}{\leq} ||u_i - u||_{L^{p^*}(\mathbb{R}^n)} + ||u_i||_{L^{p^*}(\mathbb{R}^n)} \\
&\leq ||u_i - u||_{L^{p^*}(\mathbb{R}^n)} + C\,||Du_i||_{L^p(\mathbb{R}^n)} \\
&\leq ||u_i - u||_{L^{p^*}(\mathbb{R}^n)} + C\,||Du_i - Du||_{L^p(\mathbb{R}^n)} + C\,||Du||_{L^p(\mathbb{R}^n)} \\
&\to 0 + 0 + C\,||Du||_{L^p(\mathbb{R}^n)}\,,
\end{aligned}
$$

which completes the proof, in case, we can show the following: We omitted one point above; why should L^{p^*}-limit also be u?
Claim: L^{p^*} limit in (2.10) must be u.
Reason: Assume the contrary:

$$u_i \to g \quad \text{in } L^{p^*}(\mathbb{R}^n).$$

Choose a further subsequence

$$u_i \to g$$

pointwise a.e. and by our earlier choices

$$u_i \to u \text{ a.e.},$$

a contradiction. □

Corollary 2.44.

$$u \in W^{1,p}(\mathbb{R}^n) \Rightarrow u \in L^p(\mathbb{R}^n) \cap L^{p^*}(\mathbb{R}^n).$$

Lemma 2.45 (Generalized Hölder)**.** *Let*

$$\frac{1}{p_1} + \ldots + \frac{1}{p_m} = 1$$

and suppose that $u_1 \in L^{p_1}(\Omega), \ldots, u_m \in L^{p_m}(\Omega)$. *Then*

$$\int_\Omega |u_1 \cdot \ldots \cdot u_m|\, dx \leq \prod_{i=1}^m \Big(\int_\Omega |u_i|^{p_i}\, dx \Big)^{1/p_i}.$$

Theorem 2.46 (Sobolev's inequality, $1 \leq p < n$, Ω)**.** *Let* $1 \leq p < n$. *Then there exists* $C = C(n,p)$ *such that*

$$\Big(\int_\Omega |u|^{p^*}\, dx \Big)^{1/p^*} \leq C \Big(\int_\Omega |Du|^p\, dx \Big)^{1/p}$$

for any $u \in W_0^{1,p}(\Omega)$.

Proof. Idea. Similarly as before, we can concentrate on $u \in C_0^\infty(\Omega)$ and then obtain the general case by approximation. Now, u can be extended by zero to have $u \in C_0^\infty(\mathbb{R}^n)$. Then we can apply Theorem 2.43 to obtain the result. $\square$

Remark 2.47 (Warning)**.** *The above theorem does not hold without assumption $u \in W_0^{1,p}(\Omega)$ on zero boundary values: consider a constant function.*

Corollary 2.48. *Let $1 \le p < n$. Then there exists $C = C(n,p)$ such that*

$$\Big(\fint_{B(x,r)} |u|^{p^*}\,dy\Big)^{1/p^*} \le Cr\Big(\fint_{B(x,r)} |Du|^p\,dy\Big)^{1/p}.$$

for any $u \in W_0^{1,p}(B(x,r))$.

Theorem 2.49 (Sobolev's inequality, $n < p < \infty$, Ω)**.** *Let $n < p < \infty$ and $|\Omega| < \infty$. Then there exists $C = C(n,p)$ such that*

$$\operatorname*{ess\,sup}_{\Omega} |u| \le C|\Omega|^{(p-n)/pn}\Big(\int_\Omega |Du|^p\,dx\Big)^{1/p}$$

for any $u \in W_0^{1,p}(\Omega)$.

Proof. This result is proven later in the section of Morrey's inequality. $\square$

Corollary 2.50. *Let $n < p < \infty$. Then there exists $C = C(n,p)$ such that*

$$\begin{aligned}\operatorname*{ess\,sup}_{B(x,r)} |u| &\le Cr^{(p-n)/p}\Big(\int_{B(x,r)} |Du|^p\,dy\Big)^{1/p}\\ &\le Cr\Big(\fint_{B(x,r)} |Du|^p\,dy\Big)^{1/p},\end{aligned}$$

for any $u \in W_0^{1,p}(B(x,r))$.

Corollary 2.51. *Let $n < p < \infty$. Then there exists $C = C(n,p)$ such that*

$$\Big(\int_{B(x,r)} |u|^q\,dy\Big)^{1/q} \le Cr^{1-n/p+n/q}\Big(\int_{B(x,r)} |Du|^p\,dy\Big)^{1/p}$$

i.e.

$$\Big(\fint_{B(x,r)} |u|^q\,dy\Big)^{1/q} \le Cr\Big(\fint_{B(x,r)} |Du|^p\,dy\Big)^{1/p}$$

for any $q \in (0,\infty]$ and $u \in W_0^{1,p}(B(x,r))$.

The previous three results also extend to the case $p = \infty$.

Theorem 2.52. *Let $p = n > 1$. Then for any $q \in (0, \infty)$ there exists $C = C(n, q)$ such that*

$$\Big(\int_{B(x,r)} |u|^q \, dy\Big)^{1/q} \le C r^{p/q} \Big(\int_{B(x,r)} |Du|^p \, dy\Big)^{1/p}$$

i.e.

$$\Big(⨍_{B(x,r)} |u|^q \, dy\Big)^{1/q} \le C r \Big(⨍_{B(x,r)} |Du|^p \, dy\Big)^{1/p},$$

for $u \in W_0^{1,p}(B(x,r))$.

Proof. Exercise. □

2.9.2. *Poincare's inequalities.* We denote $u_{B(x,r)} = ⨍_{B(x,r)} u \, dy$.

Observe in particular that the constant in the next estimate is independent of p.

Theorem 2.53. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, and $1 \le p < \infty$. Then there is a constant $C = C(n)$ such that*

$$\int_\Omega |u|^p \, dx \le C^p \operatorname{diam}(\Omega)^p \int_\Omega |Du|^p \, dx,$$

for every $u \in W_0^{1,p}(\Omega)$.

Proof. By approximation, we may assume $u \in C_0^\infty(\Omega)$. Set/choose

$$\begin{aligned} r &= \operatorname{diam}(\Omega) \\ y &= (y_1, \dots, y_n) \in \Omega, \\ \Omega &\subset \prod_{j=1}^n [y_j - r, y_j + r] \end{aligned}$$

Similarly as in the proof of Theorem 2.43

$$\begin{aligned} |u(x)| &\le \int_{y_1 - r}^{y_1 + r} |Du(t_1, x_2, \dots, x_n)| \, dt_1 \\ &\overset{\text{Hölder}}{\le} (2r)^{(p-1)/p} \Big(\int_{y_1 - r}^{y_1 + r} |Du(x_1, \dots, x_n)|^p \, dx_1\Big)^{1/p}, \end{aligned}$$

so that

$$|u(x)|^p \le (2r)^{(p-1)} \int_{y_1 - r}^{y_1 + r} |Du(x_1, \dots, x_n)|^p \, dx_1.$$

Using this

$$
\begin{aligned}
\int_\Omega |u|^p\,dx &\le \int_{y_n-r}^{y_n+r}\cdots\int_{y_1-r}^{y_1+r} |u|^p\,dx_1\ldots dx_n \\
&\le (2r)^p \int_{y_n-r}^{y_n+r}\cdots\int_{y_1-r}^{y_1+r} |Du|^p\,dx_1\ldots dx_n \\
&\le (2r)^p \int_\Omega |Du|^p\,dx.
\end{aligned}
$$

The case $u \in W_0^{1,p}(\Omega)$ again by approximation. □

For simplicity, we next work in cubes:

$$
Q = [a_1, b_1] \times \ldots \times [a_n, b_n] \subset \mathbb{R}^n, \quad (b_1 - a_1) = \ldots = (b_n - a_n),
$$

$$
l(Q) = (b_1 - a_1) = \text{side length of the cube,}
$$

and

$$
Q(x, l) = \{y \in \mathbb{R}^n \,:\, |y_i - x_i| \le \frac{l}{2}, i = 1, \ldots, n\}.
$$

Observe that $|Q| = l^n$ and $\operatorname{diam}(Q) = \sqrt{n}\,l$.

Theorem 2.54 ($1 \le p < \infty$)**.** *Let $1 \le p < \infty$, $Q \subset \mathbb{R}^n$ and $u \in W^{1,p}(Q)$. Then*

$$
\int_Q |u - u_Q|^p\,dx \le l^p n^p \int_Q |Du|^p\,dx.
$$

Proof. By approximation argument, we may again concentrate on $u \in C^\infty(\mathbb{R}^n)$. Let $x, y \in Q$ and approximate

$$
\begin{aligned}
|u(x) - u(y)| &\le |u(x) - u(x_1, \ldots, x_{n-1}, y_n)| + \ldots + |u(x_1, y_2, \ldots, y_n) - u(y)| \\
&\le \sum_{i=1}^n \int_{a_i}^{b_i} |Du(x_1, \ldots, x_{i-1}, t, y_{i+1}, \ldots, y_n|\,dt.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&|u(x) - u(y)|^p \\
&\le \Big(\sum_{i=1}^n \int_{a_i}^{b_i} |Du(x_1, \ldots, x_{i-1}, t, y_{i+1}, \ldots, y_n|\,dt\Big)^p \\
&\overset{\text{Hölder}}{\le} \Big(\sum_{i=1}^n (b_i - a_i)^{(p-1)/p}\Big(\int_{a_i}^{b_i} |Du(x_1, \ldots, x_{i-1}, t, y_{i+1}, \ldots, y_n)|^p\,dt\Big)^{1/p}\Big)^p \\
&\overset{*}{\le} n^{p-1} l^{p-1} \sum_{i=1}^n \int_{a_i}^{b_i} |Du(\ldots)|^p\,dt
\end{aligned}
$$

where at $*$ we used $(c_1 + \ldots + c_n)^p = (\frac{n}{n}c_1 + \ldots + \frac{n}{n}c_n)^p \overset{\text{convexity}}{\leq} \sum_{i=1}^n (nc_i)^p/n$.

Now

$$
\begin{aligned}
\int_Q |u - u_Q|^p \, dx &= \int_Q \left| u(x) - \fint_Q u(y) \, dy \right|^p dx \\
&\leq \int_Q \left| \fint_Q u(x) - u(y) \, dy \right|^p dx \\
&\overset{\text{Hölder}}{\leq} \int_Q \fint_Q |u(x) - u(y)|^p \, dy \, dx \\
&\leq \frac{n^{p-1} l^{p-1}}{|Q|} \int_Q \int_Q \sum_{i=1}^n \int_{a_i}^{b_i} |Du(\ldots)|^p \, dt \, dy \, dx \\
&\overset{\text{Fub+recall } (\ldots)}{=} \frac{n^{p-1} l^{p-1}}{|Q|} l^{n+1} \sum_{i=1}^n \int_Q |Du(z)|^p \, dz \\
&\leq n^p l^p \int_Q |Du(z)|^p \, dz.
\end{aligned}
$$

The general case $u \in W^{1,p}(Q)$ again follows by approximation. $\square$

Theorem 2.55 ($1 \leq p < n$)**.** *Let $1 \leq p < n$ and $u \in W^{1,p}(B(x,r))$. Then there exists a constant $C = C(n,p) > 0$ such that*

$$
\Big(\int_{B(x,r)} \left| u - u_{B(x,r)} \right|^{p^*} dy \Big)^{1/p^*} \leq C \Big(\int_{B(x,r)} |Du|^p \, dy \Big)^{1/p}
$$

i.e.

$$
\Big(\fint_{B(x,r)} \left| u - u_{B(x,r)} \right|^{p^*} dy \Big)^{1/p^*} \leq Cr \Big(\fint_{B(x,r)} |Du|^p \, dy \Big)^{1/p}.
$$

Similarly to the above it holds that

$$
\Big(\fint_{B(x,r)} \left| u - u_{B(x,r)} \right|^{p} dy \Big)^{1/p} \leq Cr \Big(\fint_{B(x,r)} |Du|^p \, dy \Big)^{1/p},
$$

for $1 < p < \infty$.

We do not prove the result in this form, but prove a weaker result in cubes with a bigger cube on the right hand side:

Theorem 2.56 ($1 \leq p < n$)**.** *Let $u \in W^{1,p}(2Q)$, $Q := Q(z,l) \subset \mathbb{R}^n$ and $2Q := Q(z,2l)$. Then there exists a constant $C = C(n,p) > 0$ such that*

$$
\Big(\int_Q |u - u_Q|^{p^*} dy \Big)^{1/p^*} \leq C \Big(\int_{2Q} |Du|^p \, dy \Big)^{1/p}
$$

i.e.

$$\Big(\fint_Q |u-u_Q|^{p^*}\,dy \Big)^{1/p^*} \le Cl\Big(\fint_{2Q} |Du|^p\,dy \Big)^{1/p}.$$

Proof. Let $\eta \in C_0^\infty(\mathbb{R}^n)$ be a cut-off function such that

$$0 \le \eta \le 1, \quad |D\eta| \le \frac{C}{l},$$

and

$$\eta(x) = \begin{cases} 1 & x \in Q \\ 0 & x \in \mathbb{R}^n \setminus 2Q. \end{cases}$$

Then $(u-u_Q)\eta \in W_0^{1,p}(2Q)$ and

$$\begin{aligned}
&\Big(\int_Q |u-u_Q|^{p^*}\,dx \Big)^{1/p^*} \\
&\quad \overset{\operatorname{spt}\eta \subset 2Q}{\le} \Big(\int_{2Q} |(u-u_Q)\eta|^{p^*}\,dx \Big)^{1/p^*} \\
&\quad \overset{\text{Sobo ineq}}{\le} \Big(\int_{2Q} |D((u-u_Q)\eta)|^p\,dx \Big)^{1/p} \\
&\quad \le C\Big(\int_{2Q} \eta^p |Du|^p\,dx \Big)^{1/p} + C\Big(\int_{2Q} |D\eta|^p |u-u_Q|^p\,dx \Big)^{1/p} \\
&\quad \le C\Big(\int_{2Q} |Du|^p\,dx \Big)^{1/p} + \frac{C}{l}\Big(\int_{2Q} |u-u_Q|^p\,dx \Big)^{1/p}.
\end{aligned}$$

Further we may change u_Q to u_{2Q} as

$$
\begin{aligned}
&\Big(\int_{2Q} |u-u_Q|^p\,dx\Big)^{1/p} \\
&\qquad = \Big(\int_{2Q} |u-u_Q+u_{2Q}-u_{2Q}|^p\,dx\Big)^{1/p} \\
&\qquad \leq C\Big(\int_{2Q} |u-u_{2Q}|^p\,dx\Big)^{1/p} + C\Big(\int_{2Q} |u_{2Q}-u_Q|^p\,dx\Big)^{1/p} \\
&\qquad \leq C\Big(\int_{2Q} |u-u_{2Q}|^p\,dx\Big)^{1/p} + C\Big(\int_{2Q} \Big|u_{2Q}-\fint_Q u\,dy\Big|^p\,dx\Big)^{1/p} \\
&\qquad \overset{\text{Poincaré}}{\leq} Cl\Big(\int_{2Q} |Du|^p\,dx\Big)^{1/p} + \ldots \\
&\qquad \overset{\text{Hölder}}{\leq} Cl\Big(\int_{2Q} |Du|^p\,dx\Big)^{1/p} + C\Big(\int_{2Q}\fint_{2Q} |u-u_{2Q}|^p\,dy\,dx\Big)^{1/p} \\
&\qquad \overset{\text{Poincaré}}{\leq} Cl\Big(\int_{2Q} |Du|^p\,dx\Big)^{1/p},
\end{aligned}
$$

where we used the facts that $\fint_Q \leq C\fint_{2Q}$ and $\int_{2Q} 1\,dx = |2Q|$. Combining the above estimates, l will cancel out, and we obtain the claim. $\square$

Remark 2.57 (Warning)**.** *The global version*

$$\int_\Omega |u-u_\Omega|^p\,dy \leq C\int_\Omega |Du|^p\,dy.$$

does not (in contrast with Sobolev's inequality) hold without regularity assumptions on Ω. Exercise.

2.9.3. *Morrey's inequality, $p>n$.*

Theorem 2.58. *Let $u \in W^{1,p}(\mathbb{R}^n)$, $p>n$. Then there exists $C = C(n,p)$ such that*

$$|u(x)-u(y)| \leq C|x-y|^{1-n/p}\,||Du||_{L^p(\mathbb{R}^n)}$$

for almost every $x, y \in \mathbb{R}^n$.

Proof. Let $u \in C^\infty(\mathbb{R}^n)\cap W^{1,p}(\mathbb{R}^n)$ and $x, y \in Q := Q(x_0,l)$. Again

$$
\begin{aligned}
|u(x)-u(y)| &= \Big|\int_0^1 \frac{\partial u}{\partial t}(y+t(x-y))\,dt\Big| \\
&\leq \Big|\int_0^1 Du((1-t)y+tx)\cdot(x-y)\,dt\Big|.
\end{aligned}
$$

By using this

$$
\begin{aligned}
|u(y)-u_Q| &= \left| u(y) - \fint_Q u\,dx \right| \\
&\leq \fint_Q |u(y)-u(x)|\,dx \\
&\leq \fint_Q \left| \int_0^1 Du((1-t)y+tx)\cdot(x-y)\,dt \right| dx \\
&\overset{\text{def of grad}}{\leq} \frac{1}{|Q|}\sum_{i=1}^n \int_Q \int_0^1 \left| \frac{\partial u}{\partial z_i}((1-t)y+tx) \right| \underbrace{|(x-y)_i|}_{\leq l}\,dt\,dx \\
&\overset{\text{Fub}}{\leq} \frac{1}{l^{n-1}}\sum_{i=1}^n \int_0^1 \int_{Q(x_0,l)} \left| \frac{\partial u}{\partial z_i}((1-t)y+tx) \right| dx\,dt.
\end{aligned}
$$

Then we change variables $z=(1-t)y+tx$ i.e. $z_0=(1-t)y+tx_0$ and $dz=t^n\,dx$

$$
\begin{aligned}
&\frac{1}{l^{n-1}}\sum_{i=1}^n \int_0^1 \int_{Q(x_0,l)} \left| \frac{\partial u}{\partial z_i}((1-t)y+tx) \right| dx\,dt \\
&\quad\leq \frac{1}{l^{n-1}}\sum_{i=1}^n \int_0^1 \frac{1}{t^n}\int_{Q(z_0,tl)} \left| \frac{\partial u}{\partial z_i}(z) \right| dz\,dt \\
&\quad\overset{\text{Hölder}}{\leq} \frac{1}{l^{n-1}}\sum_{i=1}^n \int_0^1 \frac{1}{t^n}\Big(\int_{Q(z_0,tl)} \left| \frac{\partial u}{\partial z_i}(z) \right|^p dz \Big)^{1/p} |Q(z_0,tl)|^{(p-1)/p}\,dt \\
&\quad\overset{Q((1-t)y+tx_0,tl)\subset Q(x_0,l)}{\leq} \frac{n}{l^{n-1}}\,||Du||_{L^p(Q(x_0,l)}\int_0^1 \frac{1}{t^n}|Q(x_0,l)|^{(p-1)/p}\,dt \\
&\quad\leq \frac{n}{l^{n-1}}\,||Du||_{L^p(Q(x_0,l))}\int_0^1 \frac{1}{t^n}(tl)^{n(p-1)/p}\,dt \\
&\quad\leq nl^{(p-n)/p}\,||Du||_{L^p(Q(x_0,l))}\int_0^1 t^{-n/p}\,dt,
\end{aligned}
$$

where we also used $n(p-1)/p-n+1=(np-n-np+p)/p=(p-n)/p$ and $n(p-1)/p-n=(np-n-np)/p=-n/p$. Since, and here we use the fact $n<p$,

$$
\int_0^1 t^{-n/p}\,dt = (1-\frac{n}{p})^{-1} = p/(p-n),
$$

we get by combining the estimates that

$$\begin{aligned}|u(y) - u_Q| &\le \frac{np}{p-n} l^{(p-n)/p} \, ||Du||_{L^p(Q(x_0,l))} \\ &\le \frac{np}{p-n} l^{1-n/p} \, ||Du||_{L^p(Q(x_0,l))} .\end{aligned}$$

To establish the final estimate, we write

$$\begin{aligned}|u(x) - u(y)| &\le |u(x) - u_Q| + |u_Q - u(y)| \\ &\le \frac{2np}{p-n} l^{1-n/p} \, ||Du||_{L^p(Q(x_0,l))} .\end{aligned}$$

for every $x, y \in Q$. Hence, as for every $x, y \in \mathbb{R}^n$ there is $Q(x_0, l)$ such that $l = 2|x-y|$ and $x, y \in Q(x_0, l)$, we finally have

$$|u(x) - u(y)| \le C|x-y|^{1-n/p} \, ||Du||_{L^p(\mathbb{R}^n)} ,$$

for $u \in C^\infty(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$.

We extend this result to $u \in W^{1,p}(\mathbb{R}^n)$ by approximation: Let u_ε be a standard mollification of u. Then by the above

$$|u_\varepsilon(x) - u_\varepsilon(y)| \le C|x-y|^{1-n/p} \, ||Du_\varepsilon||_{L^p(\mathbb{R}^n)} .$$

By passing to the limit $\varepsilon \to 0$ and using the results, proved for approximations, we get for almost every $x, y \in \mathbb{R}^n$ (at Lebesgue points of u to be more precise)

$$|u(x) - u(y)| \le C|x-y|^{1-n/p} \, ||Du||_{L^p(\mathbb{R}^n)} . \qquad \square$$

Remark 2.59. *By Morrey's inequality every $u \in W^{1,p}(\mathbb{R}^n)$ can be redefined in a set of measure zero to be* Hölder-continuous.

Remark 2.60. *Let $p > n$. In the open set Ω the above only holds locally in the sense that*

$$u \in W^{1,p}(\Omega) \Rightarrow u \in C^{0,1-n/p}_{loc}(\Omega).$$

Ex: Find an example showing that global implication is false.

2.9.4. *Lipschitz functions and $W^{1,\infty}$.*

Theorem 2.61. *A function $u : \mathbb{R}^n \to \mathbb{R}$ has a Lipschitz continuous representative if and only if $u \in W^{1,\infty}(\mathbb{R}^n)$.*

Proof. *"$\Leftarrow$":* Let $u \in W^{1,\infty}(\mathbb{R}^n)$ and $\operatorname{spt} u$ is compact (if not, we may multiply by a cut-off function). By our results for approximations

$$\begin{aligned}u_\varepsilon &\in C_0^\infty(\mathbb{R}^n) \\ u_\varepsilon &\to u \quad \text{a.e. } \mathbb{R}^n \\ ||u_\varepsilon||_{L^\infty(\mathbb{R}^n)} &\le ||u||_{L^\infty(\mathbb{R}^n)} ,\end{aligned}$$

(the third one immediately follows from the def of mollification.) Thus we may estimate

$$\begin{aligned}|u_\varepsilon(x) - u_\varepsilon(y)| &= \left| \int_0^1 Du_\varepsilon(y + t(x-y)) \cdot (x-y)\, dt \right| \\ &\le ||Du_\varepsilon||_{L^\infty(\mathbb{R}^n)} |x-y| \\ &\le ||Du||_{L^\infty(\mathbb{R}^n)} |x-y|.\end{aligned}$$

Then, we pass to the limit $\varepsilon \to 0$ and, since the left hand side converges almost everywhere we obtain that

$$|u(x) - u(y)| \le ||Du||_{L^\infty(\mathbb{R}^n)} |x-y|.$$

"⇒": Suppose that u is Lipschitz continuous i.e.

$$|u(x) - u(y)| \le L|x-y|$$

for all $x, y \in \mathbb{R}^n$. We utilize the difference quotiens and estimate

$$\left|D_j^{-h} u(x)\right| = \left|\frac{u(x - he_j) - u(x)}{h}\right| \le L$$

and thus $\left|\left|D_j^{-h} u(x)\right|\right|_{L^2(\Omega)} \le L|\Omega|^{\frac{1}{2}}$ for a bounded Ω . Since L^2 is reflexive there exists a subsequence $h_i \to 0$ and functions $v_j \in L^\infty(\Omega)$ such that

$$D_j^{-h_i} u \to v_j \quad \text{weakly in } L^2(\Omega).$$

Thus

$$\begin{aligned}\int_\Omega u \frac{\partial \varphi}{\partial x_j}\, dx &\overset{\text{def}}{=} \int_\Omega (\lim_{h_i \to 0} D_j^{h_i} \varphi) u\, dx \\ &\overset{\text{DOM}}{=} \lim_{h_i \to 0} \int_\Omega (D_j^{h_i} \varphi) u\, dx \\ &= \lim_{h_i \to 0} \int_\Omega \varphi D_j^{-h_i} u\, dx \\ &= \int_\Omega v_j \varphi\, dx\end{aligned}$$

for every $\varphi \in C_0^\infty(\Omega)$. Thus

$$\frac{\partial u}{\partial x_j} = -v_j \quad \text{in the weak sense.}$$

□

2.10. **Compactness theorem.** Recall Remark 2.23 showing that Sobolev space is not compact. However, Sobolev space embeds compactly to suitable L^p spaces. This is sometimes useful for example in the existence proofs.

Theorem 2.62 (Rellich-Kontrachov compactness thm)**.** *Let B be a ball, $u_i \in W^{1,p}(B)$, $1 \le p < n$ and $||u_i||_{W^{1,p}(B)} < C < \infty$ for each $i = 1, 2, \ldots$. Then for each $1 \le q < p^*$ there exists a subsequence and a limit $u \in W^{1,p}(B)$ such that*

$$u_i \to u \quad \text{in } L^q(B).$$

We don't work out a detailed proof, but remark that the proof is based on the following steps:

- By approximation, it holds that
$$(u_i)_\varepsilon \to u_i \quad \text{in } L^q(B) \quad \text{as } \varepsilon \to 0, \text{ uniformly in } i.$$
- Thus it suffices to prove the result for mollified functions. We show for mollified functions that
$$|(u_i)_\varepsilon| \le \frac{C}{\varepsilon^n}, \quad |D(u_i)_\varepsilon| \le \frac{C}{\varepsilon^{n+1}}.$$
- Arzela-Ascoli's compactness result completes the proof.

Remark 2.63. *The case $p > n$ is easier. Why?*

3. Uniformly elliptic linear PDEs

We consider the second order linear elliptic equations in the divergence form, and the (Dirichlet) boundary value problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open set, $u : \Omega \to \mathbb{R}$ is the (a priori unknown) solution to the problem, and $g : \overline{\Omega} \to \mathbb{R}$ and $f : \Omega \to \mathbb{R}$. Finally, L denotes a second order partial differential equation of the form

$$Lu(x) = -\sum_{i,j=1}^{n} D_i(a_{ij}(x) D_j u(x)) + \sum_{i=1}^{n} b_i(x) D_i u(x) + c(x)u(x)$$

for given coefficients a_{ij}, b_i and c.

Example 3.1. *Let $\mathcal{A} = I$. Then $\lambda = \Lambda$ and*

$$-\operatorname{div}(\mathcal{A}(x)Du) = -\operatorname{div}(Du) = -\sum_{i=1}^{n} D_i D_i u = -\Delta u$$

ie. we obtain the Laplacian studied in the course PDE1.

Example 3.2. *Let* $b_i = 0$ *and* $c = 0$ *and* $\mathcal{A}(x) = [a_{ij}]_{i,j=1,2,\dots,n}$ *. Then*

$$Lu = -\sum_{i,j=1}^{n} D_i(a_{ij}(x)D_j u(x)) = -\sum_{i=1}^{n} D_i(\sum_{j=1}^{n} a_{ij}(x)D_j u(x))$$
$$= -\operatorname{div}(\mathcal{A}(x)Du).$$

This explains, why we say that the equation is in the divergence form.

We always assume that $\mathcal{A}$ is a symmetric matrix ie. $a_{ij} = a_{ji}$.

Remark 3.3. *Observe that here* $\mathcal{A}$ *does not depend on* u *or* Du*. If it did, for example* $\mathcal{A} = |Du|^{p-2}I$*, yielding the so called* p*-Laplacian*

$$\Delta_p u := \operatorname{div}(|Du|^{p-2}Du) = 0,$$

the equation could be nonlinear. Our operator L *instead is linear,* $\tilde{a}, \tilde{b} \in \mathbb{R}$

$$L(\tilde{a}u + \tilde{b}v) = \tilde{a}Lu + \tilde{b}Lv.$$

3.1. **Physical interpretation.** As mentioned above our divergence form operator models diffusion as a physical interpretation. Consider chemical which flows and diffusion takes place from the higher concentration to lower. To be more precise

- t time
- $x \in \Omega \subset \mathbb{R}^n$ location
- $u(x,t)$ chemical concentration at place x at time t
- $b \in \mathbb{R}^n$ velocity
- a diffusion coefficient, constant for simplicity.

Also, we do not worry about smoothness etc. in the formal argument below. In any subdomain $\Omega' \subset \Omega$ the the total amount of chemical $\int_{\Omega'} u(x,t)\,dx$ only changes because of the inward and outward flux through the boundary

$$\underbrace{\frac{\partial}{\partial t}\int_{\Omega'} u(x,t)\,dx}_{\text{total change}} = \underbrace{-\int_{\partial\Omega'} bu \cdot \nu\,dS}_{\text{flow}} \underbrace{-\int_{\partial\Omega'} -aDu \cdot \nu\,dS}_{\text{diffusion}} \tag{3.11}$$

where $\nu = (\nu_1, \dots, \nu_n)$ is the outward unit normal vector. The minus sign in front of the integral on the right hand side is due to the fact that we are using outward vector. It is natural to assume that the diffusion is comparable to the difference in concentration of the chemical and thus to $-aDu$. Recall Gauss-Green theorem

$$\int_{\Omega'} D_i u\,dx = \int_{\partial\Omega'} u\nu_i\,dx, \quad i = 1, 2 \dots, n.$$

which implies the divergence theorem for $F : \Omega' \to \mathbb{R}^n$ $F = (F_1, \dots, F_n)$

$$\int_{\partial\Omega'} F \cdot \nu \, dS = \sum_{i=1}^{n} \int_{\partial\Omega'} F_i \nu_i \, dS \overset{\text{G-G}}{=} \sum_{i=1}^{n} \int_{\Omega'} D_i F_i \, dx = \int_{\Omega'} \operatorname{div} F_i \, dx.$$

Thus

$$\begin{aligned}\frac{\partial}{\partial t} \int_{\Omega'} u(x,t) \, dx &= -\int_{\partial\Omega'} bu \cdot \nu \, dS + \int_{\partial\Omega'} aDu \cdot \nu \, dS \\ &\overset{\text{div-thm}}{=} -\int_{\Omega'} \operatorname{div}(bu) \, dx + \int_{\partial\Omega'} a \operatorname{div}(Du) \, dx.\end{aligned}$$

Finally taking the time derivative inside the integral and using the fact that the above argument holds for all $\Omega' \subset \Omega$ we get

$$u_t = a\Delta u - \operatorname{div}(bu).$$

If we have reached an equilibrium, then $\frac{\partial}{\partial t} \int_{\Omega'} u(x,t) \, dx = 0$ and we end up with

$$0 = a\Delta u - \operatorname{div}(bu) = a\Delta u - \operatorname{div}(b)u - b \cdot Du,$$

which is a special case of the equations we are studying. Moreover, if we had a source/sink of the chemical then this would add

$$+ \int_{\Omega'} f \, dx$$

on the RHS of (3.11) where f is given. Moreover, decay (or creation) of the chemical would be modelled by adding

$$- \int_{\Omega'} cu \, dx$$

on the RHS of (3.11). Thus we would have

$$\begin{aligned} 0 &= \underbrace{a\Delta u}_{\text{diffusion}} \underbrace{- \operatorname{div}(bu)}_{\text{transport}} + \underbrace{f}_{\text{source/sink}} \underbrace{-cu}_{\text{decay}} \\ &= a\Delta u - b \cdot Du - (\operatorname{div}(b)u + c)u + f.\end{aligned}$$

Finally, if the diffusion coefficient is not the same constant to all the directions, i.e. we have an anisotropic medium, then we replace aDu by more general divergence form operator and end up with

$$-\sum_{i,j=1}^{n} D_i(a_{ij} D_j u) + b \cdot Du + (\operatorname{div}(b)u + c)u = f.$$

PUNCHLINE: Our equation models general diffusion, transport, decay, and source/sink.

Definition 3.4 (uniformly elliptic)**.** *PDE is uniformly elliptic if there exists constants*

$$0 < \lambda \leq \Lambda < \infty$$

such that

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2$$

for a.e. $x \in \Omega$ *and* $\xi \in \mathbb{R}^n$*.*

Our **standing assumptions** are (unless otherwise stated)

$$\begin{aligned} &a_{ij}, c_i, b_i \in L^\infty(\Omega), \quad \text{uniform ellipticity} \\ &\mathcal{A} \text{ symmetric}, \quad \Omega \text{ open, bounded.} \end{aligned} \tag{3.12}$$

Intuitively, uniform ellipticity tells us how degenerate the diffusion determined by the diffusion coefficients to each direction can be: diffusion does not extinct or blow up. This helps in existence, regularity etc. Uniform ellipticity tells that real (due to symmetry) eigenvalues $\lambda_i(x)$ of $\mathcal{A}$ satisfy $\lambda \leq \lambda_i(x) \leq \Lambda$.

3.2. Weak solutions.

Example 3.5. $x \in (0,2) = \Omega$, $b = 0 = c$, $a = 1$ *and*

$$f(x) = \begin{cases} 1 & x \in (0,1] \\ 2 & x \in (1,2). \end{cases}$$

Consider the problem

$$\begin{cases} Lu = f, & x \in \Omega \\ u(0) = 0 = u(2). \end{cases}$$

Then solving formally in $(0,1]$ *and* $(1,2)$ *as well as requiring that the solution is in* C^1*, from the equation*

$$Lu = -u'' = \begin{cases} 1, & x \in (0,1] \\ 2, & x \in (1,2) \end{cases}$$

we obtain

$$u(x) = \begin{cases} -\frac{x^2}{2} + 1.25x & x \in (0,1] \\ -x^2 + 2.25x - 0.5, & x \in (1,2). \end{cases}$$

Clearly, this is not in C^2*. Is this a unique solution in some sense? Even more irregular examples are possible, see Example 3.11.*

In the spirit of Hilbert's 20th problem, to guarantee the existence of solutions, we can extend the class of functions to be studied. These less than C^2 regular solutions are called weak solutions (in contrast with classical solutions that are C^2 and satisfy the equation pointwise).

We work in the spirit of Sobolev spaces, test the equation with smooth test functions and integrate by part to get rid of the second derivatives, so that only $u \in W^{1,2}(\Omega)$ is needed in the weak definition.

Let $u \in C^2(\Omega)$, $a_{ij} \in C^1(\Omega)$, $f \in C(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$. Then starting from $Lu = f$ we can calculate

$$\begin{aligned}\int_\Omega f\varphi\,dx &= \int_\Omega \big(-\sum_{i,j=1}^n D_i(a_{ij}D_ju(x)) + \sum_{i=1}^n b_i(x)D_iu + cu\big)\varphi\,dx \\ &\overset{\text{int by parts}}{=} \int_\Omega \big(\sum_{i,j=1}^n a_{ij}D_juD_i\varphi + \sum_{i=1}^n b_i(x)D_iu\varphi + cu\varphi\big)\,dx.\end{aligned} \tag{3.13}$$

On the other hand, if

$$\begin{aligned}0 &= \int_\Omega \big(\sum_{i,j=1}^n a_{ij}D_juD_i\varphi + \sum_{i=1}^n b_iD_iu\varphi + cu\varphi - f\varphi\big)\,dx \\ &\overset{\text{int by parts}}{=} \int_\Omega \big(-\sum_{i,j=1}^n D_i(a_{ij}D_ju) + \sum_{i=1}^n b_iD_iu + cu\varphi - f\big)\varphi\,dx\end{aligned}$$

for every $\varphi \in C_0^\infty(\Omega)$, then by fundamental lemma in calc var Lemma 2.9, it holds for $x \in \Omega$ that

$$-\sum_{i,j=1}^n D_i(a_{ij}D_ju) + \sum_{i=1}^n b_iD_iu + cu\varphi - f = 0.$$

Observe that the right hand side of (3.13) makes sense even with weaker assumptions, for example,

$$a_{ij}, b_i, c \in L^\infty(\Omega) \quad \text{and} \quad f \in L^2(\Omega)$$

and

$$u \in W^{1,2}_{\text{loc}}(\Omega).$$

Definition 3.6 (Weak solution, local)**.** *The function $u \in W^{1,2}_{loc}(\Omega)$ is a weak solution to $Lu = f$ if*

$$\int_\Omega \big(\sum_{i,j=1}^n a_{ij}D_juD_i\varphi + \sum_{i=1}^n b_iD_iu\varphi + cu\varphi\big)\,dx = \int_\Omega f\varphi\,dx$$

for every $\varphi \in C_0^\infty(\Omega)$.

Remark 3.7 (Warning)**.** *This definition is useful when studying local properties such as local regularity of solutions. However, the solutions are not uniquely identified without fixing boundary values.*

Definition 3.8 (Weak solution to the boundary value problem)**.** *Let* $g \in W^{1,2}(\Omega)$. *The function* $u \in W^{1,2}(\Omega)$ *is a weak solution to*

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

if $u - g \in W_0^{1,2}(\Omega)$ *and*

$$\int_\Omega \big(\sum_{i,j=1}^n a_{ij} D_j u D_i \varphi + \sum_{i=1}^n b_i D_i u \varphi + cu\varphi \big)\, dx = \int_\Omega f\varphi\, dx$$

for every $\varphi \in C_0^\infty(\Omega)$.

Remark 3.9. *In the literature, the sums are sometimes dropped for brevity*

$$\int_\Omega (a_{ij} D_j u D_i \varphi + b_i D_i u \varphi + cu\varphi)\, dx = \int_\Omega f\varphi\, dx.$$

Example 3.10. *Let us check that*

$$u(x) = \begin{cases} -\frac{x^2}{2} + 1.25x, & x \in (0,1] \\ -x^2 + 2.25x - 0.5, & x \in (1,2). \end{cases}$$

is a weak solution to Example 3.5. First task is to show that $u \in W^{1,2}(\Omega)$ *and* $u \in W_0^{1,2}(\Omega)$, *which is left as an exercise.*

Then, let $\varepsilon > 0$, $\varphi \in C_0^\infty((0,2)))$ *and write*

$$\begin{aligned}\int_{(0,2)} f\varphi\, dx &\overset{DOM}{\leftarrow} \int_{(0,1-\varepsilon)\cup(1+\varepsilon,2)} f\varphi\, dx \overset{class.\ sol\ x\neq 1}{=} -\int_{(0,1-\varepsilon)\cup(1+\varepsilon,2)} (au')'\varphi\, dx \\ &= \int_{(0,1-\varepsilon)\cup(1+\varepsilon,2)} (au')\varphi'\, dx - a(1-\varepsilon)u'(1-\varepsilon) + 0 - 0 + a(1+\varepsilon)u'(1+\varepsilon) \\ &\overset{DOM,\ cancellation}{\to} \int_{(0,2)} (au')\varphi'\, dx\end{aligned}$$

as $\varepsilon \to 0$. *Above the use of DOM can be justified, and at the last step cancellation*

$$-a(1-\varepsilon)u'(1-\varepsilon) + a(1+\varepsilon)u'(1+\varepsilon) \to 0$$

is important.

Since here $u \in C^1, a \in C$ no confusion arises even if we immediately write

$$\begin{aligned}\int_{(0,2)} f\varphi\,dx &= -\int_{(0,1)\cup(1,2)} (au')'\varphi\,dx\\ &= \int_{(0,2)} au'\varphi'\,dx\\ &\quad - (a(1)u'(1)\varphi(1) - a(0)u'(0)\varphi(0)) - (a(2)u'(2)\varphi(2) - a(1)u'(1)\varphi(1))\\ &= \int_{(0,2)} au'\varphi'\,dx,\end{aligned}$$

as we did with the weak derivatives.
PUNCHLINE: *The solution above is not in C^2 so it is not a classical solution but it is a weak solution.*

Look next at the example $\Omega = B(0,1) \subset \mathbb{R}^n$, $n > 1$. The first task, if no boundary conditions are considered, is to show that $u \in W^{1,2}_{loc}(\Omega)$. Suppose then that there would only be a singularity at the origin and everything is smooth elsewhere. We get by Gauss' theorem that

$$\begin{aligned}\int_{B(0,1)\setminus\overline{B}(0,\varepsilon)} f\varphi\,dx &\overset{class.,\,x\neq 0}{=} -\int_{B(0,1)\setminus\overline{B}(0,\varepsilon)} \operatorname{div}(\mathcal{A}Du)\varphi\,dx\\ &= \int_{\partial\left((B(0,1)\setminus\overline{B}(0,\varepsilon)\right)} \mathcal{A}Du\cdot\nu\,dS - \int_{B(0,1)\setminus\overline{B}(0,\varepsilon)} \mathcal{A}Du\cdot D\varphi\,dx\\ &= \int_{\partial B(0,\varepsilon)} \mathcal{A}Du\cdot\nu\,dS - \int_{B(0,1)\setminus\overline{B}(0,\varepsilon)} \mathcal{A}Du\cdot D\varphi\,dx,\end{aligned}$$

where $\varphi \in C_0^\infty(B(0,1))$. Then it remains to verify the following convergences

$$\begin{gathered}\int_{\partial B(0,\varepsilon)} \mathcal{A}Du\cdot\nu\,dS \to 0,\\ \int_{B(0,1)\setminus\overline{B}(0,\varepsilon)} \mathcal{A}Du\cdot D\varphi\,dx \to \int_{B(0,1)} \mathcal{A}Du\cdot D\varphi\,dx\\ \int_{B(0,1)\setminus\overline{B}(0,\varepsilon)} f\varphi\,dx \to \int_{B(0,1)} f\varphi\,dx\end{gathered}$$

as $\varepsilon \to 0$ in order to show that u is a weak solution. However, in Example 3.12 below we follow a slightly different strategy

Example 3.11. $x \in (0,2) = \Omega$, $f = 1$, $b = 0 = c$,

$$a(x) = \begin{cases} 1, & x \in (0,1] \\ 2, & x \in (1,2) \end{cases}$$

Consider the problem

$$\begin{cases} Lu = f, & x \in \Omega \\ u(0) = 0 = u(2). \end{cases}$$

Then solving formally in $(0,1]$ *and* $(1,2)$ *as well as requiring suitable conditions in the middle , we obtain*

$$u(x) = \begin{cases} -\frac{x^2}{2} + \frac{5}{6}x & x \in (0,1] \\ -\frac{x^2}{4} + \frac{5}{12}x + \frac{1}{6}, & x \in (1,2). \end{cases}$$

Ex: Show that this is a weak solution to the above problem.
PUNCHLINE: *The above solution is not in* C^2 *or even* C^1 *so the weak solution does not need to have classical first derivatives. This also highlights that the regularity of the coefficients affects the regularity of the solution.*

Example 3.12. *The next example is from Serrin (Pathological solutions of elliptic differential equations, 1964), which he gives for any* n *but here for simplicity* $n = 2$*. Let* $\alpha \in (0,1)$*, and*

$$\mathcal{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \frac{x_1^2 + \alpha^2 x_2^2}{|x|^2} & (1-\alpha^2)\frac{x_1 x_2}{|x|^2} \\ (1-\alpha^2)\frac{x_1 x_2}{|x|^2} & \frac{\alpha^2 x_1^2 + x_2^2}{|x|^2} \end{pmatrix}.$$

Then coefficients are always bounded and

$$\alpha^2 |\xi|^2 \leq \sum_{i,j=1}^{2} a_{ij}(x)\xi_i \xi_j \leq |\xi|^2,$$

(ex) and

$$u : B(0,1) \to \mathbb{R}, \quad u(x) = |x|^{\alpha - 1} x_1$$

with $x = (x_1, x_2)$ *is a weak solution i.e.*

$$\int_{B(0,1)} \mathcal{A}(x) Du(x) \cdot D\varphi(x)\, dx = \int_{B(0,1)} \sum_{i,j=1}^{n} a_{ij} D_j u D_i \varphi \, dx = 0$$

for all $\varphi \in C_0^\infty(\Omega)$*.*

To see this, first show that $u \in W^{1,2}(B(0,1))$ *(ex). Then show that* u *is a classical solution (ex) to*

$$-\sum_{i,j=1}^{n} D_i(a_{ij} D_j u) = 0 \quad in \quad B(0,1) \setminus \{0\},$$

and thus

$$\int_{B(0,1)} \sum_{i,j=1}^{n} a_{ij} D_j u D_i \varphi \, dx = 0$$

for every $\varphi \in C_0^\infty(B(0,1) \setminus \{0\})$*. Then define a cut off function* $\eta \in C_0^\infty(B(0,2r))$ *such that*

$$0 \le \eta \le 1, \quad \eta = 1 \text{ in } B(0,r), \quad |D\eta| \le \frac{2}{r}.$$

Let $\varphi \in C_0^\infty(B(0,1))$ *and observe that* $(1-\eta)\varphi \in C_0^\infty(B(0,1) \setminus \{0\})$*. Thus*

$$\begin{aligned} 0 &= \int_{B(0,1)} \sum_{i,j=1}^{n} a_{ij} D_j u D_i((1-\eta)\varphi) \, dx \\ &= \int_{B(0,1)} \sum_{i,j=1}^{n} (1-\eta) a_{ij} D_j u D_i \varphi \, dx - \int_{B(0,1)} \sum_{i,j=1}^{n} \varphi a_{ij} D_j u D_i \eta \, dx. \end{aligned} \tag{3.14}$$

We will show that the the first term on the RHS converges to $\int_{B(0,1)} \sum_{i,j=1}^{n} a_{ij} D_j u D_i \varphi \, dx$ *and the second converges to* 0*:*

$$\begin{aligned} &\Big|\int_{B(0,1)} \sum_{i,j=1}^{n} \varphi a_{ij} D_j u D_i \eta \, dx\Big| \\ &\quad \le \max_{i,j} ||a_{ij}||_{L^\infty(B(0,1))} \, ||\varphi||_{L^\infty(B(0,1))} \sum_{i,j=1}^{n} \int_{B(0,1)} |D_j u||D_i \eta| \, dx \\ &\quad \overset{\text{def of } \eta}{\le} \max_{i,j} ||a_{ij}||_{L^\infty(B(0,1))} \, ||\varphi||_{L^\infty(B(0,1))} \frac{2}{r} \sum_{i,j=1}^{n} \int_{B(0,2r)} |D_j u| \, dx \\ &\quad \overset{\text{Hölder}}{\le} C \frac{2}{r} \sum_{i,j=1}^{n} \Big(\int_{B(0,2r)} |D_j u|^2 \, dx \Big)^{1/2} \Big(\int_{B(0,2r)} 1 \, dx \Big)^{1/2} \\ &\quad \le \frac{C}{r} \Big(\int_{B(0,2r)} |Du|^2 \, dx \Big)^{1/2} r^{n/2} \\ &\quad \le C r^{\frac{n-2}{2}} \Big(\int_{B(0,2r)} |Du|^2 \, dx \Big)^{1/2} \to 0 \end{aligned}$$

as $r \to 0$*.*

Next we aim at using DOM for the first term on the RHS (3.14) to have

$$\int_{B(0,1)} \sum_{i,j=1}^{n} (1-\eta) a_{ij} D_j u D_i \varphi \, dx \to \int_{B(0,1)} \sum_{i,j=1}^{n} a_{ij} D_j u D_i \varphi \, dx$$

as $r \to 0$. *This is justified by the facts*

$$\lim_{r\to 0}(1-\eta) = 1 \quad \text{for almost every } x \in B(0,1)$$

and

$$|(1-\eta)a_{ij}D_j u D_i\varphi| \le \max_{i,j} ||a_{ij}||_{L^\infty(B(0,1))} |||D\varphi|||_{L^\infty(B(0,1))} |Du| \in L^1.$$

PUNCHLINE: The previous example shows that a weak solution with just bounded coefficients is not better than Hölder continuous and to prove higher regularity for solutions, we need more assumptions on the coefficients later.

Example 3.13. *A modification of Serrin's example also shows that* $W^{1,2}_{loc}(\Omega)$ *is an essential assumption in the definition. Indeed, otherwise we might lose uniqueness, local boundedness of solutions and maximum principles which all are essential and desirable features of the theory.*

Consider $n = 2$, $0 < \varepsilon < 1$, $x = (x_1, x_2)$ *and* $u : B(0,1) \to \mathbb{R}$

$$u(x) = x_1|x|^{-1-\varepsilon}.$$

This is a classical solution to

$$-\sum_{i,j=1}^{n} D_i(a_{ij}D_j u) = 0 \qquad \text{in } B(0,1) \setminus \{0\}$$

with

$$\mathcal{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \frac{\alpha^2 x_1^2 + x_2^2}{|x|^2} & (\alpha^2-1)\frac{x_1x_2}{|x|^2} \\ (\alpha^2-1)\frac{x_1x_2}{|x|^2} & \frac{x_1^2+\alpha^2x_2^2}{|x|^2} \end{pmatrix},$$

$$\alpha = \frac{1}{\varepsilon}.$$

The coefficient are again bounded, and uniformly elliptic with the constants $\lambda = 1$ *and* $\Lambda = \alpha^2$ *(ex). It holds that (ex)*

$$u \in W^{1,p}(B(0,1)) \quad \text{if } p < \frac{2}{1+\varepsilon}.$$

Observe that $p < 2$ *when* $0 < \varepsilon < 1$ *and that*

$$u \notin W^{1,2}(B(0,1)).$$

In a similar way as in the previous example we see that

$$\int_{B(0,1)} \sum_{i,j=1}^{n} a_{ij}D_j u D_i\varphi \, dx = 0 \tag{3.15}$$

for every $\varphi \in C_0^\infty(B(0,1))$.

However, without details we state that u is not locally bounded and does not have the standard maximum principle that we will encounter later. Nor is there uniqueness with the fixed boundary values: indeed solve $v \in W^{1,2}(B(0,1))$ with the boundary values u on $\partial B(0,1)$. Then$u - v = 0$ on $\partial\Omega$, and $u - v$ solves (3.15), but $u - v \not\equiv 0$.

PUNCHLINE: Without the assumption $u \in W^{1,2}_{\text{loc}}(\Omega)$ in the definition of the weak solution, we might lose uniqueness, local boundedness of solutions and maximum principles which all are essential and desirable features of the theory that we will establish later.

3.3. **Existence: Hilbert space approach.** For simplicity, let $b_i = 0$ and that we look for solutions with zero boundary values i.e.

$$u \in W_0^{1,2}(\Omega).$$

The Riesz representation theorem can be used to prove existence for weak solutions to

$$-\sum_{i,j=1}^{n} D_i(a_{ij} D_j u) + cu\varphi = f$$

To this end, we define

$$\langle u, v\rangle := \int_\Omega \big(\sum_{i,j=1}^{n} a_{ij} D_j u D_i v + cuv\big)\, dx.$$

and will show that this is an inner product in $W_0^{1,2}(\Omega)$.

Lemma 3.14. *There is $c_0 \le 0$ such that if $c \ge c_0$, then $\langle \cdot, \cdot\rangle$ is an inner product in $W_0^{1,2}(\Omega)$.*

Proof. We intend to show that $\langle u, u\rangle = 0$ implies $u = 0$ a.e. The other properties of inner product are easier (ex).

If $c \ge c_0 \ge 0$, then the proof is immediate, but we can improve the bound for c_0:

$$\begin{aligned}
\langle u, u\rangle &= \int_\Omega \big(\sum_{i,j=1}^{n} a_{ij} D_j u D_i u + cu^2\big)\, dx \\
&\overset{\text{ellipticity}}{\ge} \int_\Omega \lambda |Du|^2 + c_0 u^2\, dx \\
&\overset{\text{Sob.-Poincaré, Thm 2.53}}{\ge} \int_\Omega \frac{\lambda}{2} |Du|^2 + \big(\frac{\lambda}{\mu 2} + c_0\big) u^2\, dx \\
&\ge \alpha\, ||u||^2_{W^{1,2}(\Omega)},
\end{aligned}$$

where $\alpha = \min\{\lambda/2, (c_0 + \lambda/(2\mu))\}$, and μ originates from $\int_\Omega u^2\,dx \le \mu \int_\Omega |Du|^2\,dx$, $\mu = c\,\mathrm{diam}(\Omega)^2$. Furthermore, we require $c_0 + \lambda/(2\mu) > 0$ which gives the condition for c_0. $\square$

Remark 3.15. *If we set*

$$|||u||| := \sqrt{\langle u, u\rangle},$$

then by the above proof $|||u||| \ge c\,||u||_{W_0^{1,2}(\Omega)}$. *On the other hand*

$$\begin{aligned}
|||u|||^2 &= \int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j u D_i u + cu^2\Big)\,dx \\
&\overset{elliptic}{\le} \Lambda \int_\Omega |Du|^2\,dx + ||c||_{L^\infty(\Omega)} \int_\Omega u^2\,dx \\
&\le C\,||u||^2_{W_0^{1,2}(\Omega)}.
\end{aligned}$$

Thus the new norm $|||\cdot|||$ *is equivalent to* $||\cdot||_{W_0^{1,2}(\Omega)}$.

Lemma 3.16. *Let* $\hat{W}_0^{1,2}(\Omega)$ *be* $W_0^{1,2}(\Omega)$ *with the new inner product* $\langle\cdot,\cdot\rangle$. *Then*

$$F(v) = \int_\Omega f v\,dx$$

is a bounded linear functional in $\hat{W}_0^{1,2}(\Omega)$.

Proof.

$$\begin{aligned}
|F(v)| &= \left|\int_\Omega f v\,dx\right| \\
&\overset{\text{Hölder}}{\le} \Big(\int_\Omega f^2\,dx\Big)^{1/2} \Big(\int_\Omega v^2\,dx\Big)^{1/2} \\
&\le ||f||_{L^2(\Omega)}\,||v||_{W_0^{1,2}(\Omega)} \\
&\le C\,||f||_{L^2(\Omega)}\,|||v|||,
\end{aligned}$$

where at the last step, we used the equivalence of the norms. $\square$

Theorem 3.17. *There is a constant* $c_0 \le 0$ *such that if* $c_0 \ge c$ *then* $Lu = f$ *has a unique weak solution* $u \in W_0^{1,2}(\Omega)$ *for every* $f \in L^2(\Omega)$.

Proof. By the previous lemma

$$F(v) = \int_\Omega f v\,dx$$

is a bounded linear functional in $\hat{W}_0^{1,2}(\Omega)$. Moreover, $\hat{W}^{1,2}(\Omega)$ is a Banach space since the norms $||\cdot||_{W^{1,2}(\Omega)}$ and $|||\cdot|||$ are equivalent. By

Riesz representation theorem for Hilbert spaces, there exists a unique $u \in \hat{W}_0^{1,2}(\Omega)$ such that

$$\begin{aligned} F(v) &= \langle u, v \rangle \\ &= \int_\Omega \sum_{i,j=1}^n a_{ij} D_j u D_i v + cuv \, dx \end{aligned}$$

for every $v \in \hat{W}_0^{1,2}(\Omega)$. By the equivalence of norms we have shown that there is a unique $u \in W_0^{1,2}(\Omega)$ such that

$$\int_\Omega f\varphi \, dx = \int_\Omega \sum_{i,j=1}^n a_{ij} D_j u D_i \varphi + cu\varphi \, dx$$

for every $\varphi \in C_0^\infty(\Omega)$. □

Example 3.18. *Let $f \in L^2(\Omega)$. Then the Poisson problem*

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique weak solution.

Example 3.19. *Consider $\Omega = (0,2)$, $c = 0 = b$, $f = 1$ and*

$$a(x) = \begin{cases} x & x \in (0,1] \\ 1 & x \in (1,2) \end{cases}$$

and a problem

$$\begin{cases} Lu = f, & x \in \Omega \\ u(0) = 0 = u(2). \end{cases}$$

Observe that this is not uniformly elliptic.

Then by solving in $(0,1)$ and $(1,2)$ respectively the equation

$$1 = f = Lu = -(a(x)u'(x))'$$

we obtain

$$u(x) = \begin{cases} -x + c_1 \ln(x) + c_2, & x \in (0,1] \\ -\frac{1}{2}x^2 + c_3 x + c_4, & x \in (1,2) \end{cases}$$

One might then suggest

$$u(x) = \begin{cases} -x, & x \in (0,1] \\ -\frac{1}{2}x^2 + 2.5x - 3, & x \in (1,2) \end{cases}$$

as a weak solution by using the boundary conditions and requiring continuity at $x = 1$. However, this is not a weak solution (ex).

Example 3.20. *Also it is clear that if we consider for example the coefficient*

$$a(x) = \begin{cases} 1, & x \in (0,1] \\ 0, & x \in (1,2) \end{cases}$$

we lose the uniqueness because now we do not have much info about the second derivative in the interval $(1,2)$*.*

PUNCHLINE: Uniform ellipticity is essential for the existence and uniqueness.

Remark 3.21.

- *Let* $g \in W^{1,2}(\Omega)$ *and consider the problem*

  $$\begin{cases} Lu = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

  Then the problem

  $$\begin{cases} Lv = f - Lg & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

  has a solution (Lg *defines a bounded linear functional in the Sobolev space, and our proof extends to this setting as such). For example, for* $L = -\Delta$ *we have*

  $$\int_\Omega Dv \cdot D\varphi \, dx = \int_\Omega f\varphi \, dx - \int_\Omega Dg \cdot D\varphi \, dx.$$

  Thus $u = v + g$ *is a solution to the first problem.*
- *Also observe that no regularity assumptions on* $\partial\Omega$ *is needed.*
- *If we included* $+\sum_{i=1}^n b_i D_i u$ *to our operator, then* L *would not define an inner product. In this case, finding the element* u *as above is still based on Riesz representation theorem but requires more work. This is called* Lax-Milgram theorem.

Example 3.22. *Consider*

$$\begin{cases} -u''(x) - \pi^2 u(x) = 1, & x \in (0,1) \\ u(0) = 0 = u(1) \end{cases}$$

does not have a solution implying, that the condition on c *is necessary. Indeed, let* $v(x) = \sin(\pi x)$*, then*

$$\begin{aligned} B[u,v] &= \int_0^1 u'v' - \pi^2 uv \, dx \\ &\overset{\text{int by parts}}{=} \int_0^1 -uv'' - \pi^2 uv \, dx \end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 u\pi^2 \sin(\pi x) - \pi^2 u \sin(\pi x)\, dx \\
&= 0 \neq \int_0^1 1 \cdot \sin(\pi x)\, dx \\
&= -(\cos(\pi 1) - \cos(0))/\pi = -(-1-1)/\pi = 2/\pi,
\end{aligned}
$$

i.e. we have found a test function $v \in W_0^{1,2}((0,1))$ for which the weak definition does not hold no matter what u is. Later we show in Lemma 3.29 that the class of test functions can be extended from $C_0^\infty(\Omega)$ to $W_0^{1,2}(\Omega)$, and thus there is no weak solution to the above problem.

Looking at the proof of existence result, it fails because $B[u,v]$ is no longer positive definite i.e. there exists $u \in W_0^{1,2}$ such that $B[u,u] < 0$. In particular, $B[u,v]$ no longer gives an inner product.

On the hand, the homogenous problem

$$
\begin{cases} -u''(x) - \pi^2 u(x) = 0, & x \in (0,1) \\ u(0) = 0 = u(1) \end{cases}
$$

has infinitely many solutions

$$
u(x) = a \sin(\pi x), \quad a \in \mathbb{R}.
$$

PUNCHLINE: Lower pound on c is necessary.

3.4. **Existence: variational method.** The existence can be shown by studying the corresponding variational integral. The variational integral related to PDE

$$
-\sum_{i,j=1}^n D_i(a_{ij} D_j u) + cu = f
$$

is

$$
I(v) = \frac{1}{2} \int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j v D_i v + cv^2\Big)\, dx - \int_\Omega f v \, dx
$$

The PDE $-\sum_{i,j=1}^n D_i(a_{ij} D_j u) + cu = f$ is called the Euler-Lagrange equation of this variational integral.

Example 3.23. *The variational integral corresponding to the Poisson equation $-\Delta u = f$ is*

$$
\frac{1}{2} \int_\Omega |Dv|^2 \, dx - \int_\Omega f v \, dx.
$$

Definition 3.24. *A function $u \in W_0^{1,2}(\Omega)$ is a minimizer to the variational integral if*

$$I(u) \leq I(v)$$

for every $v \in W_0^{1,2}(\Omega)$.

Definition 3.25. *Let $g \in W^{1,2}(\Omega)$. A function $u \in W^{1,2}(\Omega)$ with $u-g \in W_0^{1,2}(\Omega)$ is a minimizer to the variational integral with boundary values if*

$$I(u) \leq I(v)$$

for every $v \in W^{1,2}(\Omega)$ such that $v - g \in W_0^{1,2}(\Omega)$.

Theorem 3.26 (Dirichlet principle)**.** *If $u \in W_0^{1,2}(\Omega)$ is a minimizer to the variational integral $I(u)$, then it is a weak solution to the corresponding Euler-Lagrange equation.*

Proof. Let $\varphi \in C_0^\infty(\Omega)$ and $\varepsilon > 0$. Now

$$\begin{aligned} I(u) &\overset{u+\varepsilon\varphi \in W_0^{1,2}(\Omega)}{\leq} I(u+\varepsilon\varphi) \\ &= \frac{1}{2}\int_\Omega \sum_{i,j=1}^n a_{ij} D_j(u+\varepsilon\varphi) D_i(u+\varepsilon\varphi) + c(u+\varepsilon\varphi)^2 \, dx - \int_\Omega f(u+\varepsilon\varphi) \, dx \\ &=: i(\varepsilon). \end{aligned}$$

We utilize the fact that if u is a minimizer, then $i(\varepsilon)$ has a minimum at $\varepsilon = 0$ so that

$$i'(0) = 0.$$

Then

$$\begin{aligned} i(\varepsilon) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^n & a_{ij}(D_j u D_i u + \varepsilon D_j u D_i \varphi + \varepsilon D_j \varphi D_i u + \varepsilon^2 D_j \varphi D_i \varphi) \\ &+ \frac{1}{2}\int_\Omega c(u^2 + 2\varepsilon u\varphi + \varepsilon^2\varphi^2) \, dx - \int_\Omega f(u+\varepsilon\varphi) \, dx. \end{aligned}$$

and

$$\begin{aligned} i'(\varepsilon) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^n & a_{ij}(D_j u D_i \varphi + D_j \varphi D_i u + 2\varepsilon D_j\varphi D_i\varphi) + c(2u\varphi + 2\varepsilon\varphi^2) \, dx \\ &- \int_\Omega f\varphi \, dx. \end{aligned}$$

From this

$$i'(0) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^n a_{ij}(D_j u D_i\varphi + D_j\varphi D_i u) + c2u\varphi\,dx - \int_\Omega f\varphi\,dx$$

$$\overset{a_{ij}=a_{ji}}{=} \int_\Omega \sum_{i,j=1}^n a_{ij} D_j u D_i \varphi + cu\varphi\,dx - \int_\Omega f\varphi\,dx = 0. \qquad \square$$

Lemma 3.27. *Let $f \in L^2(\Omega)$. There is a constant c_0 such that the variational integral $I(v)$ is bounded from below in $W_0^{1,2}(\Omega)$ if $c \geq c_0$.*

Further, we have the estimate

$$\int_\Omega |Dv|^2\,dx + \int_\Omega v^2\,dx \leq c_1 + c_2 I(v),$$

where $c_1, c_2 > 0$ are independent of v.

Proof. By Young's inequality $\int_\Omega |\sqrt{\varepsilon} f v/\sqrt{\varepsilon}|\,dx \leq \frac{\varepsilon}{2}\int_\Omega v^2\,dx + \frac{1}{2\varepsilon}\int_\Omega f^2\,dx$, and thus

$$I(v) \overset{\text{ell}}{\geq} \int_\Omega \frac{\lambda}{2}|Dv|^2 + \frac{c_0}{2}v^2\,dx - \int_\Omega |f||v|\,dx$$

$$\overset{\text{Young}}{\geq} \int_\Omega \frac{\lambda}{2}|Dv|^2 + \frac{c_0}{2}v^2\,dx - \frac{\varepsilon}{2}\int_\Omega v^2\,dx - \frac{1}{2\varepsilon}\int_\Omega f^2\,dx$$

$$\overset{\text{Poincaré,Thm 2.53}}{\geq} \frac{\lambda}{4}\int_\Omega |Dv|^2\,dx + \frac{1}{2}\Big(\frac{\lambda}{2\mu} + c_0 - \varepsilon\Big)\int_\Omega v^2\,dx - \frac{1}{2\varepsilon}\int_\Omega f^2\,dx$$

where we choose $c_0 > -\lambda/(2\mu)$ and ε such that $\frac{\lambda}{2\mu} + c_0 - \varepsilon \geq 0$, so that inequality holds for every $v \in W_0^{1,2}(\Omega)$. Recall that μ is the constant in Poincaré's inequality.

The estimate in the claim is also build in the above proof. $\square$

Next we show existence of a minimizer. As shown above, minimizer is also a solution to the Euler-Lagrange equation. The following proof does not use Hilbert space structure (unlike the first proof) and works in the context of nonlinear equations as well.

Theorem 3.28. *There is a constant c_0 such that if $c \geq c_0$, then for any $f \in L^2(\Omega)$, the variational integral $I(v)$ has a minimizer $u \in W_0^{1,2}(\Omega)$.*

Proof. By the previous lemma $I(v)$ is bounded from below and thus

$$\inf_{v \in W_0^{1,2}(\Omega)} I(v)$$

is a finite number. By the definition of inf there exists a minimizing sequence $u_k \in W_0^{1,2}(\Omega)$ such that

$$I(u_k) \to \inf_{v \in W_0^{1,2}(\Omega)} I(v)$$

as $k \to \infty$. Since the finite limit exists, we also have

$$I(u_k) \le M$$

for some $M < \infty$. By this and the estimate in the previous lemma, we have

$$\int_\Omega |u_k|^2 \, dx + \int_\Omega |Du_k|^2 \, dx \le c_1 + c_2 M.$$

Since u_k and Du_k are bounded in $L^2(\Omega)$, there is a subsequence, still denoted by u_k such that

$$\begin{aligned} u_k &\to u \quad \text{weakly in } L^2(\Omega), \\ Du_k &\to Du \quad \text{weakly in } L^2(\Omega)^n. \end{aligned}$$

Since the space $W_0^{1,2}(\Omega)$ is closed under weak convergence so that $u \in W_0^{1,2}(\Omega)$.

Next we show

$$I(u) \le \liminf_k I(u_k).$$

To establish this, observe that a similar argument as in Lemma 3.27 implies

$$\begin{aligned} &\int_\Omega \sum_{i,j=1}^n a_{ij} D_j(u_k - u) D_i(u_k - u) + c(u_k - u)^2 \, dx \\ &\quad \overset{\text{ell}}{\ge} \int_\Omega \lambda |D(u_k - u)|^2 + c(u_k - u)^2 \, dx \ge 0. \end{aligned}$$

from which it follows that

$$\begin{aligned} &\int_\Omega \sum_{i,j=1}^n a_{ij} D_j u_k D_i u_k + c u_k^2 \, dx \\ &\quad \ge 2 \int_\Omega \sum_{i,j=1}^n a_{ij} D_j u_k D_i u + c u_k u \, dx \\ &\qquad - \int_\Omega \sum_{i,j=1}^n a_{ij} D_j u D_i u + c u^2 \, dx. \end{aligned}$$

Using this, we get

$$\begin{aligned}\liminf_k \int_\Omega \sum_{i,j=1}^n a_{ij} D_j u_k D_i u_k + c(u_k)^2 \, dx \\ &\geq 2 \liminf_k \int_\Omega \sum_{i,j=1}^n a_{ij} D_j u_k D_i u + c u_k u \, dx \\ &\quad - \int_\Omega \sum_{i,j=1}^n a_{ij} D_j u D_i u + c(u)^2 \, dx \\ &= \int_\Omega \sum_{i,j=1}^n a_{ij} D_j u D_i u + c(u)^2 \, dx,\end{aligned}$$

since $D_j u_k \to D_j u$ weakly in $L^2(\Omega)$. Combining this to the fact that weak convergence implies

$$\lim_k \int_\Omega f u_k \, dx = \int_\Omega f u \, dx.$$

we obtain $I(u) \leq \liminf_k I(u_k)$.

Since we originally chose u_k so that $\lim_k I(u_k) = \inf_{v \in W_0^{1,2}(\Omega)} I(v)$, we finally obtain

$$\begin{aligned} I(u) &\leq \liminf_k I(u_k) \\ &= \lim_k I(u_k) \\ &= \inf_{v \in W_0^{1,2}(\Omega)} I(v). \end{aligned}$$

Thus $u \in W_0^{1,2}(\Omega)$ is a minimizer to the variational integral. $\square$

3.5. **Uniqueness and comparison principle.** In this section we consider

$$Lu = -\sum_{i,j=1}^n D_i(a_{ij}(x) D_j u(x)) + c(x)u(x) = f.$$

We start by showing that we can extend the class $C_0^\infty(\Omega)$ of test functions to $W_0^{1,2}(\Omega)$.

Lemma 3.29. *If $u \in W_0^{1,2}(\Omega)$ is a weak solution to $Lu = f$, then*

$$\int_\Omega \sum_{i,j=1}^n a_{ij} D_j u D_i v + cuv \, dx = \int_\Omega f v \, dx$$

for every $v \in W_0^{1,2}(\Omega)$.

Proof. Let $v \in W_0^{1,2}(\Omega)$. By definition of $W_0^{1,2}(\Omega)$, we may take a sequence $\varphi_k \in C_0^\infty(\Omega)$ such that

$$\varphi_k \to v \quad \text{in } W^{1,2}(\Omega).$$

By using this, (3.12), and Hölder's inequality, we obtain

$$\begin{aligned}
&\Big| \int_\Omega \sum_{i,j=1}^n \big(a_{ij} D_j u D_i v + cuv - fv\big)\, dx \Big| \\
&\quad = \Big| \int_\Omega \sum_{i,j=1}^n \big(a_{ij} D_j u D_i (v-\varphi_k) + cu(v-\varphi_k) - f(v-\varphi_k)\big)\, dx \\
&\qquad + \int_\Omega \sum_{i,j=1}^n a_{ij} D_j u D_i \varphi_k + cu\varphi_k - f\varphi_k \, dx \Big| \\
&\quad \le \sum_{i,j=1}^n ||a_{ij}||_{L^\infty(\Omega)} \int_\Omega |D_j u D_i (v-\varphi_k)|\, dx \\
&\qquad + \int_\Omega |cu(v-\varphi_k)| + |f(v-\varphi_k)|\, dx + 0 \\
&\quad \le \sum_{i,j=1}^n ||a_{ij}||_{L^\infty(\Omega)} \Big(\int_\Omega |D_j u|^2\, dx\Big)^{1/2} \Big(\int_\Omega |D_i(v-\varphi_k)|^2\, dx\Big)^{1/2} \\
&\qquad + ||c||_{L^\infty(\Omega)} \Big(\int_\Omega u^2\, dx\Big)^{1/2} \Big(\int_\Omega |v-\varphi_k|^2\, dx\Big)^{1/2} \\
&\qquad + \Big(\int_\Omega f^2\, dx\Big)^{1/2} \Big(\int_\Omega |v-\varphi_k|^2\, dx\Big)^{1/2} \\
&\quad \to 0
\end{aligned}$$

as $k \to \infty$. $\square$

Theorem 3.30 (Uniqueness)**.** *Let $u_1, u_2 \in W_0^{1,2}(\Omega)$ be two weak solutions. There is c_0 such that if $c \ge c_0$ it holds almost everywhere that*

$$u_1 = u_2.$$

Proof. By the previous lemma,

$$\int_\Omega \sum_{i,j=1}^n a_{ij} D_j u_1 D_i v + cu_1 v\, dx = \int_\Omega fv\, dx$$

$$\int_\Omega \sum_{i,j=1}^n a_{ij} D_j u_2 D_i v + cu_2 v\, dx = \int_\Omega fv\, dx$$

for every $v \in W_0^{1,2}(\Omega)$. By subtracting the equations

$$\int_\Omega \sum_{i,j=1}^n a_{ij} D_j(u_1 - u_2) D_i v + c(u_1 - u_2) v \, dx = 0.$$

Now we choose $v = (u_1 - u_2) \in W_0^{1,2}(\Omega)$ and estimate

$$\begin{aligned} 0 &= \int_\Omega \sum_{i,j=1}^n a_{ij} D_j(u_1 - u_2) D_i(u_1 - u_2) + c(u_1 - u_2)^2 \, dx \\ &\geq \int_\Omega \lambda |D_i(u_1 - u_2)|^2 + c(u_1 - u_2)^2 \, dx. \end{aligned}$$

Then

$$\int_\Omega c|u_1 - u_2|^2 \, dx \geq -\frac{\lambda}{2\mu} \int_\Omega |u_1 - u_2|^2 \, dx$$

with the choice $c \geq -\lambda/(2\mu)$. Combining the facts and recalling Poincaré's inequality $\int_\Omega v^2 \, dx \leq \mu \int_\Omega |Dv|^2 \, dx$ we have

$$\begin{aligned} 0 &\geq \int_\Omega \frac{\lambda}{2} |D_i(u_1 - u_2)|^2 + \Big(\frac{\lambda}{2\mu} - \frac{\lambda}{2\mu}\Big)(u_1 - u_2)^2 \, dx \\ &= \frac{\lambda}{2} \int_\Omega |D_i(u_1 - u_2)|^2 \, dx. \end{aligned}$$

Using Poincaré's inequality, we see that $u_1 = u_2$ a.e. □

Example 3.31. *The uniform ellipticity was utilized again: Choose* $\Omega = (0, 2)$, $b = 0 = c$,

$$f(x) = a(x) = \begin{cases} 1, & x \in (0, 1] \\ 0, & x \in [1, 2), \end{cases}$$

and consider the problem

$$\begin{cases} Lu = f, & \textit{in } (0, 2), \\ u(0) = 0 = u(2). \end{cases}$$

Then

$$u_1(x) = \begin{cases} -0.5x^2 + x, & x \in (0, 1] \\ -x^2 + 2.5x - 1, & x \in [1, 2) \end{cases}$$

and

$$u_2(x) = \begin{cases} -0.5x^2 + x, & x \in (0, 1] \\ 1.5 - x, & x \in [1, 2) \end{cases}$$

are weak solutions to $Lu = f$. *(Ex)*

Theorem 3.32 (Comparison principle)**.** *Let $u, w \in W^{1,2}(\Omega)$ be weak solutions and $(u-w)_+ \in W_0^{1,2}(\Omega)$. Then there is c_0 such that if $c \geq c_0$ it holds that*

$$u \leq w \quad \text{in } \Omega.$$

Proof. The idea is the same as in the proof of the uniqueness. First

$$\int_\Omega \sum_{i,j=1}^n a_{ij} D_j u D_i v + cuv \, dx = \int_\Omega fv \, dx$$

$$\int_\Omega \sum_{i,j=1}^n a_{ij} D_j w D_i v + cwv \, dx = \int_\Omega fv \, dx$$

for every $v \in W_0^{1,2}(\Omega)$. By subtracting the equations

$$\int_\Omega \sum_{i,j=1}^n a_{ij} D_j (u-w) D_i v + c(u-w)v \, dx = 0.$$

Now we choose $v = (u-w)_+ \in W_0^{1,2}(\Omega)$ and estimate

$$\begin{aligned} 0 &= \int_\Omega \sum_{i,j=1}^n a_{ij} D_j (u-w) D_i (u-w)_+ + c(u-w)_+^2 \, dx \\ &\geq \int_\Omega \lambda |D_i (u-w)_+|^2 + c(u-w)_+^2 \, dx. \end{aligned}$$

Since

$$\int_\Omega c(u-w)_+^2 \, dx \geq -\frac{\lambda}{2\mu} \int_\Omega (u-w)_+^2 \, dx$$

with choice $c \geq -\lambda/(2\mu)$. Combining the facts and recalling Poincaré's inequality $\int_\Omega v^2 \, dx \leq \mu \int_\Omega |Dv|^2 \, dx$ we have

$$\begin{aligned} 0 &\geq \int_\Omega \frac{\lambda}{2} |D_i (u-w)_+|^2 + \Big(\frac{\lambda}{2\mu} - \frac{\lambda}{2\mu} \Big) (u-w)_+^2 \, dx \\ &= \frac{\lambda}{2} \int_\Omega |D_i (u-w)_+|^2 \, dx. \end{aligned}$$

Again using Poincaré's inequality, we see that $(u-w)_+ = 0$ a.e., that is $u \leq w$ a.e. □

PUNCHLINE: The same technique that gave us uniqueness also gives us the comparison principle. On the other hand, comparison implies uniqueness for solutions.

Remark 3.33. *By analyzing the above proof, we see that also the following holds: Let $u, w \in W^{1,2}(\Omega)$ and u and w be sub- and supersolutions respectively ie.*

$$\int_\Omega \sum_{i,j=1}^n a_{ij} D_j u D_i v + cuv\, dx \le \int_\Omega fv\, dx$$
$$\int_\Omega \sum_{i,j=1}^n a_{ij} D_j w D_i v + cwv\, dx \ge \int_\Omega fv\, dx$$

for every $v \ge 0$, $v \in W_0^{1,2}(\Omega)$, and $(u-w)_+ \in W_0^{1,2}(\Omega)$. Then

$$u \le w \quad \text{in } \Omega.$$

PUNCHLINE: For the comparison principle (but not for uniqueness), it is enough to have sub- and supersolution in the correct order.

3.6. **Regularity.**

3.6.1. *Local L^2-regularity.* In the previous sections, we relaxed the concept of a solution and observed that weak solutions are not necessarily C^2. Next we study what is the natural regularity class and which conditions are needed to have a better regularity.

First we motivate our approach by a formal calculation. Let $f \in L^2(\Omega)$ and u be a solution with zero bdr values to a Poisson equation

$$-\Delta u = f$$

in Ω. Then

$$\begin{aligned}
\int_\Omega f^2\, dx &= \int_\Omega (\Delta u)^2\, dx \\
&= \int_\Omega \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} \sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2}\, dx \\
&= \sum_{i,j=1}^n \int_\Omega \frac{\partial^2 u}{\partial x_i^2} \frac{\partial^2 u}{\partial x_j^2}\, dx \\
&\overset{\text{int by parts}}{=} -\sum_{i,j=1}^n \int_\Omega \frac{\partial^3 u}{\partial x_i^2 \partial x_j} \frac{\partial u}{\partial x_j}\, dx \\
&\overset{\text{int by parts}}{=} \sum_{i,j=1}^n \int_\Omega \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j}\, dx \\
&= \int_\Omega \left|D^2 u\right|^2 dx,
\end{aligned}$$

where we denoted

$$D^2u = \begin{pmatrix} \frac{\partial^2 u}{\partial x_1^2} & \cdots & \frac{\partial^2 u}{\partial x_1 \partial x_n} \\ \frac{\partial^2 u}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 u}{\partial x_2 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 u}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 u}{\partial x_n^2} \end{pmatrix}$$

and $|D^2u|^2 = \sum_{i,j=1}^n \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2$.

1. guess: The L^2 norm of second derivatives is estimated in terms of L^2-norm of f.

Then let us differentiate the Poisson equation

$$\frac{\partial f}{\partial x_k} = -\frac{\partial}{\partial x_k}\Delta u = -\Delta \frac{\partial u}{\partial x_k}.$$

Denote $\overline{f} := \frac{\partial f}{\partial x_k}$ and $\overline{u} := \frac{\partial u}{\partial x_k}$ ie.

$$-\Delta \overline{u} = \overline{f}.$$

Now we may apply the previous calculation to have

2. guess: L^2-norm of the third derivatives of u can be estimated in terms of L^2-norm of the first derivatives of f.

3. guess: A solution u has two more derivatives than f and L^2-norm of the kth derivatives of u can be estimated in terms of L^2-norm of the $k-2$ derivatives of f.

Next we make these formal calculations rigorous for

$$Lu = -\sum_{i,j=1}^n D_i(a_{ij}D_j u) + \sum_{i=1}^n b_i D_i u + cu.$$

with the uniform ellipticity condition, and open, bounded Ω.

Idea: We establish this by roughly speaking replacing derivatives of the formal calculation by difference quotients, and carefully deriving estimates for these.

Theorem 3.34 (local L^2-regularity)**.** *Let*

$$a_{ij} \in C^1(\Omega), b_i \in L^\infty(\Omega), c \in L^\infty(\Omega)$$

and

$$f \in L^2(\Omega).$$

Further, let $u \in W^{1,2}(\Omega)$ be a weak solution to $Lu = f$. Then

$$u \in W^{2,2}_{loc}(\Omega)$$

and for any $\Omega' \Subset \Omega$

$$||u||_{W^{2,2}(\Omega')} \leq C\big(\,||f||_{L^2(\Omega)} + ||Du||_{L^2(\Omega)} + ||u||_{L^2(\Omega)}\,\big),$$

where C *may depend on* Ω', Ω *and* a_{ij}, b_i, c, *but not on* u.

Remark 3.35.

- *Observe that* C *is uniform over all the boundary values, since we didn't assume zero bdr values this time.*
- *It follows from the theorem that*

$$Lu = f\, a.e.,$$

because if $u \in W^{2,2}_{loc}(\Omega)$ *then*

$$\int_\Omega \Big(\sum_{i,j=1}^{n} a_{ij} D_j u D_i \varphi + \sum_{i=1}^{n} b_i D_i u \varphi + cu\varphi - f\varphi\Big)\, dx$$

$$\overset{\text{weak deriv.}}{=} -\int_\Omega \Big(\sum_{i,j=1}^{n} D_i(a_{ij} D_j u) + \sum_{i=1}^{n} b_i D_i u + cu - f\Big)\varphi\, dx$$

holds for every $\varphi \in C_0^\infty(\Omega)$. *Then the fundamental lemma in the calculus of variations, Lemma 2.9, implies the claim. Such solutions are sometimes called strong solutions.*
- *Our earlier examples show that some regularity assumption on* a_{ij} *is needed.*

Proof of Theorem 3.34. Let $\Omega' \Subset \Omega'' \Subset \Omega$ and choose a test function $\eta \in C_0^\infty(\Omega),\ 0 \leq \eta \leq 1$ such that

$$\eta(x) = \begin{cases} 1 & x \in \Omega' \\ 0 & x \in \Omega \setminus \Omega'' \end{cases}$$

Since u is a weak solution, for every $v \in W_0^{1,2}(\Omega)$

$$\int_\Omega \sum_{i,j=1}^{n} a_{ij} D_j u D_i v\, dx = \int_\Omega \overline{f} v\, dx$$

where $\overline{f} = f - \sum_{i=1}^n b_i D_i u - cu$. We choose a test function, for $h > 0$ small enough

$$v = -D_k^{-h}(\eta^2 D_k^h u)$$

where

$$D_k^h u(x) = \frac{u(x + he_k) - u(x)}{h}$$

is the difference quotient introduced in Section 2.8.

Let

$$A := -\int_\Omega \sum_{i,j=1}^n a_{ij} D_j u D_i\big(D_k^{-h}(\eta^2 D_k^h u)\big)\, dx$$

and

$$B := -\int_\Omega \overline{f} D_k^{-h}(\eta^2 D_k^h u)\, dx.$$

We first estimate A using $D_i D_k^{-h}(\eta^2 D_k^h u) = D_k^{-h} D_i(\eta^2 D_k^h u)$ at the first step, as well as the standard rules of calculus

$$\begin{aligned}
A &= -\int_\Omega \sum_{i,j=1}^n a_{ij} D_j u D_i D_k^{-h}(\eta^2 D_k^h u)\, dx \\
&\overset{\text{int by parts for } D_k^h}{=} \int_\Omega \sum_{i,j=1}^n D_k^h\big(a_{ij} D_j u\big) D_i(\eta^2 D_k^h u)\, dx \\
&= \int_\Omega \sum_{i,j=1}^n \big(D_k^h a_{ij} D_j u + a_{ij} D_k^h D_j u\big)\big(2\eta D_i \eta D_k^h u + \eta^2 D_k^h D_i u\big)\, dx \\
&= \int_\Omega \sum_{i,j=1}^n \big(D_k^h a_{ij} D_j u(2\eta D_i \eta D_k^h u) + D_k^h a_{ij} D_j u(\eta^2 D_k^h D_i u) + a_{ij} D_k^h D_j u(2\eta D_i \eta D_k^h u)\big)\, dx \\
&\quad + \int_\Omega \sum_{i,j=1}^n a_{ij} D_k^h D_j u(\eta^2 D_k^h D_i u)\, dx \\
&= A_1 + A_2.
\end{aligned}$$

Then since $|D\eta|, |a_{ij}|, \big|D_k^h a_{ij}\big| \le C$ and $\eta^2 \le C\eta$, we have

$$\begin{aligned}
|A_1| &\le C \int_\Omega \eta\Big(|Du|\big|D_k^h u\big| + |Du|\big|D_k^h Du\big| + \big|D_k^h Du\big|\big|D_k^h u\big|\Big)\, dx \\
&\overset{\text{Young}}{\le} \varepsilon \int_\Omega \eta^2 \big|D_k^h Du\big|^2\, dx + C(\varepsilon) \int_{\Omega''} \big(|Du|^2 + \big|D_k^h u\big|^2\big)\, dx \\
&\le \varepsilon \int_\Omega \eta^2 \big|D_k^h Du\big|^2\, dx + C(\varepsilon) \int_\Omega |Du|^2\, dx
\end{aligned}$$

where at the last step we used Theorem 2.40: $\int_{\Omega''} \left|D_k^h u\right|^2 dx \leq \int_\Omega |Du|^2 dx$. By uniform ellipticity

$$\begin{aligned} A_2 &= \int_\Omega \sum_{i,j=1}^n a_{ij} D_k^h D_j u(\eta^2 D_k^h D_i u)\, dx \\ &\geq \lambda \int_\Omega \eta^2 \left|D_k^h Du\right|^2 dx. \end{aligned}$$

It remains to estimate B. We calculate

$$\begin{aligned} |B| &= \left|\int_\Omega \overline{f} v\, dx\right| \\ &= \left|\int_\Omega \overline{f} D_k^{-h}(\eta^2 D_k^h u)\, dx\right| \\ &= \left|\int_\Omega (f - \sum_{i=1}^n b_i D_i u - cu) D_k^{-h}(\eta^2 D_k^h u)\, dx\right| \\ &\overset{\text{Young}}{\leq} C(\varepsilon) \int_\Omega \left(|f|^2 + |Du|^2 + |u|^2\right) dx \\ &\quad + \varepsilon \int_\Omega \left|D_k^{-h}(\eta^2 D_k^h u)\right|^2 dx. \end{aligned}$$

Next we estimate the last integral again by Theorem 2.40

$$\begin{aligned} \int_\Omega \left|D_k^{-h}\left(\eta^2 D_k^h u\right)\right|^2 dx &\leq \int_\Omega \left|D\left(\eta^2 D_k^h u\right)\right|^2 dx \\ &\leq \int_\Omega \left|2\eta D\eta D_k^h u + \eta^2 D D_k^h u\right|^2 dx \\ &\leq \int_\Omega \left|2\eta D\eta D_k^h u + \eta^2 D_k^h Du\right|^2 dx \\ &\overset{|\eta||D\eta| \leq C,\, \eta^4 \leq C\eta^2}{\leq} C \int_\Omega \eta^2 \left|D_k^h u\right|^2 dx + C \int_\Omega \eta^2 \left|D_k^h Du\right|^2 dx \\ &\overset{\text{Thm 2.40}}{\leq} C \int_\Omega |Du|^2 dx + C \int_\Omega \eta^2 \left|D_k^h Du\right|^2 dx. \end{aligned}$$

Thus

$$|B| \leq C(\varepsilon) \int_\Omega \left(|f|^2 + |Du|^2 + |u|^2\right) dx + \varepsilon \int_\Omega \eta^2 \left|D_k^h Du\right|^2 dx.$$

Combining the estimates with the fact

$$A_2 - |A_1| \leq |A| = |B|$$

we have

$$\lambda \int_\Omega \eta^2 \left|D_k^h Du\right|^2 dx - \varepsilon \int_\Omega \eta^2 \left|D_k^h Du\right|^2 dx - C(\varepsilon) \int_\Omega |Du|^2 dx$$
$$\leq C(\varepsilon) \int_\Omega \left(|f|^2 + |Du|^2 + |u|^2\right) dx + \varepsilon \int_\Omega \eta^2 \left|D_k^h Du\right|^2 dx$$

ie.

$$\lambda \int_\Omega \eta^2 \left|D_k^h Du\right|^2 dx - 2\varepsilon \int_\Omega \eta^2 \left|D_k^h Du\right|^2 dx$$
$$\leq C(\varepsilon) \int_\Omega \left(|f|^2 + |Du|^2 + |u|^2\right) dx + C(\varepsilon) \int_\Omega |Du|^2 dx.$$

Choosing $\varepsilon = \lambda/4$ and recalling $\eta = 1$ in Ω', we have

$$\frac{\lambda}{2} \int_{\Omega'} \eta^2 \left|D_k^h Du\right|^2 dx \leq C \int_\Omega \left(|f|^2 + |Du|^2 + |u|^2\right) dx.$$

This implies by Theorem 2.41 that $D_i u \in W^{1,2}_{\text{loc}}(\Omega)$ and thus $u \in W^{2,2}_{\text{loc}}(\Omega)$. □

We can also obtain $\int_\Omega |u|^2 \, dx$ instead of $\int_\Omega |Du|^2 \, dx$ on the right hand side of the estimate in the previous theorem.

Lemma 3.36 (Caccioppoli's ie)**.** *Let u, a_{ij}, b_i, c and f be as in the previous theorem. Then*

$$\int_{\Omega'} |Du|^2 \, dx \leq C \int_\Omega \left(|u|^2 + f^2\right) dx$$

for $\Omega' \Subset \Omega$.

Proof. Choose a test function $v = \eta^2 u$, where η is the same cut-off function as in the proof of the previous theorem so that

$$\int_\Omega \sum_{i,j=1}^n a_{ij} D_j u D_i(\eta^2 u) + \sum_{i=1}^n b_i D_i u(\eta^2 u) + cu(\eta^2 u) \, dx$$
$$= \int_\Omega \sum_{i,j=1}^n a_{ij} D_j u(2\eta D_i \eta u + \eta^2 D_i u) + \sum_{i=1}^n b_i D_i u(\eta^2 u) + cu(\eta^2 u) \, dx$$
$$= \int_\Omega \sum_{i,j=1}^n \eta^2 a_{ij} D_j u D_i u \, dx$$
$$+ \int_\Omega \sum_{i,j=1}^n 2 a_{ij} u \eta D_j u D_i \eta + \sum_{i=1}^n b_i D_i u(\eta^2 u) + cu(\eta^2 u) \, dx$$
$$= A_1 + A_2.$$

By the uniform ellipticity
$$A_1 \geq \lambda \int_\Omega \eta^2 |Du|^2 \, dx$$
Recalling that $|a_{ij}|, \eta, |D\eta| \leq C$ and using Young's inequality yields
$$|A_2| \leq \varepsilon \int_\Omega \eta^2 |Du|^2 \, dx + C \int_\Omega u^2 \, dx.$$
Finally, again by Young's inequality
$$\left| \int_\Omega f \eta^2 u \, dx \right| \leq C \int_\Omega f^2 \, dx + C \int_\Omega u^2 \, dx.$$
Combining the above estimates with the PDE itself we have
$$\lambda \int_\Omega \eta^2 |Du|^2 \, dx \leq \varepsilon \int_\Omega \eta^2 |Du|^2 \, dx + C \int_\Omega u^2 + f^2 \, dx.$$
By choosing $\varepsilon = \lambda/2$ we can "absorb" the first integral on the RHS into the left, and the proof is complete. $\square$

By adjusting the proof of Theorem 3.34 slightly to obtain some domain $\tilde{\Omega}$, $\Omega'' \Subset \tilde{\Omega} \Subset \Omega$ on the right in the estimate, we could combine Theorem 3.34 with Caccioppoli's inequality and have the following corollary.

Corollary 3.37. *Let*
$$a_{ij} \in C^1(\Omega), b_i \in L^\infty(\Omega), c \in L^\infty(\Omega)$$
and
$$f \in L^2(\Omega).$$
Further, let $u \in W^{1,2}(\Omega)$ be a weak solution to $Lu = f$. Then
$$u \in W^{2,2}_{loc}(\Omega)$$
and for any $\Omega' \Subset \Omega$
$$||u||_{W^{2,2}(\Omega')} \leq C\big(||f||_{L^2(\Omega)} + ||u||_{L^2(\Omega)} \big),$$
where C may depend on Ω', Ω and a_{ij}, b_i, c, but not on u.

By a similar argument as above combined with the induction, we could prove the following higher regularity result if the coefficient and data are smooth enough. For details, see Evans: PDE p. 316. Consider $\Delta u = f$, and suppose then that $f \in W^{1,2}(\Omega)$. By the above $u \in W^{2,2}_{\text{loc}}(\Omega)$, and thus by the weak definition
$$\int Du \cdot D\frac{\partial \varphi}{\partial x_i} \, dx = \int f \frac{\partial \varphi}{\partial x_i} \, dx.$$

so that

$$-\int D\frac{\partial u}{\partial x_i}\cdot D\varphi\,dx = -\int \frac{\partial f}{\partial x_i}\varphi\,dx.$$

Thus $\frac{\partial u}{\partial x_i}$ is a weak solution with the RHS $\frac{\partial f}{\partial x_i} \in L^2$. Thus $u \in W^{3,2}_{\text{loc}}(\Omega)$. Iterating further, and using a generalized Sobolev imbedding gives that u is smooth.

PUNCHLINE: The solution u has roughly speaking 2 derivatives more than f.

Theorem 3.38 (Local smoothness). *Let*

$$a_{ij}, b_i, c \in C^\infty(\Omega)$$

and

$$f \in C^\infty(\Omega).$$

Further, let $u \in W^{1,2}(\Omega)$ *be a weak solution to* $Lu = f$. *Then*

$$u \in C^\infty(\Omega).$$

3.6.2. *Global L^2-regularity.* Also a global regularity result holds.

Definition 3.39. *We denote $\partial\Omega \in C^k(\Omega)$, if for each point $x_0 \in \partial\Omega$ there is a $r > 0$ and a C^k-function*

$$\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$$

such that upon relabeling and reorienting the coordinate axes if necessary it holds that

$$\Omega \cap B(x_0, r) = \{x \in B(x_0; r) \,:\, x_n > \gamma(x_1, \ldots, x_{n-1})\}.$$

We also denote for bounded Ω

$$C^1(\overline{\Omega}) = \{u \in C^1(\Omega) \,:\, D^\alpha u,\ |\alpha| \leq 1, \text{ is uniformly continuous in } \Omega\}.$$

Theorem 3.40 (Global regularity). *Let*

$$a_{ij} \in C^1(\overline{\Omega}), b_i \in L^\infty(\Omega), c \in L^\infty(\Omega)$$

and

$$f \in L^2(\Omega).$$

Further, assume that $\partial\Omega \in C^2(\Omega)$, let $g \in W^{2,2}(\Omega)$ and $u \in W^{1,2}(\Omega)$ be a weak solution to

$$\begin{cases} Lu = f & \text{in } \Omega, \\ u - g \in W^{1,2}_0(\Omega). \end{cases}$$

Then

$$u \in W^{2,2}(\Omega)$$

and

$$||u||_{W^{2,2}(\Omega)} \leq C\big(||f||_{L^2(\Omega)} + ||u||_{L^2(\Omega)} + ||g||_{W^{2,2}(\Omega)} \big),$$

where C may depend on Ω and a_{ij}, b_i, c, but not on u.

Remark 3.41 (Warning)**.** *One might be tempted to think that all kind of properties of the boundary value function are inherited by the solution as long as the boundary has the same regularity. This is false however!*

Let $\alpha \in (\frac{1}{2}, 1)$ and

$$\Omega = \{z \in \mathbb{C} \, : \, |z| > 0, \ \operatorname{Arg} z \in (-\frac{\pi}{2\alpha}, \frac{\pi}{2\alpha})\}.$$

Denote $z = x + iy = re^{i\theta}$ with $\theta \in (-\pi, \pi]$. Since

$$\log z = i\theta + \log r$$

$$z^\alpha := e^{\alpha \log(z)} = e^{\alpha \log r} e^{i\alpha\theta} = r^\alpha(\cos(\alpha\theta) + i \sin(\alpha\theta)).$$

We take for granted that z^α is an analytic function in Ω, and thus its real part

$$u(x, y) := \operatorname{Re} z^\alpha = r^\alpha \cos(\alpha\theta)$$

is a harmonic function. Then for $x > 0$ it holds that

$$u(x, 0) = |x|^\alpha$$

even if

$$u \equiv 0 \text{ on } \partial\Omega.$$

A harmonic function is actually locally but not necessarily globally Lipschitz.

The similar phenomenon happens even if the boundary is smooth. Indeed, consider the upper half plane and

$$\begin{cases} \Delta u = 0 & (x, y) \in \mathbb{R}^2_+ \\ u(x, 0) = g(x), \end{cases} \tag{3.16}$$

where $g(x) = |x|$ close to 0 and continued in a suitable bounded and smooth fashion to the whole of $\mathbb{R}$. Then $y \mapsto u(0, y)$ is only Hölder continuous close to $y = 0$.

Similarly, if $g \in C^2(\mathbb{R})$ in (3.16) it does not always follow that $u \in C^2(\overline{\mathbb{R}}^2_+)$.

3.6.3. *Local L^p-regularity.*

Example 3.42 (Calderón-Zygmund type inequality)**.** *First consider a classical approach to the problem*

$$\Delta u = f$$

where $f \in L^p$, $2 \le p < \infty$. A solution u is of the form (we say nothing about a domain or uniqueness as this is just the idea on general level)

$$u(x) = C \int \frac{f(y)}{|x-y|^{n-2}}\, dy.$$

One of the questions in the regularity theory of PDEs is, does u have the second derivatives in L^p i.e.

$$\frac{\partial^2 u}{\partial x_i \partial x_j} \in L^p?$$

If we formally differentiate u, we get

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = C \int_{\mathbb{R}^n} f(y) \underbrace{\frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{|x-y|^{n-2}}}_{|\cdot| \le C/|x-y|^n}\, dy.$$

It follows that $\int f(y) \frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{|x-y|^{n-2}}\, dy$ defines (the precise definitions are beyond our scope here) a singular integral $Tf(x)$. A typical theorem in the theory of singular integrals says

$$||Tf||_p \le C\, ||f||_p$$

and thus we can deduce that $\frac{\partial^2 u}{\partial x_i \partial x_j} \in L^p$. Working further, we get that $u \in W^{2,p}$. This was established by Calderón and Zygmund (1952, Acta Math.) and thus the above inequality is often called the Calderón-Zygmund inequality.

Theorem 3.43 (Local L^p-regularity)**.** *Let $1 < p < \infty$ and*

$$f \in L^p(\Omega).$$

Further, let $u \in W^{1,2}(\Omega)$ be a weak solution to $\Delta u = f$. Then

$$u \in W^{2,p}_{loc}(\Omega)$$

and for any $\Omega' \Subset \Omega$

$$||u||_{W^{2,p}(\Omega')} \le C\big(||f||_{L^p(\Omega)} + ||u||_{L^p(\Omega)} \big),$$

where C may depend on p, n, Ω', and Ω but not on u.

3.6.4. *C^α regularity using De Giorgi's method.* For expository reasons we only consider the Laplacian. Nonetheless, the method also applies to the uniformly elliptic equation with bounded measurable coefficients. Recall that Example 3.12 shows that this is the best we can hope for under such weak assumptions on the coefficients.

Lemma 3.44 (Caccioppoli's inequality). *Let $u \in W^{1,2}_{loc}(\Omega)$ be a weak solution to $\Delta u = 0$ in Ω. Then there exists a constant $c = c(n)$ such that*

$$\int_{B(x_0,R)} \eta^2 |D(u-k)_+|^2 \, dx \le \Big(\frac{C}{R-r}\Big)^2 \int_{B(x_0,R)} |(u-k)_+|^2 \, dx,$$

where $k \in \mathbb{R}$, $0 < r < R < \infty$ s.t. $B(x_0,R) \subset \Omega$ and $u_+ = \max(u,0)$.

Proof. Let $\eta \in C_0^\infty(B(x_0,R))$ cut-off function s.t.

$$0 \le \eta \le 1, \quad \eta = 1 \text{ in } B(x_0,r), \quad |D\eta| \le \frac{C}{R-r}$$

and a test function

$$\varphi = (u-k)_+\eta^2 \in W_0^{1,2}(B(x_0,R)).$$

Since $D\varphi = D(u-k)_+\eta^2 + (u-k)_+ 2\eta D\eta$ we obtain using weak formulation

$$\int_\Omega Du \cdot D(u-k)_+\eta^2 \, dx = -2\int_\Omega Du \cdot (u-k)_+\eta D\eta \, dx. \tag{3.17}$$

Recall

$$D(u-k)_+ = \begin{cases} Du & \text{a.e. } \{u > k\} \\ 0 & \text{a.e. } \{u \le k\}. \end{cases} \tag{3.18}$$

Thus a.e.

$$|D(u-k)_+|^2 = Du \cdot D(u-k)_+$$

and combining this with (3.17), we get

$$\begin{aligned}
\int_{B(x_0,R)} &|D(u-k)_+|^2\eta^2\,dx \\
&= \int_{B(x_0,R)} Du\cdot D(u-k)_+\eta^2\,dx \\
&\leq 2\int_{B(x_0,R)} |Du|(u-k)_+\eta|D\eta|\,dx \\
&\overset{(3.18)}{=} 2\int_{B(x_0,R)} |D(u-k)_+|(u-k)_+\eta|D\eta|\,dx \\
&\overset{\text{Young}}{\leq} \varepsilon\int_{B(x_0,R)} |D(u-k)_+|^2\eta^2\,dx + C(\varepsilon)\int_{B(x_0,R)} (u-k)_+|D\eta|\,dx.
\end{aligned}$$

From this the result follows by absorbing the first term on the RHS into the left, and recalling the definition of η. $\square$

Theorem 3.45 (ess sup-estimate). *Let $u\in W^{1,2}_{loc}(\Omega)$ be a weak solution to $\Delta u = 0$ in Ω. Then there exists $c=c(n)$ such that*

$$\operatorname*{ess\,sup}_{B(x_0,\frac{r}{2})} u \leq k_0 + C\Big(\fint_{B(x_0,r)} |(u-k_0)_+|^2\,dx\Big)^{1/2}$$

where $k_0\in\mathbb{R}$ and $B(x_0,r)\subset\Omega$.

Proof. Let $0<r/2<\rho<r$ and $\eta\in C_0^\infty(B(x_0,r))$

$$0\leq\eta\leq 1,\quad \eta=1 \text{ in } B(x_0,\rho),\quad |D\eta|\leq\frac{C}{r-\rho},$$

and use test function $v=(u-k)_+\eta$. The proof will be based on the use of Sobolev-Poincare, Caccioppoli and iteration. To be more precise,

$$\begin{aligned}
\int_{B(x_0,r)} |Dv|^2\,dx &\leq \int_{B(x_0,r)} |D((u-k)_+\eta)|^2\,dx \\
&\leq \int_{B(x_0,r)} |D(u-k)_+\eta + (u-k)_+D\eta|^2\,dx \\
&\leq c\int_{B(x_0,r)} |D(u-k)_+\eta|^2 + |(u-k)_+D\eta|^2\,dx \\
&\overset{\text{Cacc, def. of }\eta}{\leq} \frac{C}{(r-\rho)^2}\int_{B(x_0,r)} |(u-k)_+|^2\,dx.
\end{aligned}\tag{3.19}$$

Further,

$$\begin{aligned}\Big(\fint_{B(x_0,\rho)} |(u-k)_+|^{2^*}\,dx\Big)^{2/2^*} &\le C\Big(\fint_{B(x_0,r)} v^{2^*}\,dx\Big)^{2/2^*}\\ &\overset{v\in W_0^{1,2}(B(x_0,r)),\text{ Sobo ie}}{\le} Cr^2\fint_{B(x_0,r)} |Dv|^2\,dx.\end{aligned} \tag{3.20}$$

Combining (3.20) and (3.19), we get

$$\Big(\fint_{B(x_0,\rho)} |(u-k)_+|^{2^*}\,dx\Big)^{2/2^*} \le \frac{Cr^2}{(r-\rho)^2}\fint_{B(x_0,r)} |(u-k)_+|^2\,dx. \tag{3.21}$$

Define

$$A(k,\rho) = B(x_0,\rho)\cap\{x\in\Omega\,:\,u(x)>k\}$$

and observe

$$\begin{aligned}&\fint_{B(x_0,\rho)} |(u-k)_+|^2\,dx\\ &= \frac{1}{|B(x_0,\rho)|}\int_{A(k,\rho)} |(u-k)_+|^2\,dx\\ &\overset{\text{Hölder}}{\le} \frac{1}{|B(x_0,\rho)|}\Big(\int_{A(k,\rho)} |(u-k)_+|^{2^*}\,dx\Big)^{2/2^*} |A(k,\rho)|^{1-\frac{2}{2^*}}\\ &= \Big(\frac{1}{|B(x_0,\rho)|}\int_{A(k,\rho)} |(u-k)_+|^{2^*}\,dx\Big)^{2/2^*}\Big(\frac{|A(k,\rho)|}{|B(x_0,\rho)|}\Big)^{1-\frac{2}{2^*}}\\ &\le \Big(\fint_{B(x_0,\rho)} |(u-k)_+|^{2^*}\,dx\Big)^{2/2^*}\Big(\frac{|A(k,\rho)|}{|B(x_0,\rho)|}\Big)^{1-\frac{2}{2^*}}\\ &\overset{(3.21)}{\le} \frac{Cr^2}{(r-\rho)^2}\fint_{B(x_0,r)} |(u-k)_+|^2\,dx\Big(\frac{|A(k,\rho)|}{|B(x_0,\rho)|}\Big)^{1-\frac{2}{2^*}}.\end{aligned} \tag{3.22}$$

If $h<k$, then

$$\begin{aligned}(k-h)^2|A(k,\rho)| &= \int_{A(k,\rho)} (k-h)^2\,dx\\ &\overset{u>k\text{ in }A(k,\rho)}{\le} \int_{A(k,\rho)} (u-h)^2\,dx\\ &\overset{h<k}{\le} \int_{A(h,\rho)} (u-h)^2\,dx\\ &\le \int_{B(x_0,\rho)} (u-h)^2\,dx.\end{aligned}$$

By this, denoting

$$u(h,\rho) := \Big(\fint_{B(x_0,\rho)} |(u-h)_+|^2 \, dx \Big)^{1/2}, \tag{3.23}$$

we get

$$|A(k,\rho)| \le \frac{1}{(k-h)^2} \int_{B(x_0,\rho)} (u-h)^2 \, dx = \frac{|B(x_0,\rho)|}{(k-h)^2} u(h,\rho)^2. \tag{3.24}$$

Using this with (3.22), we get

$$\begin{aligned} u(k,\rho)^2 &\overset{(3.22)}{\le} C\Big(\frac{r}{r-\rho}\Big)^2 u(k,r)^2 \Big(\frac{|A(k,\rho)|}{|B(x_0,\rho)|}\Big)^{1-\frac{2}{2^*}} \\ &\overset{(3.24)}{\le} C\Big(\frac{r}{r-\rho}\Big)^2 u(k,r)^2 \Big(\frac{u(h,\rho)}{k-h}\Big)^{2(1-\frac{2}{2^*})}. \end{aligned} \tag{3.25}$$

This implies

$$\begin{aligned} u(k,\rho) &\overset{u(k,r)\,\le\, u(h,r)}{\le} C\frac{r}{r-\rho} u(h,r)^{1+1-\frac{2}{2^*}} (k-h)^{-(1-\frac{2}{2^*})} \\ &= C\frac{r}{r-\rho} u(h,r)^{1+\theta}(k-h)^{-\theta}, \end{aligned} \tag{3.26}$$

where $\theta := 1 - \frac{2}{2^*}$.
Auxiliary claim: For $k \in \mathbb{R}$ it holds that

$$u(k_0+d, r/2) = 0$$

where $d^\theta := C2^{(1+\theta)^2/\theta+1} u(k_0,r)^\theta$ and c, θ are as above.
Proof of the auxiliary claim: Let

$$\begin{aligned} k_j &= k_0 + d(1-2^{-j}) \\ \rho_j &= r/2 + 2^{-j-1} r, \quad j = 0,1,2,\dots \end{aligned}$$

so that $\rho_0 = r$, $\rho_j \searrow r/2$ and $k_j \nearrow k+d$ as $j \to \infty$. Then we show by induction that

$$u(k_j,\rho_j) \le 2^{-\mu j} u(k_0,r), \quad j = 0,1,2,\dots \tag{3.27}$$

where $\mu = (1+\theta)/\theta$.

Indeed $j = 0$ immediately follows since $\rho_0 = r$.

Assume then that the claim holds for some j, and observe that

$$\begin{aligned} \rho_j - \rho_{j+1} &= (2^{-j-1} - 2^{-j-2})r = 2^{-j-2} r, \\ k_{j+1} - k_j &= (-2^{j-1} + 2^{-j})d = 2^{j-1} d, \\ \rho_j &\le r. \end{aligned}$$

Using these with (3.26), we have

$$
\begin{aligned}
u(k_{j+1},\rho_{j+1}) &\leq \frac{C\rho_j}{\rho_j-\rho_{j+1}}(k_{j+1}-k_j)^{-\theta}u(k_j,\rho_{j+1})^{1+\theta}\\
&\leq \frac{Cr}{2^{-j-2}r}(2^{-j-1}d)^{-\theta}2^{-\mu j(1+\theta)}u(k_0,r)^{1+\theta},
\end{aligned}
$$

where we used induction assumption to estimate $u(k_j,\rho_{j+1})$. Then recalling the shorthand notations , we get

$$
\begin{aligned}
u(k_{j+1},\rho_{j+1}) &\leq C2^{j+2+\theta(j+1)-\underbrace{\frac{(\theta+1)^2}{\theta}}_{\mu(1+\theta)}j}\underbrace{(C2^{\frac{(\theta+1)^2}{\theta}+1}u(k_0,r)^{\theta})^{-1}}_{d^{-\theta}}u(k_0,r)^{1+\theta}\\
&= 2^{1+(1+\theta)(j+1)-\frac{(\theta+1)^2}{\theta}j-\frac{(\theta+1)^2}{\theta}-1}u(k_0,r)\\
&= 2^{(\theta+1)(j+1)\underbrace{(1-\frac{\theta+1}{\theta})}_{-\frac{1}{\theta}}}u(k_0,r)\\
&= 2^{-\mu(j+1)}u(k_0,r),
\end{aligned}
$$

and thus the induction is complete. Further, this implies

$$
\lim_{j\to\infty}u(k_j,\rho_j)=0
$$

and

$$
\begin{aligned}
0\leq u(k_0+d,\frac{r}{2}) &\overset{k_j\,\leq\,k_0+d}{=} \Big(\fint_{B(x_0,\frac{r}{2})}|(u-k_j)_+|^2\,dx\Big)^{\frac{1}{2}}\\
&\overset{\frac{r}{2}\,\leq\,\rho_j}{\leq} \Big(\frac{B(x_0,\rho_j)}{B(x_0,\frac{r}{2})}\int_{B(x_0,\rho_j)}|(u-k_j)_+|^2\,dx\Big)^{\frac{1}{2}}\leq Cu(k_j,\rho_j)\to 0.
\end{aligned}
$$

It follows that

$$
u(k_0+d,\frac{r}{2})=0,
$$

and this ends the proof of the auxiliary claim.

By using the auxiliary claim, we now finish the proof of the essup-estimate. Indeed,

$$
0=u(k_0+d,\frac{r}{2})=\Big(\fint_{B(x_0,\frac{r}{2})}|(u-(k_0+d))_+|^2\,dx\Big)^{\frac{1}{2}},
$$

where $d^\theta=C2^{(1+\theta)^2/\theta+1}u(k_0,r)^\theta$. Thus a.e. in $B(x_0,\frac{r}{2})$ it holds that

$$
u\leq k_0+d=k_0+C\Big(\fint_{B(x_0,r)}|(u-k_0)_+|^2\,dx\Big)^{\frac{1}{2}}
$$

from which the claim

$$\operatorname*{ess\,sup}_{B(x_0,\frac{r}{2})} u \le k_0 + C\Big(\fint_{B(x_0,r)} |(u-k_0)_+|^2\,dx\Big)^{\frac{1}{2}}$$

follows. $\square$

Corollary 3.46. *Let u be a weak solution to $\Delta u = 0$ in Ω. Then there exists $C = C(n)$ such that*

$$\operatorname*{ess\,sup}_{B(x_0,\frac{r}{2})} |u| \le c\Big(\fint_{B(x_0,r)} |u|^2\,dx\Big)^{\frac{1}{2}}$$

for all $B(x_0,\frac{r}{2}) \subset \Omega$.

Proof. Choose $k_0 = 0$ in the previous result and observe that

$$\operatorname*{ess\,sup}_{B(x_0,\frac{r}{2})} u \le C\Big(\fint_{B(x_0,r)} |u_+|^2\,dx\Big)^{\frac{1}{2}}.$$

Since $-u$ is also a solution, we obtain

$$-\operatorname*{ess\,inf}_{B(x_0,\frac{r}{2})} u = \operatorname*{ess\,sup}_{B(x_0,\frac{r}{2})}(-u) \le C\Big(\fint_{B(x_0,r)} |(-u)_+|^2\,dx\Big)^{\frac{1}{2}}.$$

Combining the estimates

$$\operatorname*{ess\,sup}_{B(x_0,\frac{r}{2})} |u| \le \max\{\operatorname*{ess\,sup}_{B(x_0,\frac{r}{2})} u, -\operatorname*{ess\,inf}_{B(x_0,\frac{r}{2})} u\} \le C\Big(\fint_{B(x_0,r)} |u|^2\,dx\Big)^{\frac{1}{2}}. \quad \square$$

The above result implies that (unlike Sobolev functions in general) are locally bounded.

Next lemma is needed in order to prove Hölder continuity for weak solutions.

Lemma 3.47. *[Measure decay] Let u be a weak solution to $\Delta u = 0$ in Ω, $B(x_0,2r) \Subset \Omega$,*

$$m(2r) = \operatorname*{ess\,inf}_{B(x_0,2r)} u, \quad M(2r) = \operatorname*{ess\,sup}_{B(x_0,2r)} u,$$

and

$$|A(k_0,r)| \le \gamma |B(x_0,r)|, \quad 0 < \gamma < 1,$$

where $A(k_0,r) = B(x_0,r) \cap \{x \in \Omega : u(x) > k_0\}$ and $k_0 = (m(2r) + M(2r))/2$. Then

$$\lim_{k \nearrow M(2r)} |A(k,r)| = 0.$$

We postpone the proof and go to the proof of the Hölder continuity immediately.

Theorem 3.48 (Hölder-continuity)**.** *If u be a weak solution to $\Delta u = 0$ in Ω, then u is locally Hölder-continuous (or has such a representative to be more precise).*

Proof. Let

$$k_0 = \frac{m(2r) + M(2r)}{2}$$

similarly to the previous lemma. Without loss of generality we may assume that

$$|B(x_0, r) \cap \{x \in \Omega \,:\, u(x) > k_0\}| = |A(k_0, r)| \leq \frac{1}{2}|B(x_0, r)|, \quad (3.28)$$

since otherwise if $|A(k_0, r)| > \frac{1}{2}|B(x_0, r)|$ it holds that

$$|\{x \in B(x_0, r) \,:\, -u(x) \leq -k_0\}| > \frac{1}{2}|B(x_0, r)|$$
$$|\{x \in B(x_0, r) \,:\, -u(x) > -k_0\}| < \frac{1}{2}|B(x_0, r)|$$

and the argument below works for $-u$ and $-k_0$ instead. Here we need that both u and $-u$ are solutions. Using the esssup-estimate with

$$k_l = M(2r) - 2^{-(l+1)}(M(2r) - m(2r))$$

we have

$$\begin{aligned} M(\frac{r}{2}) &\leq k_l + C\Big(⨍_{B(x_0,r)} \underbrace{|(u - k_l)_+|^2}_{\leq (M(2r) - k_l)^2} \, dx \Big)^{\frac{1}{2}} \\ &\leq k_l + C(M(2r) - k_l)\Big(\frac{|A(k_l, r)|}{|B(x_0, r)|}\Big)^{\frac{1}{2}} \end{aligned}$$

since the integrand can only be nonzero in the set $A(k_l, r)$. By Lemma 3.47, we may choose l large enough so that

$$C\Big(\frac{|A(k_l, r)|}{|B(x_0, r)|}\Big)^{\frac{1}{2}} < \frac{1}{2}.$$

This fixes l. Combining the previous two estimates we obtain

$$\begin{aligned} M(\frac{r}{2}) &\leq k_l + \frac{1}{2}(M(2r) - k_l) \\ &\leq M(2r) - 2^{-(l+1)}(M(2r) - m(2r)) \\ &\qquad\qquad + \frac{1}{2}\big(M(2r) - (M(2r) - 2^{-(l+1)}(M(2r) - m(2r)))\big) \\ &\leq M(2r) - 2^{-(l+2)}(2 - 1)(M(2r) - m(2r)). \end{aligned}$$

From this we get

$$\begin{aligned} M(\frac{r}{2}) - m(\frac{r}{2}) &\le M(\frac{r}{2}) - m(2r) \\ &\le (1 - 2^{-(l+2)})(M(2r) - m(2r)). \end{aligned}$$

Using the notation $\operatorname{osc}_{B(x_0,2r)} u := M(2r) - m(2r) = \operatorname{ess\,sup}_{B(x_0,2r)} u - \operatorname{ess\,inf}_{B(x_0,2r)} u$ and $\lambda := (1 - 2^{-(l+2)}) < 1$, the above reads as

$$\operatorname{osc}_{B(x_0,\frac{r}{2})} u \le \lambda \operatorname{osc}_{B(x_0,2r)} u.$$

The Hölder continuity follows from this by a standard iteration. To be more precise, choose $j \in \mathbb{N}$ such that

$$4^{j-1} \le \frac{R}{r} < 4^j.$$

Then

$$\begin{aligned} \operatorname{osc}_{B(x_0,\frac{r}{2})} u &\le \lambda^{j-1} \operatorname{osc}_{B(x_0,4^{j-1}r)} u \\ &\overset{4^{j-1}r \le R}{\le} \lambda^{j-1} \operatorname{osc}_{B(x_0,R)} u \\ &\le \lambda^{j-1} \operatorname{osc}_{B(x_0,R)} u \\ &\le C\Big(\frac{r}{R}\Big)^{\alpha} \operatorname{osc}_{B(x_0,R)} u, \end{aligned} \tag{3.29}$$

where we denoted $\alpha = -\log\lambda/\log 4 \in (0,1)$ and observed

$$\begin{aligned} &\frac{R}{r} < 4^j \quad \Rightarrow \quad 4^{-j\alpha} \le \Big(\frac{R}{r}\Big)^{-\alpha} = \Big(\frac{r}{R}\Big)^{\alpha} \\ &\lambda^{j-1} = 4^{\log_4(\lambda^{j-1})} = 4^{(j-1)\log(\lambda)/\log(4)} = 4^{(1-j)\alpha} \le 4^{\alpha}\Big(\frac{r}{R}\Big)^{\alpha}. \end{aligned}$$

Let $y \in \Omega$ s.t. $|x_0 - y| \le \frac{1}{8}\operatorname{dist}(x_0,\partial\Omega)$, $R = \operatorname{dist}(x_0,\partial\Omega)$, $r = |x_0 - y|$ (actually only for a.e. point but then the below deduction can be used to define the Hölder continuous representative). Then by the estimate

(3.29)

$$
\begin{aligned}
|u(x_0) - u(y)| &\leq \operatorname{osc}_{B(x_0,2r)} u \\
&\leq C\Big(\frac{2r}{\frac{3}{8}R}\Big)^{\alpha} \operatorname{osc}_{B(x_0,\frac{3}{8}R)} u \\
&\leq C\Big(\frac{2|x_0-y|}{\frac{3}{8}R}\Big)^{\alpha} \operatorname{osc}_{B(x_0,\frac{3}{8}R)} u \\
&\leq C|x_0-y|^{\alpha} R^{-\alpha} 2 \operatorname*{ess\,sup}_{B(x_0,\frac{3}{8}R)} |u| \\
&\overset{\text{esssup-est}}{\leq} C|x_0-y|^{\alpha} R^{-\alpha}\Big(\fint_{B(0,\frac{6}{8}R)} |u|^2\,dx\Big)^{\frac{1}{2}} \\
&\leq C|x_0-y|^{\alpha}.
\end{aligned}
$$

□

It remains to prove the lemma used above.

Proof of Lemma 3.47. The proof is based on deriving an estimate containing $|A(h,r)| - |A(k,r)|$ by using Poincare's and Caccioppoli's inequalities. To this end, we let $k > h > k_0$ and define an auxiliary function

$$
v(x) = \begin{cases} k-h, & u(x) \geq k, \\ u(x)-h, & h < u(x) < k, \\ 0, & u(x) \leq h. \end{cases}
$$

It immediately follows

$$
\begin{aligned}
\{x \in B(x_0,r) \,:\, v(x) = 0\} &= |\{x \in B(x_0,r) \,:\, u(x) \leq h\}| \\
&= |B(x_0,r)| - |\{x \in B(x_0,r) \,:\, u(x) > h\}| \\
&= |B(x_0,r)| - |A(h,r)| \\
&\overset{k_0<h}{\geq} |B(x_0,r)| - |A(k_0,r)| \\
&\overset{\text{assump.}}{\geq} (1-\gamma)|B(x_0,r)|.
\end{aligned}
$$

From this denoting $v_{B(x_0,r)} := \fint_{B(x_0,r)} v\,dx$, we get

$$
\begin{aligned}
k - h - v_{B(x_0,r)} &= \fint_{B(x_0,r)} (k-h-v)\,dx \\
&\geq (1-\gamma) \fint_{\{x\in B(x_0,r)\,:\,v(x)=0\}} (k-h-v)\,dx \\
&= (1-\gamma)(k-h).
\end{aligned}
$$

Integrating this over $A(k,r)$ and using Poincare's inequality, we obtain

$$\begin{aligned}
(k-h)|A(k,r)| &\le \frac{1}{1-\gamma}\int_{A(k,r)} (k-h-v_{B(x_0,r)})\,dx \\
&\overset{v=k-h \text{ in } A(k,r)}{=} \frac{1}{1-\gamma}\int_{A(k,r)} (v-v_{B(x_0,r)})\,dx \\
&\le \frac{1}{1-\gamma}\int_{B(x_0,r)} (v-v_{B(x_0,r)})\,dx \\
&\overset{\text{Hölder}}{\le} \frac{1}{1-\gamma}|B(x_0,r)|\Big(\fint_{B(x_0,r)} \left|v-v_{B(x_0,r)}\right|^{\frac{n}{n-1}}\,dx\Big)^{\frac{n-1}{n}} \\
&\overset{\text{Poinc.:Thm 2.55, } p=1}{\le} \frac{1}{1-\gamma}|B(x_0,r)|cr\fint_{B(x_0,r)} |Dv|\,dx.
\end{aligned}$$

Then using the above estimate and by Hölder's as well as Caccioppoli's inequalities

$$\begin{aligned}
(k-h)|A(k,r)| &\le \frac{1}{1-\gamma}Cr\int_{B(x_0,r)} |Dv|\,dx \\
&\overset{\text{def. } v}{\le} \frac{1}{1-\gamma}Cr\int_{A(h,r)\setminus A(k,r)} |Du|\,dx \\
&\overset{\text{Hölder}}{\le} \frac{1}{1-\gamma}Cr\Big(\int_{A(h,r)} |Du|^2\,dx\Big)^{1/2}(|A(h,r)|-|A(k,r)|)^{\frac{1}{2}} \\
&\overset{\text{Cacc.}}{\le} C\Big(\int_{A(h,2r)} |u-h|^2\,dx\Big)^{1/2}(|A(h,r)|-|A(k,r)|)^{\frac{1}{2}},
\end{aligned} \tag{3.30}$$

where at the last step we observed that $r/(2r-r) \le C$. Next replace k and h in the above inequality by k_j and k_{j-1} where

$$k_j = M(2r) - 2^{-(j+1)}(M(2r)-m(2r)).$$

Then since

$$\begin{aligned}
k_j - k_{j-1} &= (M(2r) - 2^{-(j+1)}(M(2r)-m(2r))) \\
&\quad - \big(M(2r) - 2^{-j}(M(2r)-m(2r))\big) \\
&= 2^{-(j+1)}(M(2r)-m(2r)),
\end{aligned}$$

we get by (3.30) that

$$\begin{aligned}
2^{-(j+1)}(M(2r)-m(2r))|A(k_j,r)| &= (k_j-k_{j-1})|A(k_j,r)| \\
&\le C\Big(\int_{A(k_{j-1},2r)} |u-k_{j-1}|^2\,dx\Big)^{1/2}(|A(k_{j-1},r)|-|A(k_j,r)|)^{\frac{1}{2}}.
\end{aligned}$$

Then observing $u-k_{j-1} \le M(2r)-k_{j-1} = M(2r)-\big(M(2r)-2^{-j}(M(2r)-m(2r))\big) = 2^{-j}(M(2r)-m(2r))$ and using the above estimate, we obtain

$$\begin{aligned}
&2^{-(j+1)}(M(2r)-m(2r))|A(k_j,r)| \\
&\qquad \le C(M(2r)-k_{j-1})|A(k_{j-1},2r)|^{\frac{1}{2}}(|A(k_{j-1},r)|-|A(k_j,r)|)^{\frac{1}{2}} \\
&\qquad \le C2^{-j}(M(2r)-m(2r))|A(k_{j-1},2r)|^{\frac{1}{2}}(|A(k_{j-1},r)|-|A(k_j,r)|)^{\frac{1}{2}}.
\end{aligned}$$

Cancelling $2^{-j}(M(2r)-m(2r))$ on both sides and choosing $l, l \ge j$, we end up with

$$|A(k_l,r)| \le |A(k_j,r)| \le C|B(x_0,2r)|^{\frac{1}{2}}(|A(k_{j-1},r)|-|A(k_j,r)|)^{\frac{1}{2}}.$$

Taking squares and summing over j, this gives by telescoping

$$\begin{aligned}
l|A(k_l,r)|^2 &= \sum_{j=1}^{l} |A(k_l,r)|^2 \\
&\le C|B(x_0,2r)| \sum_{j=1}^{l} (|A(k_{j-1},r)|-|A(k_j,r)|) \\
&\le C|B(x_0,2r)|(|A(k_0,r)|-|A(k_l,r)|) \le cC|B(x_0,2r)||B(x_0,r)|.
\end{aligned}$$

Dividing by l, we finally get

$$\lim_{l\to\infty} |A(k_l,r)| = 0. \qquad \square$$

3.7. **Weak and strong max principles.** In this section we consider

$$Lu = -\sum_{i,j=1}^{n} D_i(a_{ij}(x)D_ju(x)) + c(x)u(x) = 0.$$

For the next theorem, we define

$$\sup_{\partial\Omega} u := \inf\{l \in \mathbb{R} \,:\, (u-l)_+ \in W_0^{1,2}(\Omega)\}.$$

Theorem 3.49 (Weak max principle)**.** *Let $u \in W^{1,2}(\Omega)$ be a weak solution to $-\sum_{i,j=1}^{n} D_i(a_{ij}(x)D_ju(x)) + c(x)u(x) = 0$, with $c \ge 0$. Then*

$$\operatorname*{ess\,sup}_{\Omega} u \le \sup_{\partial\Omega} u_+.$$

Proof. Set $M := \sup_{\partial\Omega} u_+ \ge 0$. It holds that $(u-M)_+ \in W_0^{1,2}(\Omega)$. To see this, choose decreasing sequence $l_i \to M$ so that $(u-l_i)_+ = (u_+-l_i)_+ \in W_0^{1,2}(\Omega)$. Then since Ω is bounded, it follows that $u-l_i \to u-M \quad \text{in } W^{1,2}(\Omega)$. By it holds ,

$$(u-l_i)_+ \to (u-M)_+ \quad \text{in } W^{1,2}(\Omega)$$

and thus the claim $(u-M)_+ \in W_0^{1,2}(\Omega)$ follows.

We may use $v = (u-M)_+$ as a test function in

$$\int_\Omega \sum_{i,j=1}^n a_{ij} D_j u D_i v + cuv \, dx = 0$$

$$\int_\Omega \sum_{i,j=1}^n a_{ij} D_j M D_i v + cMv \, dx \geq 0,$$

where $M, c, v \geq 0$ was used. We subtract these to have

$$\lambda \int_\Omega |D(u-M)_+|^2 + c(u-M)_+^2 \, dx \leq 0.$$

From this it follows that $u \leq M$ a.e. □

PUNCHLINE: Roughly speaking the max principle says that the largest value is attained at the boundary or at least a solution cannot obtain strict maximum inside the domain. This is tightly connected to the comparison principles too: a comparison with a constant.

Remark 3.50 (Warning)**.** *If there is f on the right hand side, the form of the max-principle changes (ex).*

3.7.1. *Strong maximum principle.* Strong maximum principle for weak solutions follows Harnack type arguments that we have not proven yet. Nonetheless, we show that due to the classical theory this is something to be expected anyway.

Recall

$$C^1(\overline{\Omega}) = \{u \in C^1(\Omega) \, : \, D^\alpha u \text{ is uniformly continuous for all } |\alpha| \leq 1\}.$$

The argument does not rely on divergence form. For simplicity of notation we consider

$$Lu = -\sum_{i,j=1}^n a_{ij} D_i D_j u = 0.$$

By interior ball condition for Ω at $x_0 \in \partial\Omega$, we mean that there is a ball $B \subset \Omega$ such that $x_0 \in \partial B$.

Lemma 3.51 (Hopf)**.** *Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfy $Lu \leq 0$, and suppose that there is x_0 satisfying interior ball condition for B and*

$$u(x_0) > u(x) \quad \textit{for all } x \in \Omega.$$

Then

$$\frac{\partial u}{\partial \nu}(x_0) > 0$$

where ν is exterior unit normal for B at x_0.

Proof. We may assume that $B = B(0, r)$ and $u(x_0) \geq 0$. Set for $\gamma > 0$

$$v(x) = e^{-\gamma|x|^2} - e^{-\gamma r^2}, \quad x \in B(0, r).$$

Then

$$D_j v = -2x_j \gamma e^{-\gamma|x|^2}$$

and

$$D_i D_j v = (-2\delta_{ij}\gamma + 4\gamma^2 x_i x_j) e^{-\gamma|x|^2}.$$

Thus

$$\begin{aligned} Lv &= -\sum_{i,j=1}^{n} a_{ij} D_i D_j v \\ &= -\sum_{i,j=1}^{n} a_{ij}(-2\delta_{ij}\gamma + 4\gamma^2 x_i x_j) e^{-\gamma|x|^2} \\ &\overset{\text{ell}}{\leq} (2\gamma \sum_{i=1}^{n} a_{ii} - 4\gamma^2 \lambda |x|^2) e^{-\gamma|x|^2}. \end{aligned}$$

Thus for large enough γ, we have

$$Lv \leq (2\gamma \sum_{i=1}^{n} a_{ii} - 4\gamma^2 \lambda |x|^2) e^{-\gamma|x|^2} \leq 0, \quad x \in B(0, r) \setminus \overline{B}(0, \frac{r}{2}).$$

By the assumption $u(x_0) > u(x)$ for all $x \in \Omega$, for small enough $\varepsilon > 0$, it holds that

$$u(x_0) \geq u(x) + \varepsilon v(x)$$

on $\partial B(0, r/2) \subset \Omega$. The same holds on $\partial B(0, r)$ since there $v = 0$. We have

$$L(u + \varepsilon v - u(x_0)) = Lu + \varepsilon Lv \leq 0,$$

and therefore the weak maximum principle for classical solutions (ex.) implies

$$u + \varepsilon v - u(x_0) \leq 0 \quad \text{in } B(0, r) \setminus \overline{B}(0, r/2).$$

But

$$u(x_0) + \varepsilon v(x_0) - u(x_0) = 0$$

so that

$$\frac{\partial(u + \varepsilon v - u(x_0))}{\partial \nu}(x_0) \geq 0.$$

This yields

$$\frac{\partial u}{\partial \nu}(x_0) \geq -\varepsilon \frac{\partial v}{\partial \nu}(x_0) = -\varepsilon \frac{x_0}{r} Dv(x_0) = -\varepsilon \frac{x_0}{r}(-2x_0 \gamma e^{-\gamma |x_0|^2}) > 0. \quad \square$$

Remark 3.52. *The nontrivial point on Hopf's lemma is that the inequality* $\frac{\partial u}{\partial \nu}(x_0) > 0$ *is strict!*

Theorem 3.53 (Strong max principle)**.** *Let* $u \in C^2(\Omega) \cap C(\overline{\Omega})$ *satisfy*

$$Lu \leq 0,$$

and let Ω *be a bounded, open and connected set. Then if* u *attains its max at the interior of* Ω*, it follows that*

$$u \equiv \sup_{\Omega} u.$$

Proof. Let $M := \max_{\overline{\Omega}} u$ and

$$C = \{x \in \Omega \,:\, u(x) = M\},$$
$$V = \{x \in \Omega \,:\, u(x) < M\}.$$

Let us make a counter proposition that V is not empty. Take a point $y \in V$ with $\operatorname{dist}(y, C) < \operatorname{dist}(y, \partial\Omega)$, which exist since $\operatorname{dist}(C, V) = 0$ by continuity of u. Let

$$B = B(y, r) \subset V$$

be a largest possible ball in V centered at y. Then B touches C at some point x_0, and thus V satisfies interior ball condition at this point. By Hopf's lemma,

$$\frac{\partial u}{\partial \nu}(x_0) > 0$$

but this is a contradiction since x_0 is a max point for u implying $Du(x_0) = 0$. $\square$

4. Linear parabolic equations

Next we study generalizations of the heat equation. We denote

$$\Omega_T = \Omega \times (0, T)$$

and

$$\partial_p \Omega_T = (\overline{\Omega} \times \{0\}) \cup (\partial\Omega \times [0, T]).$$

Definition 4.1 (parabolic Sobolev space)**.** *The Sobolev space*

$$L^2(0,T;W^{1,2}(\Omega)),$$

consists of all measurable(in Ω_T) functions $u(x,t)$ such that $u(x,t)$ belongs to $W^{1,2}(\Omega)$ for almost every $0<t<T$, (u(x,t) is measurable as a mapping from $(0,T)$ to $W^{1,2}(\Omega)$, and the norm

$$\left(\iint_{\Omega_T} (|u(x,t)|^2 + |Du(x,t)|^2)\,dx\,dt\right)^{1/2}$$

is finite. The definition of the space $L^2(0,T;W_0^{1,2}(\Omega))$ is analogous.

The notation above refers to Banach valued functions $(0,T)\mapsto W^{1,2}(\Omega)$ and thus refers to Bochner integration theory. However, we do not pursue this analysis here.

Definition 4.2. *The space*

$$C(0,T;L^2(\Omega))$$

consists of all measurable functions $u:\Omega_T\to\mathbb{R}$ such that

$$||u||_{C(0,T;L^2(\Omega))} := \max_{t\in[0,T]}\left(\int_\Omega |u(x,t)|^2\,dx\right)^{1/2} < \infty$$

and for any $\varepsilon>0$ and $t_1\in[0,T]$ there is $\delta>0$ such that if $|t_1-t_2|\le\delta$, where $t_2\in[0,T]$, then

$$\left(\int_\Omega |u(x,t_1)-u(x,t_2)|^2\,dx\right)^{1/2}\le\varepsilon.$$

Theorem 4.3. *The space $C^\infty(\Omega_T)$ is dense in $L^2(0,T;W^{1,2}(\Omega))$.*

Proof. The space $W^{1,2}(\Omega)$ is separable (not proven here). The proof consists of three steps. First, by separability, we can approximate any function $u\in L^2(0,T;W^{1,2}(\Omega))$, denoted by $u(t)=u(x,t)$, with simple functions. By modifying the simple functions in the set where the norm is large compared to the norm of the original function, and using Lebesgue's dominated convergence theorem, we obtain a L^2-convergent sequence. Finally, we mollify the simple function.

Next we work out the details. Utilizing the separability of $W^{1,2}(\Omega)$, we can choose a countable dense set

$$\{a_k\}_{k=1}^\infty \subset u(0,T).$$

We define for $k=1,\dots,n$

$$\mathcal{F}_k^n = \{f\in W^{1,2}(\Omega)\,:\,||f-a_k||_{W^{1,2}(\Omega)} = \min_{1\le i\le n}||f-a_i||_{W^{1,2}(\Omega)}\}$$

and

$$B_k^n = u^{-1}(\mathcal{F}_k^n), \quad D_1^n = B_1^n, \quad D_k^n = B_k^n \setminus (\cup_{i=1}^{k-1} B_i^n) \quad \text{for} \quad k = 2, 3, \ldots.$$

It follows from the measurability of $u(t)$ that the sets D_k^n are measurable, and thus

$$u_n(t) = \sum_{k=1}^{n} a_k \chi_{D_k^n}(t)$$

is a simple function. Because $\{a_k\}_{k=1}^{\infty}$ is a dense set, it follows that a.e.

$$u_n(t) \to u(t) \quad \text{in} \quad W^{1,2}(\Omega) \qquad \text{as} \quad n \to \infty.$$

In order to use Lebesgue's dominated convergence theorem, we modify u_n whenever $||u_n(t)||_{W^{1,2}(\Omega)}$ is large compared to $||u(t)||_{W^{1,2}(\Omega)}$, and define

$$v_n(t) = \begin{cases} u_n(t), & \text{if} \quad ||u_n(t)||_{W^{1,2}(\Omega)} \le 2\,||u(t)||_{W^{1,2}(\Omega)}\,, \\ 0, & \text{if} \quad ||u_n(t)||_{W^{1,2}(\Omega)} > 2\,||u(t)||_{W^{1,2}(\Omega)}\,. \end{cases}$$

If $||u(t)||_{W^{1,2}(\Omega)} = 0$, then $v_n(t) = 0$ and if $||u(t)||_{W^{1,2}(\Omega)} > 0$, then $v_n(t) = u_n(t)$ for n large enough. We deduce

$$v_n(t) \to u(t) \quad \text{in} \quad W^{1,2}(\Omega), \qquad \text{and} \qquad ||v_n(t)||_{W^{1,2}(\Omega)} \le 2\,||u(t)||_{W^{1,2}(\Omega)}\,.$$

Thus Lebesgue's dominated convergence theorem implies

$$\int_0^T ||v_n(t) - u(t)||^2_{W^{1,2}(\Omega)}\,dt \to 0, \quad \text{as} \quad n \to \infty.$$

Next we denote

$$\hat{D}_k^n = D_k^n \setminus \Big\{ t \in (0,T) \,:\, ||u_n(t)||_{W^{1,2}(\Omega)} > 2\,||u(t)||_{W^{1,2}(\Omega)} \Big\}$$

and get

$$v_n(t) = \sum_{k=1}^{n} a_k \chi_{\hat{D}_k^n}(t).$$

We have shown earlier, using approximations that $C^\infty(\Omega)$ is dense in $W^{1,2}(\Omega)$, and hence we can choose $\varphi_k \in C^\infty(\Omega)$ such that

$$||\varphi_k - a_k||^2_{W^{1,2}(\Omega)} < \frac{\varepsilon}{T}.$$

This implies

$$\int_0^T \left|\left| \sum_{k=1}^{n} a_k \chi_{\hat{D}_k^n}(t) - \sum_{k=1}^{n} \varphi_k \chi_{\hat{D}_k^n}(t) \right|\right|^2_{W^{1,2}(\Omega)} dt < \varepsilon.$$

Finally, we may mollify in t with a mollification parameter δ_n (this follows from the approximation results applied in 1D) such that for each $k = 1, \dots, n$

$$\int_0^T \left|\chi_{\hat{D}_k^n}(t) - (\chi_{\hat{D}_k^n})_{\delta_n}(t)\right|^2 dt < \frac{\varepsilon}{n\,||\varphi_k||_{W^{1,2}(\Omega)}}.$$

Accomplishing this approximation for each $k = 1, 2, \dots, n$, we obtain the desired smooth function

$$\sum_{k=1}^{n} \varphi_k (\chi_{\hat{D}_k^n})_{\delta_n}(t), \tag{4.31}$$

which completes the proof. $\square$

Lemma 4.4. *Let $u \in L^2(\Omega_T)$, extend u as zero to $(-\infty, 0)$ and (T, ∞) and set*

$$u_\varepsilon(x,t) = \int_{\mathbb{R}} u(x, t-s)\eta_\varepsilon(s)\,ds$$

where η_ε is a standard mollifier. Then

$$u_\varepsilon \to u \quad \textit{in } L^2(\Omega_T)$$

Proof. By repeating the argument in the previous proof (cf. (4.31)), we can produce a smooth approximation g such that

$$\Big(\int_{\Omega_T} |u-g|^2\,dy\Big)^{1/2} < \delta/3.$$

We extend u by zero to $(-\infty, 0)$ and (T, ∞), and denote by u_ε a standard mollification in the time direction. Similarly as for space mollifications

$$\begin{aligned}
|u_\varepsilon(x,t)| &= \left|\int_{t-\varepsilon}^{t+\varepsilon} \eta_\varepsilon(t-s)u(x,s)\,ds\right| \\
&\le \int_{t-\varepsilon}^{t+\varepsilon} \eta_\varepsilon(t-s)^{1/2}\eta_\varepsilon(t-s)^{1/2}|u(x,s)|\,ds \\
&\overset{\text{Hölder}}{\le} \underbrace{\Big(\int_{t-\varepsilon}^{t+\varepsilon} \eta_\varepsilon(t-s)\,ds\Big)^{1/2}}_{1} \Big(\int_{t-\varepsilon}^{t+\varepsilon} \eta_\varepsilon(t-s)|u(x,s)|^2\,ds\Big)^{1/2}.
\end{aligned}$$

and

$$\begin{aligned}
\int_{\Omega_T} |u_\varepsilon(x,t)|^2\,dx\,dt &\le \int_{\Omega_T} \int_{t-\varepsilon}^{t+\varepsilon} \eta_\varepsilon(t-s)|u(x,s)|^2\,ds\,dx\,dt \\
&= \int_0^T \int_\Omega \int_{\mathbb{R}} \eta_\varepsilon(t-s)|u(x,s)|^2\,ds\,dx\,dt \\
&\overset{\text{Fubini}}{=} \int_{\mathbb{R}} \int_0^T \int_\Omega \eta_\varepsilon(t-s)|u(x,s)|^2\,dx\,dt\,ds \\
&= \int_{\mathbb{R}} \int_0^T \eta_\varepsilon(t-s)\,dt \int_\Omega |u(x,s)|^2\,dx\,ds \\
&\le \int_{\mathbb{R}} \int_\Omega |u(x,s)|^2\,dx\,ds \\
&= \int_{\Omega_T} |u(x,s)|^2\,dx\,ds.
\end{aligned}$$

We deduce

$$\begin{aligned}
&\Big(\int_{\Omega_T} |u-u_\varepsilon|^2\,dx\,dt\Big)^{1/2} \\
&\overset{\text{Minkowski}}{\le} \Big(\int_{\Omega_T} |u-g|^2\,dx\,dt\Big)^{1/2} + \Big(\int_{\Omega_T} |g-g_\varepsilon|^2\,dx\,dt\Big)^{1/2} + \Big(\int_{\Omega_T} |g_\varepsilon-u_\varepsilon|^2\,dx\,dt\Big)^{1/2} \\
&\qquad \le \delta/3 + \Big(\int_{\Omega_T} |g-g_\varepsilon|^2\,dx\,dt\Big)^{1/2} + \Big(\int_{\Omega_T} |g-u|^2\,dx\,dt\Big)^{1/2} \\
&\qquad \le \delta/3 + \Big(\int_{\Omega_T} |g-g_\varepsilon|^2\,dx\,dt\Big)^{1/p} + \delta/3.
\end{aligned}$$

By adjusting the argument we used in with x-approximations, we see that $g_\varepsilon \to g$ pointwise in $\Omega\times(0,T)$. Moreover, $|g-g_\varepsilon|^2 \le 4\max_{\Omega_T}|g| \in L^1(\Omega_T)$ and thus by DOM, for all small enough ε

$$\Big(\int_{\Omega_T} |g-g_\varepsilon|^2\,dx\,dt\Big)^{1/2} \le \delta/3. \qquad \square$$

Theorem 4.5. *Let $u \in L^2(\Omega_T)$ and $\frac{\partial u}{\partial t} \in L^2(\Omega_T)$. Then there is such a representative that*

$$u \in C(0,T;L^2(\Omega)).$$

Proof. By the previous lemma

$$\begin{cases} u_\varepsilon \to u, & \text{in } L^2(\Omega_T) \\ \frac{\partial u_\varepsilon}{\partial t} \to \frac{\partial u}{\partial t}, & \text{in } L^2(\Omega\times(h,T-h)), \end{cases} \tag{4.32}$$

where $\varepsilon < h$ and the proof of the second statement again follows the guidelines of the space approximations. By Fubini's theorem for a.e. x the function $t \mapsto u(x,t)$ in $L^2(0,T) \subset L^1(0,T)$. Thus for a.e. x $u_\varepsilon(x,t)$ is a smooth function so that

$$u_\varepsilon(x,t_1) - u_\varepsilon(x,t_2) = \int_{t_1}^{t_2} \frac{\partial u_\varepsilon}{\partial t}\, dt$$

and

$$||u_\varepsilon(x,t_1) - u_\varepsilon(x,t_2)||^2_{L^2(\Omega)} = \left|\left|\int_{t_1}^{t_2} \frac{\partial u_\varepsilon}{\partial t}\, dt\right|\right|^2_{L^2(\Omega)}.$$

We apply (4.32) to the RHS together with Fubini's thm and state without a proof (cf. approx section) that LHS converges for a.e. t_1, t_2. Thus

$$||u(x,t_1) - u(x,t_2)||^2_{L^2(\Omega)} \leq C(t_1 - t_2) \int_{t_1}^{t_2} \int_\Omega \left|\frac{\partial u}{\partial t}\right|^2 dx\, dt.$$

This also implies the continuity on the whole interval $[0,T]$. □

We study initial-boundary value problem for given $g : \overline{\Omega}_T \to \mathbb{R}$, $f : \Omega_T \to \mathbb{R}$

$$\begin{cases} u_t + Lu = f, & x \in \Omega_T \\ u = g, & x \in \partial_p \Omega_T. \end{cases}$$

Here

$$Lu(x,t) = -\sum_{i,j=1}^n D_i(a_{ij}(x,t) D_j u(x,t)) + \sum_{i=1}^n b_i(x,t) D_i u(x,t) + c(x,t)u(x,t).$$

Definition 4.6 (uniformly parabolic)**.** *The operator is uniformly parabolic if there are $0 < \lambda \leq \Lambda < \infty$ such that*

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^n a_{ij} \xi_i \xi_j \leq \Lambda |\xi|^2.$$

Definition 4.7 (local weak solution)**.** *A functions $u \in C_{loc}(0,T; L^2_{loc}(\Omega)) \cap L^2_{loc}(0,T; W^{1,2}_{loc}(\Omega))$ is a weak solution to the above PDE if*

$$-\int_{\Omega_T} u\varphi_t\, dx\, dt + \int_{\Omega_T} \sum_{i,j=1}^n a_{ij} D_j u D_i \varphi + \sum_{i=1}^n b_i D_i u \varphi + cu\varphi\, dx\, dt = \int_{\Omega_T} f\varphi\, dx\, dt$$

for every $\varphi \in C_0^\infty(\Omega_T)$.

Definition 4.8 (global weak solution)**.** *Let* $g \in C(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega))$. *A functions* $u \in C(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega))$ *is a weak solution with boundary values* g *to the above PDE if*

$$\begin{aligned}-\int_{\Omega_T} u\varphi_t\,dx\,dt + \int_{\Omega_T} \sum_{i,j=1}^n a_{ij}D_j u D_i\varphi + \sum_{i=1}^n b_i D_i u\varphi + cu\varphi\,dx\,dt \\ = \int_{\Omega_T} f\varphi\,dx\,dt\end{aligned}$$

for every $\varphi \in C_0^\infty(\Omega_T)$, *and*

$$u - g \in L^2(0,T;W_0^{1,2}(\Omega))$$

as well as

$$\int_\Omega u(x,0)\phi(x)\,dx = \int_\Omega g(x,0)\phi(x)\,dx \quad \textit{for every} \quad \phi \in C_0^\infty(\Omega).$$

4.1. **Existence: Galerkin method.** Let $f \in L^2(\Omega_T)$. For simplicity we only consider the problem

$$\begin{cases} u_t = \Delta u + f, & \text{in } \Omega_T \\ u = 0, & \text{on } \partial\Omega \times [0,T] \\ u(x,0) = g(x), & \text{on } \Omega, \end{cases}$$

where $g \in W_0^{1,2}(\Omega)$, but intend to use methods that also work in greater generality. In the weak form,

$$-\int_{\Omega_T} u\frac{\partial\varphi}{\partial t}\,dx\,dt + \int_{\Omega_T} Du \cdot D\varphi\,dx\,dt = \int_\Omega f\varphi\,dx\,dx \tag{4.33}$$

for every $\varphi \in C_0^\infty(\Omega_T)$.

Idea in Galerkin's method is to take a basis ω_i $i = 1, 2, \dots$ in L^2 and $W_0^{1,2}(\Omega)$ and approximate solution as

$$u_m(x,t) = \sum_{i=1}^m c_i^m(t)\omega_i(x).$$

Choosing the coefficients properly, we can show that this approximation converges to a weak solution. Galerkin's method has turned out to be useful in numerical approximations to solutions of PDEs as well.

Step 1(basis): Let

$$\omega_i, \qquad i = 1, 2, \dots$$

be orthogonal basis in $W_0^{1,2}(\Omega)$ (wrt the standard inner product in $W_0^{1,2}(\Omega)$), and orthonormal in $L^2(\Omega)$ (with respect to inner prod of L^2).

Step 2 (approx solutions): Construct approximating solutions by

$$u_m(x,t) = \sum_{i=1}^{m} c_i^m(t)\omega_i(x).$$

where the coefficients satisfy

$$\int_\Omega \frac{\partial u_m}{\partial t}\omega_k\,dx = -\int_\Omega Du_m\cdot D\omega_k\,dx + \int_\Omega f\omega_k\,dx \tag{4.34}$$

for $k = 1, 2, \ldots, m$. Then for LHS

$$\int_\Omega \frac{\partial u_m}{\partial t}\omega_k\,dx = \int_\Omega \sum_i^m \frac{\partial c_i^m}{\partial t}\omega_i\omega_k\,dx \overset{\text{orthonormality}}{=} \frac{\partial c_k^m}{\partial t}$$

and

$$-\int_\Omega Du_m\cdot D\omega_k\,dx = -\int_\Omega c_k^m(t)D\omega_k\cdot D\omega_k\,dx = -c_k^m(t)/\lambda_k.$$

Altogether, we obtain ODE

$$\frac{\partial c_k^m(t)}{\partial t} = -c_k^m(t)/\lambda_k + f_k(t),$$

where $f_k(t) = \int_\Omega f(x,t)\omega_k(x)\,dx$. It follows that

$$c_k^m(t) = e^{-t/\lambda_k}\Big(c_k + \int_0^t e^{\tau/\lambda_k}f_k(\tau)\,d\tau\Big)$$

where c_k are chosen so that

$$g(x) = \sum_{k=1}^{\infty} c_k\omega_k(x)$$

which is possible since $\omega_i,\ i = 1, 2, \ldots$ forms a basis for $W_0^{1,2}(\Omega)$.

Step 3 (uniform estimates for solutions): Multiplying (4.34) by the coefficients and summing, we obtain

$$\int_\Omega \frac{\partial u_m}{\partial t}u_m\,dx = -\int_\Omega Du_m\cdot Du_m\,dx + \int_\Omega fu_m\,dx$$

i.e.

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega u_m^2\,dx = -\int_\Omega |Du_m|^2\,dx + \int_\Omega fu_m\,dx.$$

Further, by integrating over $(0, \tau)$, we obtain

$$\begin{aligned}\frac{1}{2}\int_\Omega u_m^2(x,\tau)\,dx &- \frac{1}{2}\int_\Omega u_m^2(x,0)\,dx \\ &= -\int_{\Omega_\tau} |Du_m|^2\,dx\,dt + \int_{\Omega_\tau} f u_m\,dx\,dt.\end{aligned}$$

We further estimate by using Young's and Sobolev-Poincaré's inequalities

$$\begin{aligned}\left|\int_{\Omega_\tau} f(x,t)u_m\,dx\,dt\right| &\le C\int_{\Omega_T} f^2\,dx\,dt + \varepsilon/\mu^2\int_0^T\int_\Omega u_m^2\,dx\,dt \\ &\le C\int_{\Omega_T} f^2\,dx\,dt + \varepsilon\int_0^T\int_\Omega |Du_m|^2\,dx\,dt\end{aligned}$$

where μ is again the constant in Sobolev-Poincaré's inequality. By choosing $\varepsilon > 0$ small enough, we can absorb the gradient term and obtain an important *energy estimate*

$$\begin{aligned}\sup_{t\in[0,T]}\frac{1}{2}\int_\Omega u_m^2(x,t)\,dx + &\int_{\Omega_T} |Du_m|^2\,dx\,dt \\ &\le C\int_\Omega u_m^2(x,0)\,dx + C\int_{\Omega_T} f^2\,dx\,dt.\end{aligned} \tag{4.35}$$

Multiplying (4.34) by $\frac{\partial}{\partial t}c_k^m(t)$ and summing over k, we obtain

$$\int_\Omega \frac{\partial u_m}{\partial t}\frac{\partial u_m}{\partial t}\,dx = -\int_\Omega Du_m\cdot D\frac{\partial u_m}{\partial t}\,dx + \int_\Omega f(x,t)\frac{\partial u_m}{\partial t}\,dx$$

and again integrating over $(0, T)$ and using Fubini, we have

$$\int_{\Omega_T}\left|\frac{\partial u_m}{\partial t}\right|^2 dx\,dt = -\frac{1}{2}\int_\Omega\int_0^T \frac{\partial}{\partial t}|Du_m|^2\,dt\,dx + \int_{\Omega_T} f\frac{\partial u_m}{\partial t}\,dx\,dt.$$

Again by using Young's inequality

$$\begin{aligned}\int_{\Omega_T}\left|\frac{\partial u_m}{\partial t}\right|^2 dx\,dt + &\frac{1}{2}\Big(\int_\Omega |Du_m(x,T)|^2 - \int_\Omega |Du_m(x,0)|^2\,dx\Big) \\ &\le \varepsilon\int_{\Omega_T}\left|\frac{\partial u_m}{\partial t}\right|^2 dx\,dt + C\int_{\Omega_T} f^2\,dx\,dt.\end{aligned} \tag{4.36}$$

Combining (4.35) and (4.36), we have

$$\begin{aligned}\int_{\Omega_T} |Du_m|^2 + \left|\frac{\partial u_m}{\partial t}\right|^2 + |u_m|^2 \, dx\, dt \\ \leq C \underbrace{\int_\Omega |Du_m(x,0)|^2 \, dx}_{\to \int |Dg(x)|^2 \, dx, \text{ as } m \to \infty} + C \int_{\Omega_T} f^2 \, dx\, dt,\end{aligned} \tag{4.37}$$

where the right hand side is independent of m . Altogether, we have

$$\int_{\Omega_T} |Du_m|^2 + \left|\frac{\partial u_m}{\partial t}\right|^2 + |u_m|^2 \, dx\, dt \leq C \tag{4.38}$$

where C is independent of m.

Step 4 (taking limits): Since the estimate (4.38) is uniform in m, the sequence u_m is uniformly bounded in $L^2(0,T;W^{1,2}(\Omega))$ and $\frac{\partial u_m}{\partial t}$ in $L^2(\Omega_T)$. Thus, there exists a weak limit such that

$$u \in L^2(0,T;W_0^{1,2}(\Omega)),\ \frac{\partial u}{\partial t} \in L^2(\Omega_T).$$

Further, by Thm 4.5, $u \in C(0,T;L^2(\Omega))$.

Step 5 (weak solution): A priori, u_m satisfies the weak formulation for basis functions, so it remains first to check that u is a weak solution. To this end, let

$$h \in C_0^\infty(\Omega) \quad \text{and} \quad \psi \in C_0^\infty([0,T]),$$

and choose a sequence

$$h_j(x) := \sum_{k=1}^{j} \alpha_{kj}\omega_k(x) \to h \quad \text{in } W^{1,2}(\Omega) \text{ as } j \to \infty.$$

We multiply (4.34) by $\psi(t)$, integrate over $(0,T)$, and pass to the limit $m \to \infty$ to have

$$\begin{aligned}\int_0^T \int_\Omega \frac{\partial u}{\partial t}\omega_k \psi \, dx\, dt = -\int_0^T \int_\Omega Du \cdot D\omega_k \psi \, dx \\ + \int_0^T \int_\Omega f\omega_k\psi \, dx\, dt.\end{aligned}$$

Then, we multiply this by α_{kj}, and sum up to have

$$\int_0^T \int_\Omega \frac{\partial u}{\partial t} \sum_{k=1}^{j} \alpha_{kj}\omega_k\psi \, dx \, dt = -\int_0^T \int_\Omega Du \cdot D \sum_{k=1}^{j} \alpha_{kj}\omega_k\psi \, dx \, dt + \int_0^T \int_\Omega f(x,t) \sum_{k=1}^{j} \alpha_{kj}\omega_k\psi \, dx \, dt.$$

Then passing to the limit with j, we end up with

$$\int_0^T \int_\Omega \frac{\partial u(x,t)}{\partial t} h(x)\psi(t) \, dx \, dt = -\int_0^T \int_\Omega Du(x,t) \cdot Dh(x)\psi(t) \, dx \, dt + \int_0^T \int_\Omega f(x,t)h(x)\psi(t) \, dx \, dt.$$

By modifying the proof of Thm 4.3, see in particular (4.31), we see that by summing up the functions of the type $h(x)\psi(t)$ we may approximate functions in $L^2(0,T;W_0^{1,2}(\Omega))$. Thus, in particular,

$$\int_0^T \int_\Omega \frac{\partial u}{\partial t}\varphi \, dx \, dt = -\int_0^T \int_\Omega Du \cdot D\varphi \, dx \, dt + \int_0^T \int_\Omega f\varphi \, dx \, dt$$

for all $\varphi \in C_0^\infty(\Omega_T)$.

Step 5 (initial condition): It remains to check that the initial condition is satisfied. Similarly as above, denoting $v_j(x,t) := \sum_{k=1}^{j} \beta_{kj}(t)\omega_k(x)$, $j \le m$, and for which $v_j(x,T) = 0$ we obtain

$$\int_0^T \int_\Omega \frac{\partial u_m}{\partial t} v_j \, dx \, dt = -\int_0^T \int_\Omega Du_m \cdot Dv_j(x) \, dx \, dt + \int_0^T \int_\Omega f(x,t)v_j(x) \, dx \, dt. \tag{4.39}$$

Integrating by parts wrt t,

$$-\int_\Omega u_m(x,0)v_j(x,0) \, dx - \int_0^T \int_\Omega u_m \frac{\partial v_j}{\partial t} \, dx \, dt = -\int_0^T \int_\Omega Du_m \cdot Dv_j \, dx \, dt + \int_0^T \int_\Omega f v_j \, dx \, dt.$$

Then we pass to the limit $m \to \infty$, and then with $j \to \infty$, where we may choose β_{kj} so that $v_j(x,0) \to \phi(x) \in C_0^\infty(\Omega)$ in $L^2(\Omega)$ and v_j

converges to a suitable test function v. This produces

$$\begin{aligned}\int_\Omega g(x)\phi(x)\,dx - \int_0^T \int_\Omega u\frac{\partial v}{\partial t}\,dx\,dt \\ = -\int_0^T \int_\Omega Du\cdot Dv\,dx\,dt + \int_0^T \int_\Omega fv\,dx\,dt.\end{aligned} \tag{4.40}$$

On the other hand, passing first to the limit $m \to \infty$ and then $j \to \infty$ in (4.39), as well as integrating by parts wrt t after that, we get

$$\begin{aligned}-\int_\Omega u(x,0)\phi(x)\,dx - \int_0^T \int_\Omega u\frac{\partial v}{\partial t}\,dx\,dt \\ = -\int_0^T \int_\Omega Du\cdot Dv\,dx\,dt + \int_0^T \int_\Omega fv\,dx\,dt.\end{aligned} \tag{4.41}$$

Comparing (4.40) and (4.41), we see that u satisfies the initial condition.

We have proven the following.

Theorem 4.9. *Let $g \in W_0^{1,2}(\Omega)$ and $f \in L^2(\Omega_T)$. There exists a weak solution to the problem*

$$\begin{cases} u_t = \Delta u + f, & \text{in } \Omega_T \\ u = 0, & \text{on } \partial\Omega \times [0,T] \\ u(x,0) = g(x), & \text{on } \Omega. \end{cases}$$

Remark 4.10. *The condition $g \in W_0^{1,2}(\Omega)$ can be relaxed as well as the operator with*

$$a_{ij}, b_i, c \in L^\infty(\Omega_T), f \in L^2(\Omega_T)$$

is ok, see Evans p. 356. The method remains essentially the same.

Method also generalizes for more general bdr conditions $g \in C(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega))$.

4.2. **Standard time mollification.** Now $\frac{\partial u}{\partial t}$ exists but in more general situation (for example $u_t = \operatorname{div}(\mathcal{A}(x,t,Du))$ for a suitable nonlinear operator), u does not necessarily have time derivative. Nonetheless, it is often useful to have u in the test function, and thus we would have $\frac{\partial u}{\partial t}$ in the weak formulation, which does not necessarily exist as a function. This problem is treated by time mollification.

Let $\phi \in C_0^\infty(\Omega_T)$. Our goal is to show

$$-\int_0^T \int_\Omega u_\varepsilon \frac{\partial \phi}{\partial t}\,dz + \int_0^T \int_\Omega (Du)_\varepsilon \cdot D\phi\,dz = 0, \tag{4.42}$$

where ε in u_ε and $(Du)_\varepsilon$ denote the mollification with respect to t.

Let $\operatorname{spt}\phi(x,\cdot) \subset (\varepsilon, T-\varepsilon)$. We can use Lebesgue's dominated convergence theorem to see that $D\int = \int D$ in this case. Further, by Fubini's theorem and by taking into account the support of $\phi(x,\cdot)$, we see that

$$\begin{aligned}
&\int_0^T \int_\Omega Du(x,t) \cdot D\phi_\varepsilon \, dz \\
&\qquad = \int_0^T \int_\Omega Du(x,t) \cdot D \int_{\mathbb{R}} \phi(x,s)\eta_\varepsilon(t-s) \, ds \, dz \\
&\qquad = \int_{\mathbb{R}} \int_\Omega \int_0^T Du(x,t) \cdot D\phi(x,s)\eta_\varepsilon(t-s) \, dt \, dx \, ds \\
&\qquad = \int_{\mathbb{R}} \int_\Omega \int_0^T Du(x,t)\, \eta_\varepsilon(t-s) \, dt \cdot D\phi(x,s) \, dx \, ds.
\end{aligned} \tag{4.43}$$

Since η_ε is an even function, we have

$$\int_0^T Du(x,t)\, \eta_\varepsilon(t-s) \, dt = \int_0^T Du(x,t)\, \eta_\varepsilon(s-t) \, dt = (Du(x,s))_\varepsilon, \tag{4.44}$$

where we can restrict $\varepsilon \le s \le T-\varepsilon$. This is due to assumption $\operatorname{spt}(\phi(x,\cdot)) \subset (\varepsilon, T-\varepsilon)$. By subtracting (4.44) into (4.43), we obtain

$$\begin{aligned}
&\int_0^T \int_\Omega Du(x,t) \cdot D\phi_\varepsilon(x,t) \, dz \\
&\qquad = \int_\varepsilon^{T-\varepsilon} \int_\Omega (Du(x,s))_\varepsilon \cdot D\phi(x,s) \, dx \, ds.
\end{aligned} \tag{4.45}$$

Similarly

$$\begin{aligned}
&\int_0^T \int_\Omega u(x,t) \frac{\partial \phi_\varepsilon}{\partial t} \, dx \, dt \\
&\qquad = \int_0^T \int_\Omega \int_{\mathbb{R}} u(x,t) \frac{\partial}{\partial s}\phi(x,s)\eta_\varepsilon(t-s) \, ds \, dx \, dt \\
&\qquad = \int_\varepsilon^{T-\varepsilon} \int_\Omega \int_0^T u(x,t)\eta_\varepsilon(t-s) \, dt \frac{\partial}{\partial s}\phi(x,s) \, dx \, ds \\
&\qquad = \int_\varepsilon^{T-\varepsilon} \int_\Omega u_\varepsilon(x,s) \frac{\partial}{\partial s}\phi(x,s) \, dx \, ds.
\end{aligned} \tag{4.46}$$

The definition of a weak solution combined with (4.45) and (4.46) imply (4.42).

4.3. **Steklov averages.** Another alternative is to use Steklov averages. Let $u \in L^1(\Omega_T)$. Then the Steklov average is defined as

$$u_h = \frac{1}{h}\int_t^{t+h} u(x,\tau)\,d\tau, \qquad t \in (0, T-h).$$

Weak formulation can also be written (ex) for $0 < t_1 < t_2 < T$ as

$$\begin{aligned}\int_\Omega u(x,t_2)\varphi(x,t_2)\,dx - \int_\Omega u(x,t_1)\varphi(x,t_1)\,dx \\ - \int_{\Omega\times(t_1,t_2)} u\varphi_t\,dx\,dt + \int_{\Omega\times(t_1,t_2)} Du\cdot D\varphi\,dx\,dt = 0.\end{aligned}$$

Then choose $\varphi(s,x) \in C_0^\infty(\Omega_T)$ independent of t (this is not compactly supported in t as it is constant in t, but it does not matter). Since φ is compactly supported in s, we can choose $t_1 = s$, $t_2 = s+h$ for small enough h. Then divide by h, and observe that $\varphi_t = 0$ so that

$$\begin{aligned}0 =&\frac{1}{h}\int_\Omega (u(x,s+h) - u(x,s))\varphi(x,s)\,dx \\ &+ \frac{1}{h}\int_\Omega\int_s^{s+h} Du(x,t)\,dt\cdot D\varphi(x,s)\,dx \\ =&\int_\Omega \frac{\partial u_h(x,s)}{\partial s}\varphi(x,s)\,dx + \int_\Omega \big(Du\big)_h(x,s)\cdot D\varphi(x,s)\,dx.\end{aligned}$$

Integrate wrt s over $(0,T)$ to obtain

$$\begin{aligned}0 &= \int_{\Omega_T} \frac{\partial u_h(x,s)}{\partial s}\varphi(x,s)\,dx\,ds + \int_{\Omega_T} (Du)_h(x,s)\cdot D\varphi(x,s)\,dx\,ds \\ &= -\int_{\Omega_T} u_h\frac{\partial\varphi}{\partial s}\,dx\,ds + \int_{\Omega_T} (Du)_h\cdot D\varphi\,dx\,ds,\end{aligned} \tag{4.47}$$

for every $\varphi \in C_0^\infty(\Omega_T)$.

4.4. **Uniqueness.** In this section, similarly in the elliptic case, we simplify the treatment considering

$$Lu = -\sum_{i,j=1}^n D_i(a_{ij}D_ju) + cu$$

with $c \geq c_0$, $c_0 \in \mathbb{R}$. In the proof below, we want avoid using the time derivative of a solution and therefore use mollifications.

Theorem 4.11. *There is c_0 such that if $c \geq c_0$, then the weak solution to*

$$\begin{cases} u_t + Lu = 0, & \text{in } \Omega_T \\ u = g & \text{on } \partial_p \Omega_T. \end{cases}$$

with $g \in C(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega))$ is unique.

Proof. Let u and w be two weak solutions. Then similarly as in (4.42)

$$-\int_{\Omega_T} u_\varepsilon \frac{\partial v^\varepsilon}{\partial t}\,dx\,dt + \int_{\Omega_T} \sum_{i,j=1}^n (a_{ij} D_j u)_\varepsilon D_i v^\varepsilon + cu(v^\varepsilon)_\varepsilon \,dx\,dt = 0$$

where $\operatorname{spt} v \subset \Omega_T$, and ε is small enough, and a similar equation for w. Then by subtracting the equations, we have

$$\begin{aligned} -\int_{\Omega_T} (u-w)_\varepsilon \frac{\partial v^\varepsilon}{\partial t}\,dx\,dt + \int_{\Omega_T} \sum_{i,j=1}^n &(a_{ij}(D_j(u-w))_\varepsilon D_i(v^\varepsilon)_\varepsilon) \\ &+ c(u-w)(v^\varepsilon)_\varepsilon \,dx\,dt = 0. \end{aligned} \tag{4.48}$$

We choose

$$v^\varepsilon(x,t) = (\chi^h_{0,T}(t))_\varepsilon (u-w)_\varepsilon$$

with

$$\chi^h_{0,T} = \begin{cases} 0 & t \leq h, \\ (t-h)/h & h < t \leq 2h, \\ 1, & 2h < t \leq T-2h, \\ (-t+T-h)/h, & T-2h < t \leq T-h, \\ 0, & T-h < t. \end{cases}$$

Moreover, by density we can extend the class of test functions so that $(v^\varepsilon)_\varepsilon$ is admissible (ex). We estimate

$$\begin{aligned} &\int_{\Omega_T} (u-w)_\varepsilon \frac{\partial v^\varepsilon}{\partial t}\,dx\,dt \\ &\quad = \int_{\Omega_T} (u-w)_\varepsilon \frac{\partial (\chi_h)_\varepsilon (u-w)_\varepsilon}{\partial t}\,dx\,dt \\ &\quad = \int_{\Omega_T} (u-w)_\varepsilon \frac{\partial (\chi_h)_\varepsilon}{\partial t}(u-w)_\varepsilon + (\chi_h)_\varepsilon \frac{\partial (u-w)_\varepsilon}{\partial t}\,dx\,dt \\ &\quad = \int_{\Omega_T} (u-w)^2_\varepsilon \frac{\partial (\chi_h)_\varepsilon}{\partial t}\,dx\,dt + \int_{\Omega_T} (\chi_h)_\varepsilon \frac{1}{2}\frac{\partial (u-w)^2_\varepsilon}{\partial t}\,dx\,dt \end{aligned}$$

Then we integrate by parts and pass to the limit

$$
\begin{aligned}
&\overset{\text{int by parts}}{=} \int_{\Omega_T} (u-w)_\varepsilon^2 \frac{\partial(\chi_h)_\varepsilon}{\partial t}\,dx\,dt - \frac{1}{2}\int_{\Omega_T} \frac{\partial(\chi_h)_\varepsilon}{\partial t}(u-w)_\varepsilon^2\,dx\,dt \\
&= \frac{1}{2}\int_{\Omega_T} \frac{\partial(\chi_h)_\varepsilon}{\partial t}(u-w)_\varepsilon^2\,dx\,dt \\
&\overset{\varepsilon\to 0}{=} \frac{1}{2}\int_{\Omega_T} \frac{\partial\chi_h}{\partial t}(u-w)^2\,dx\,dt \\
&= \frac{1}{2h}\int_h^{2h}\int_\Omega (u-w)^2\,dx\,dt - \frac{1}{2h}\int_{T-2h}^{T-h}\int_\Omega (u-w)^2\,dx\,dt \\
&\overset{h\to 0}{=} 0 - \frac{1}{2}\int_\Omega (u(x,T)-w(x,T))^2\,dx,
\end{aligned}
$$

where at the last step we used continuity and the initial condition.

The other terms in (4.48) converge by similar approximation arguments as before, and combining the above calculation together with (4.48), we obtain by first letting $\varepsilon \to 0$ and then $h \to 0$

$$
\begin{aligned}
0 &= \frac{1}{2}\int_\Omega (u(x,T)-w(x,T))^2\,dx \\
&\quad + \int_{\Omega_T} \sum_{i,j=1}^n a_{ij}D_j(u-w)D_i(u-w) + c(u-w)(u-w)\,dx\,dt \\
&\overset{\text{parab}}{\geq} \frac{1}{2}\int_\Omega (u(x,T)-w(x,T))^2\,dx \\
&\quad + \int_{\Omega_T} \lambda\left|D_j(u-w)\right|^2 + c(u-w)^2\,dx\,dt \\
&\geq \frac{1}{2}\int_\Omega (u(x,T)-w(x,T))^2\,dx + \int_{\Omega_T}\Big(\frac{\lambda}{\mu}+c_0\Big)(u-w)^2\,dx\,dt
\end{aligned}
$$

where we used Sobolev-Poincaré's inequality with a constant μ. If $-\gamma := \frac{1}{2}(\lambda/\mu + c_0) > 0$ then the result is immediate. Otherwise, let us denote $\eta(T) := \int_\Omega (u(x,T)-w(x,T))^2\,dx$. Then the above estimate reads as

$$
\gamma \int_0^T \eta(t)\,dt \geq \eta(T).
$$

We can repeat the argument for a.e. $t \in (0,T)$ instead of T, and have $\gamma\int_0^t \eta(s)\,ds \geq \eta(t)$. But this is as in well-known Grönwall's inequality (proof is ex.) which now says $\eta(t) = 0$ a.e. completing the proof. $\square$

4.5. C^α **regularity using Moser's method.** For simplicity we concentrate on $\frac{\partial u}{\partial t} = \Delta u + f$, $f \in L^\infty(\Omega_T)$ but method immediately extends to more general linear PDEs.

Definition 4.12 (supersolution)**.** *A function $u \in L^2_{loc}(0,T;W^{1,2}_{loc}(\Omega))$ is a weak supersolution to $\frac{\partial u}{\partial t} = \Delta u + f$, if*

$$-\int_{\Omega_T} u\frac{\partial \varphi}{\partial t}\,dx\,dt + \int_{\Omega_T} Du\cdot D\varphi\,dx\,dt \geq \int_{\Omega_T} f\varphi\,dx\,dt$$

for every $\varphi \in C_0^\infty(\Omega_T)$, $\varphi \geq 0$.

Definition 4.13 (subsolution)**.** *A function $u \in L^2_{loc}(0,T;W^{1,2}_{loc}(\Omega))$ is a weak subsolution to $\frac{\partial u}{\partial t} = \Delta u + f$, if*

$$-\int_{\Omega_T} u\frac{\partial \varphi}{\partial t}\,dx\,dt + \int_{\Omega_T} Du\cdot D\varphi\,dx\,dt \leq \int_{\Omega_T} f\varphi\,dx\,dt$$

for every $\varphi \in C_0^\infty(\Omega_T)$, $\varphi \geq 0$.

Formally we can write for example for subsolution $\frac{\partial u}{\partial t} - \Delta u \leq f$.
Rough plan:

We will describe details, notation etc. later, but look at rough ideas to begin with.

We look at parabolic Harnack's inequality. The elliptic Harnack's inequality for positive harmonic function in Ω reads as

$$\operatorname*{ess\,sup}_B u \leq C \operatorname*{ess\,inf}_B u$$

where $2B \subset \Omega$ (local estimate). In contrast with this, in parabolic Harnack's inequality the sets on RHS/LHS are not the same. Instead, they take into account the flow of information from the past to the future. Indeed, parabolic Harnack's inequality for a positive solution to the heat equation can be stated as

$$\operatorname*{ess\,sup}_{Q^-} u \leq C \operatorname*{ess\,inf}_{Q^+} u$$

where Q^- lies in the past compared to Q^+, where the cylinder lie well within the domain (again a local estimate) . There are counterexamples (ex) showing that this so called waiting time is indispensable.

(1) (Easier part) We intend to show that a positive subsolution is bounded from above with explicit estimate

$$\operatorname*{ess\,sup}_{Q} u \leq C \int_{\tilde{Q}} u\,dx\,dt$$

where $Q, \tilde{Q}$ are parabolic cylinders .

(2) (Harder part) We partly show a lower bound for a positive weak supersolution in a form

$$\int_{Q^-} u\,dx\,dt \le C \operatorname*{ess\,inf}_{\tilde{Q}^+} u.$$

In this estimate, direction of time plays a crucial role.

Lemma 4.14 (Energy estimate)**.** *Let $u \ge 0$ be a weak subsolution to $\frac{\partial u}{\partial t} - \Delta u \le 0$ in Ω_T and $\gamma \ge 1$. Then there exists $C = C(\gamma)$ such that*

$$\int_{\Omega_T} |Du|^2 u^{\gamma-1}\eta^2\,dx\,dt + \operatorname*{ess\,sup}_{t\in(0,T)} \int_{\Omega} u^{1+\gamma}\eta^2\,dx\,dt$$
$$\le C\int_{\Omega_T} u^{1+\gamma}|D\eta|^2\,dx\,dt + C\int_{\Omega_T} u^{1+\gamma}\eta\left|\frac{\partial \eta}{\partial t}\right|dx\,dt$$

for every $\eta \in C_0^\infty(\Omega_T)$, $\eta \ge 0$.

Proof. Use (formal, details are ex.) test function $\varphi = \eta^2\chi_{0,t}^h u^\gamma$ (now γ is a power) in

$$-\int_{\Omega_T} u\frac{\partial \varphi}{\partial t}\,dx\,ds + \int_{\Omega_T} Du\cdot D\varphi\,dx\,ds \le 0.$$

First term can be estimated by integration by parts as

$$\begin{aligned}\int_{\Omega_T} u\frac{\partial\varphi}{\partial t}\,dx\,ds &= \int_{\Omega_T} u\frac{\partial(\eta^2\chi_{0,t}^h u^\gamma)}{\partial t}\,dx\,ds\\ &= \int_{\Omega_T} u\Big(\frac{\partial\eta^2\chi_{0,t}^h}{\partial t}u^\gamma + \eta^2\chi_{0,t}^h\gamma u^{\gamma-1}\frac{\partial u}{\partial t}\Big)\,dx\,ds\\ &= \frac{1}{1+\gamma}\int_{\Omega_T}\frac{\partial\eta^2\chi_{0,t}^h}{\partial t}u^{\gamma+1}\,dx\,ds.\end{aligned}$$

For the second term

$$\begin{aligned}&\int_{\Omega_T} Du\cdot D\varphi\,dx\,ds\\ &\quad= \int_{\Omega_T} Du\cdot D(\eta^2\chi_{0,t}^h u^\gamma)\,dx\,ds\\ &\quad= \int_{\Omega_T} \eta^2\chi_{0,t}^h\gamma|Du|^2u^{\gamma-1} + Du\cdot\chi_{0,t}^h D\eta^2 u^\gamma\,dx\,ds\\ &\quad= \int_{\Omega_T} \gamma\eta^2\chi_{0,t}^h|Du|^2u^{\gamma-1} + u^{(\gamma-1)/2}Du\cdot\chi_{0,t}^h(D\eta^2)u^{(\gamma+1)/2}\,dx\,ds.\end{aligned}$$

Then use Young's inequality to estimate the second term.

Finally combine the estimates and absorb the gradient term into the left, choose t suitably so that one of the terms is close to ess sup-term (detailed calculation was presented during the lecture). $\square$

Choosing $\gamma = 2q^k - 1,\ k = 0, 1, 2, \dots,\ q \geq 1$ gives the following corollary. Also observe that as γ increases, the constants in the previous lemma remain bounded. Thus we can choose the constant independent of k below.

Corollary 4.15. *Let $u \geq 0$ be a subsolution in Ω_T and $q \geq 1$. Then there exists $C = C(q)$ such that*

$$\int_{\Omega_T} \left| Du^{q^k} \right|^2 \eta^2 \, dx \, dt + \operatorname*{ess\,sup}_{t \in (0,T)} \int_\Omega u^{2q^k} \eta^2 \, dx \, dt \\ \leq C \int_{\Omega_T} u^{2q^k} |D\eta|^2 \, dx \, dt + C \int_{\Omega_T} u^{2q^k} \eta \left| \frac{\partial \eta}{\partial t} \right| dx \, dt$$

for every $\eta \in C_0^\infty(\Omega_T),\ \eta \geq 0$.

Lemma 4.16 (Parabolic Sobolev's inequality)**.** *Let $u \in L^2(0, T; W_0^{1,2}(\Omega))$ and $q = 1 + 2/n$. Then there exists $C = C(n)$ such that*

$$\int_{\Omega_T} |u|^{2q} \, dx \, dt \leq C \Big(\operatorname*{ess\,sup}_{t \in (0,T)} \int_\Omega |u|^2 \, dx \Big)^{2/n} \int_{\Omega_T} |Du|^2 \, dx \, dt$$

Proof. By Hölder's inequality for a.e. $t \in (0, T)$

$$\begin{aligned} \int_\Omega |u|^{2q} \, dx &\leq \int_\Omega |u|^{1-q+2q+(q-1)} \, dx \\ &\leq \Big(\int_\Omega |u|^{(1+q)n/(n-1)} \, dx \Big)^{(n-1)/n} \Big(\int_\Omega |u|^{(q-1)n} \, dx \Big)^{1/n} \\ &\leq \Big(\int_\Omega |u|^{(1+q)n/(n-1)} \, dx \Big)^{(n-1)/n} \Big(\int_\Omega |u|^2 \, dx \Big)^{1/n} \end{aligned}$$

Then using Sobolev's inequality with $1^* = n/(n-1)$ and 1, we have

$$\begin{aligned} \Big(\int_\Omega |u|^{(1+q)n/(n-1)} \, dx \Big)^{(n-1)/n} &\leq C \int_\Omega \left| D(|u|^{(1+q)}) \right|^1 dx \\ &= C \int_\Omega |u|^q |Du| \, dx \\ &\overset{\text{Hölder}}{\leq} C \Big(\int_\Omega |u|^{2q} \, dx \Big)^{1/2} \Big(\int_\Omega |Du|^2 \, dx \Big)^{1/2}. \end{aligned}$$

Then we combine the estimates, integrate over $(0, T)$ and estimate by ess sup to obtain the result. $\square$

For notational convenience we consider the domain around the origin. This can be done without loss of generality.

$$Q_R = B(0,R) \times (-R^2, R^2),$$

We assume that $2 < n$ and $R \le 1$.

Lemma 4.17. *Let $u \ge 0$ be a subsolution to $\frac{\partial u}{\partial t} - \Delta u \le f$ in Q_{2R} . Then there are $C = C(n)$ such that*

$$\operatorname*{ess\,sup}_{Q_{R/2}} u \le C\Big(\fint_{Q_R} u^2 \, dx \, dt \Big)^{1/2} + C \, ||f||_{L^\infty(Q_R)} \, .$$

Proof. The proof consists of several steps:
Step 1 (simplification): Set $w = u + (t_{\max} - t) \, ||f||_{L^\infty(Q_R)}$, where $t_{\max}$ is a suitable constant so that $t_{\max} - t \ge 0$, and observe that

$$\begin{aligned}
-\int_{Q_R} & w \frac{\partial \varphi}{\partial t} \, dx \, dt + \int_{Q_R} Dw \cdot D\varphi \, dx \, dt \\
&= -\int_{Q_R} u \frac{\partial \varphi}{\partial t} \, dx \, dt + \int_{Q_R} Du \cdot D\varphi \, dx \, dt - \int_{Q_R} ||f||_{L^\infty(Q_R)} \, \varphi \, dx \, dt \\
&\le \int_{Q_R} (f - ||f||_{L^\infty(Q_R)}) \varphi \, dx \, dt \le 0.
\end{aligned}$$

Thus we may concentrate on the homogenous equation $\frac{\partial w}{\partial t} - \Delta w \le 0$. If the results holds for w

$$\operatorname*{ess\,sup}_{Q_{R/2}} w \le C\Big(\fint_{Q_R} w^2 \, dx \, dt \Big)^{1/2}$$

this then implies

$$\operatorname*{ess\,sup}_{Q_{R/2}} u \le C\Big(\fint_{Q_R} u^2 \, dx \, dt \Big)^{1/2} + C \, ||f||_{L^\infty(Q_R)} \, .$$

Step 2 (reverse Hölder): By step 1, let $u \ge 0$ be a subsolution to $\frac{\partial u}{\partial t} - \Delta u \le 0$. Let ρ, σ be such that

$$\frac{R}{2} \le \rho < \sigma \le R$$

and choose a cut-off function $\eta \in C_0^\infty(Q_\sigma), \ 0 \le \eta \le 1$ such that $\eta = 1$ in Q_ρ and

$$|D\eta| + \left| \frac{\partial \eta}{\partial t} \right|^{\frac{1}{2}} \le \frac{C}{\sigma - \rho}$$

By Corollary 4.15 (the same proofs give the estimates in Q_σ instead of Ω_T) choosing $k = 0$, we have

$$\begin{aligned}
\int_{Q_\sigma} |Du|^2 \eta^2 \, dx\, dt &+ \operatorname*{ess\,sup}_{t\in(-\sigma^2,\sigma^2)} \int_{B(0,\sigma)} u^2\eta^2 \, dx\, dt \\
&\le C \int_{Q_\sigma} u^2 |D\eta|^2 \, dx\, dt + C\int_{Q_\sigma} u^2 \eta \left| \frac{\partial \eta}{\partial t} \right| dx\, dt \\
&\le \frac{C}{(\rho-\sigma)^2} \int_{Q_\sigma} u^2 \, dx\, dt.
\end{aligned}$$

By using parabolic Sobolev's inequality in Lemma 4.16 and then the previous estimate, we deduce

$$\begin{aligned}
&\Big(\int_{Q_\rho} u^{2q} \, dx\, dt \Big)^{1/q} \\
&\le C^{1/q} \operatorname*{ess\,sup}_t \Big(\int_{\sigma B} (\eta u)^2 \, dx \Big)^{2/(nq)} \Big(\int_{Q_\sigma} |D(\eta u)|^2 \, dx\, dt \Big)^{1/q} \\
&\le C^{1/q} \Big(\operatorname*{ess\,sup}_t \int_{\sigma B} \eta^2 u^2 \, dx + \int_{Q_\sigma} |D(\eta u)|^2 \, dx\, dt \Big)^{\overbrace{(2/n+1)/q}^{1}} \\
&\le C^{1/q} \Big(\operatorname*{ess\,sup}_t \int_{\sigma B} \eta^2 u^2 \, dx + \int_{Q_\sigma} |D\eta|^2 u^2 + \eta^2 |Du|^2 \, dx\, dt \Big) \\
&\le \frac{C^{1/q}}{(\sigma-\rho)^2} \int_{Q_\sigma} u^2 \, dx\, dt.
\end{aligned}$$

Similarly

$$\begin{aligned}
&\Big(\int_{Q_\rho} u^{2q^2} \, dx\, dt \Big)^{1/q^2} \\
&\overset{\text{parab Sobo}}{\le} C^{1/q^2} \operatorname*{ess\,sup}_t \Big(\int_{B(0,\sigma)} (\eta u^q)^2 \, dx \Big)^{2/(nq^2)} \Big(\int_{Q_\sigma} |D(\eta u^q)|^2 \, dx\, dt \Big)^{1/q^2} \\
&\le C^{1/q^2} \Big(\operatorname*{ess\,sup}_t \int_{B(0,\sigma)} \eta^2 u^{2q} \, dx + \int_{Q_\sigma} |D(\eta u^q)|^2 \, dx\, dt \Big)^{\overbrace{(2/n+1)/q^2}^{1/q}} \\
&\le C^{1/q^2} \Big(\operatorname*{ess\,sup}_t \int_{B(0,\sigma)} \eta^2 u^{2q} \, dx + \int_{Q_\sigma} |D\eta|^2 u^{2q} + \eta^2 |Du^q|^2 \, dx\, dt \Big)^{1/q} \\
&\overset{\text{Cor 4.15, } k=1}{\le} \Big(\frac{C^{1/q}}{(\sigma-\rho)^2} \int_{Q_\sigma} u^{2q} \, dx\, dt \Big)^{1/q}.
\end{aligned}$$

This argument in general yields

$$\begin{aligned}\Big(\frac{1}{R^{n+2}}\int_{Q_\rho} u^{2q^{k+1}}\,dx\,dt\Big)^{1/(2q^{k+1})}\\ \le \Big(\frac{C^{1/q}}{R^n(\rho-\sigma)^2}\int_{Q_\sigma} u^{2q^k}\,dx\,dt\Big)^{1/(2q^k)}.\end{aligned}\tag{4.49}$$

Step 3 (Moser's iteration): Replace in (4.49) ρ by ρ_{k+1} and σ by ρ_k where

$$\rho_k = \frac{R}{2}(1+2^{-k}),\ \ k=0,1,\dots$$

so that $\rho_k - \rho_{k+1} = \frac{R}{2}2^{-k}(1-1/2) = R2^{-k-2}$. Thus

$$\Big(\frac{1}{R^{n+2}}\int_{Q_{\rho_{k+1}}} u^{2q^{k+1}}\,dx\,dt\Big)^{1/(2q^{k+1})} \le \Big(\frac{C^{1/q}2^{2(k+2)}}{R^{n+2}}\int_{Q_{\rho_k}} u^{2q^k}\,dx\,dt\Big)^{1/(2q^k)}.$$

We iterate this

$$\begin{aligned}&\Big(\frac{1}{R^{n+2}}\int_{Q_{\rho_{k+1}}} u^{2q^{k+1}}\,dx\,dt\Big)^{1/(2q^{k+1})}\\ &\le C^{1/(2q^{k+1})}2^{(k+2)/q^k}\Big(\frac{1}{R^{n+2}}\int_{Q_{\rho_k}} u^{2q^k}\,dx\,dt\Big)^{1/(2q^k)}\\ &\le C^{1/(2q^{k+1})}2^{(k+2)/q^k}C^{1/(2q^k)}2^{((k-1)+2)/q^{k-1}}\Big(\frac{1}{R^{n+2}}\int_{Q_{\rho_{k-1}}} u^{2q^{k-1}}\,dx\,dt\Big)^{1/(2q^{k-1})}\\ &\le C^{1/(2q^{k+1})}2^{(k+2)/q^k}C^{1/(2q^k)}2^{((k-1)+2)/q^{k-1}}\Big(\frac{1}{R^{n+2}}\int_{Q_{\rho_{k-1}}} u^{2q^{k-1}}\,dx\,dt\Big)^{1/(2q^{k-1})}\\ &\dots\\ &\le C^{\gamma'}2^{\gamma^*}\Big(\fint_{Q_{\rho_0}} u^2\,dx\,dt\Big)^{1/2},\end{aligned}$$

where

$$\gamma' = \sum_{i=1}^\infty \frac{1}{2q^i}, \qquad \gamma^* = \sum_{i=0}^\infty \frac{2(i+2)}{q^i}.$$

Then let $k\to\infty$ and observe that the LHS in the above estimate converges (see Measure and integration 1) to $\operatorname{ess\,sup}_{Q_{R/2}} u$. □

Lemma 4.18 (Iteration lemma). *Let $G(s)$ be a bounded and nonnegative function for $s\in[R/2,\,R]$*

$$G(\rho) \le \theta G(\sigma) + \frac{C_0}{(\sigma-\rho)^\alpha}$$

where $\theta < 1$ and $R/2 \le \rho < \sigma \le R$. Then there is $C = C(\alpha, \theta)$ such that

$$G(\rho') \le C\Big(\frac{C_0}{(\sigma' - \rho')^\alpha}\Big),$$

where $R/2 \le \rho' < \sigma' \le R$.

Proof. Ex. □

Corollary 4.19. *Let $u \ge 0$ be a subsolution to $\frac{\partial u}{\partial t} - \Delta u \le f$ in Q_{2R}. Then there is $C = C(n, s)$ such that*

$$\operatorname*{ess\,sup}_{Q_{R/2}} u \le C\Big(\fint_{Q_R} u^s \, dx \, dt\Big)^{1/s} + C \, ||f||_{L^\infty(Q_R)} .$$

for $s > 0$.

Proof. First, we may again without loss of generality restrict ourselves to the homogenous case.

If $s \ge 2$, then the result follows directly from the previous lemma and Hölder's inequality. Let then $0 < s < 2$. Using the result of previous lemma with σ, ρ instead of $R/2$, R, and $\rho_i = \rho + 2^{-i}(\sigma - \rho)$ as well as inspecting carefully the proof, we get

$$\begin{aligned}
\Big(\frac{1}{\sigma^n} \int_{Q_{\rho_{k+1}}} & u^{1/(2q^{k+1})} \, dx \, dt\Big)^{1/(2q^{k+1})} \\
&\le \Big(\frac{C}{\sigma^n(\sigma - \rho)^2} \int_{Q_{\rho_k}} u^{2q^k} \, dx \, dt\Big)^{1/(2q^k)} \\
&\le \left(\frac{\sigma}{\sigma - \rho}\right)^{1/q^k} \Big(\frac{C}{\sigma^{n+2}} \int_{Q_{\rho_k}} u^{2q^k} \, dx \, dt\Big)^{1/(2q^k)} .
\end{aligned}$$

Iterating this, observing that $\sum_{i=0}^\infty 1/q^i = 1/(1 - 1/q) = 1/((q-1)/q) = q/(q-1) = (1 + 2/n)/(2/n) = (n+2)/2$, and then using Young's inequality to the resulting inequality we have

$$\begin{aligned}
\operatorname*{ess\,sup}_{Q_\rho} u &\le C^{\gamma'} 2^{\gamma^*} \left(\frac{\sigma}{\sigma - \rho}\right)^{\sum_{i=0}^\infty 1/q^i} \Big(\frac{1}{\sigma^{n+2}} \int_{Q_\sigma} u^2 \, dx \, dt\Big)^{1/2} \\
&\le C \left(\frac{\sigma}{\sigma - \rho}\right)^{(n+2)/2} \Big(\frac{1}{\sigma^{n+2}} \int_{Q_\sigma} u^2 \, dx \, dt\Big)^{1/2} \\
&\le C\Big(\frac{1}{(\sigma - \rho)^{n+2}} \int_{Q_\sigma} (\operatorname*{ess\,sup}_{Q_{\rho_k}} u)^{2-s} u^s \, dx \, dt\Big)^{1/2} \\
&\overset{\text{Young}}{\le} \frac{1}{2} \operatorname*{ess\,sup}_{Q_\sigma} u + C\Big(\frac{1}{(\sigma - \rho)^{n+2}} \int_{Q_\sigma} u^s \, dx \, dt\Big)^{1/s},
\end{aligned}$$

since $(2-s)/2+s/2=1$. Then use iteration Lemma 4.18 with $\rho'=R/2$ and $\sigma'=R$, we get

$$\operatorname*{ess\,sup}_{Q_{R/2}} u \le \frac{C}{(R-R/2)^{(n+2)/s}}\Big(\int_{Q_R} u^s\,dx\,dt\Big)^{1/s}. \qquad \square$$

Next we consider the second part.

Lemma 4.20. *Let $u\ge\delta>0$ be a weak supersolution to $\frac{\partial u}{\partial t}-\Delta u\ge 0$. Then $w=u^{-1}$ is a weak subsolution.*

Proof. First observe that $u^{-1}\le\delta^{-1}$ and $|Du^{-1}|=|u^{-2}Du|\le\delta^{-2}|Du|$ so that u^{-1} is in the right parabolic Sobolev space. We choose (formally) a test function $\varphi=\eta u^{-2}$ with $\eta\in C_0^\infty(\Omega_T)$, $\eta\ge 0$, and calculate

$$\begin{aligned}
0 &\le \int_{\Omega_T} -u\frac{\partial\varphi}{\partial t}+Du\cdot D\varphi\,dx\,dt\\
&= \int_{\Omega_T} -u\frac{\partial(\eta u^{-2})}{\partial t}+Du\cdot D(\eta u^{-2})\,dx\,dt\\
&= \int_{\Omega_T} -u(\frac{\partial\eta}{\partial t}u^{-2}-2\eta u^{-3}\frac{\partial u}{\partial t})+Du\cdot(u^{-2}D\eta-2\eta u^{-3}Du)\,dx\,dt\\
&= \int_{\Omega_T} -\frac{\partial\eta}{\partial t}u^{-1}-2\eta\frac{\partial u^{-1}}{\partial t}-Du^{-1}\cdot D\eta-2\eta u^{-3}|Du|^2\,dx\,dt\\
&\le \int_{\Omega_T} \frac{\partial\eta}{\partial t}u^{-1}-Du^{-1}\cdot D\eta\,dx\,dt
\end{aligned}$$

where at the last step we integrated by parts and dropped the negative term. Thus

$$0\ge\int_{\Omega_T} -\frac{\partial\eta}{\partial t}u^{-1}+Du^{-1}\cdot D\eta\,dx\,dt. \qquad \square$$

Lemma 4.21. *Let $u\ge\delta>0$ be a weak supersolution to $\frac{\partial u}{\partial t}-\Delta u\ge 0$. Then there is $C=C(n,s)$ such that*

$$\Big(\fint_{Q_R} u^{-s}\,dx\,dt\Big)^{-1/s}\le C\operatorname*{ess\,inf}_{Q_{R/2}} u.$$

for any $s>0$.

Proof. By the previous lemma, u^{-1} is a subsolution. Then by Corollary 4.19, we have

$$\operatorname*{ess\,sup}_{Q_{R/2}} u^{-1}\le C\Big(\fint_{Q_R}(u)^{-s}\,dx\,dt\Big)^{1/s}$$

so that

$$\Big(\fint_{Q_R} u^{-s} \, dx \, dt \Big)^{-1/s} \le \operatorname*{ess\,inf}_{Q_{R/2}} u.$$

□

We denote

$$\begin{aligned}
\tilde{Q} &= B(0,R) \times (-3R^2, 3R^2), \\
\tilde{Q}^+ &= B(0,R) \times (R^2, 3R^2), \\
\tilde{Q}^- &= B(0,R) \times (-3R^2, -R^2), \\
Q^+ &= B(0,R/2) \times (2R^2 - (R/2)^2, 2R^2 + (R/2)^2), \\
Q^- &= B(0,R/2) \times (-2R^2 - (R/2)^2, -2R^2 + (R/2)^2).
\end{aligned} \tag{4.50}$$

The proof of the following deep theorem can be found in Fabes and Garofalo: Parabolic B.M.O and Harnack's inequality.

Theorem 4.22. *Let $u \ge \delta > 0$ be weak supersolution to $\frac{\partial u}{\partial t} - \Delta u \ge 0$ in Q_{2R}. Then there is $s > 0$ and $C = C(n)$ such that*

$$\Big(\fint_{\tilde{Q}^-} u^s \, dx \, dt \Big)^{1/s} \le C \Big(\fint_{\tilde{Q}^+} u^{-s} \, dx \, dt \Big)^{-1/s}.$$

Combining the previous two results i.e. Lemma 4.21 and Theorem 4.22, we immediately obtain weak Harnack's inequality. One could show (not done here) that this holds for $0 < s < (n+2)/2$ with $C = C(n,s)$ and in particular with $s = 1$.

Theorem 4.23. *Let $u \ge \delta > 0$ be a weak supersolution to $\frac{\partial u}{\partial t} - \Delta u \ge 0$ in Q_{2R}. Then there is $s > 0$ and $C = C(n)$ such that*

$$\Big(\fint_{\tilde{Q}^-} u^s \, dx \, dt \Big)^{1/s} \le C \operatorname*{ess\,inf}_{Q^+} u.$$

Corollary 4.24. *Let $u \ge \delta > 0$ be a weak supersolution to $\frac{\partial u}{\partial t} - \Delta u \ge f$ in Q_{2R}. Then there is $s > 0$ and $C = C(n)$ such that*

$$\Big(\fint_{\tilde{Q}^-} u^s \, dx \, dt \Big)^{1/s} \le C \operatorname*{ess\,inf}_{Q^+} u + C \, ||f||_{L^\infty(\tilde{Q})} .$$

Proof. Observe that $u + (t - t_{\min}) \, ||f||_{L^\infty(\tilde{Q})} \ge \delta$, where $t_{\min}$ is a constant such that $t - t_{\min} \ge 0$, is a weak supersolution to $\frac{\partial u}{\partial t} - \Delta u \ge 0$ and thus by the previous theorem

$$\begin{aligned}
&\Big(\fint_{\tilde{Q}^-} (u + (t - t_{\min}) \, ||f||_{L^\infty(\tilde{Q})})^s \, dx \, dt \Big)^{1/s} \\
&\qquad \le C \operatorname*{ess\,inf}_{Q^+} (u + (t - t_{\min}) \, ||f||_{L^\infty(\tilde{Q})}).
\end{aligned}$$

This implies the result. □

Then by weak Harnack's inequality (Corollary 4.24) and local boundedness estimate (Corollary 4.19), we get for a weak solution of $\frac{\partial u}{\partial t} - \Delta u = f$ that

$$\begin{aligned}\operatorname*{ess\,sup}_{Q^-} u &\le C\Big(\fint_{\tilde{Q}^-} u^s \,dx\,dt\Big)^{1/s} + C\,||f||_{L^\infty(\tilde{Q})} \\ &\le C \operatorname*{ess\,inf}_{Q^+} u + C^2\,||f||_{L^\infty(\tilde{Q})}\,.\end{aligned}$$

This finally gives us parabolic Harnack's inequality.

Theorem 4.25 (Harnack)**.** *Let $u \ge \delta > 0$ be a weak solution to $\frac{\partial u}{\partial t} - \Delta u = f$ in $\tilde{Q}$. Then there is $C = C(n)$ such that*

$$\operatorname*{ess\,sup}_{Q^-} u \le C \operatorname*{ess\,inf}_{Q^+} u + C\,||f||_{L^\infty(\tilde{Q})}\,.$$

Remark 4.26. *The assumption $u \ge \delta > 0$ is only technical: if $u \ge 0$, we may consider $u + \delta$ and since the constant in Harnack's inequality is independent of δ, we may let $\delta \to 0$.*

Example 4.27. *"Elliptic" Harnack's ie., where we have same cylinder on both sides, does not hold in the parabolic case: the equation $\frac{\partial u}{\partial t} - u_{xx} = 0$ has a nonnegative solution in $(-R, R) \times (-R^2, R^2)$ (translated fundamental solution)*

$$u(x,t) = \frac{1}{\sqrt{t + 2R^2}} e^{-\frac{(x+\xi)^2}{4(t+2R^2)}}$$

where ξ is a constant. Let $x \in (-R/2, R/2)$, $x \neq 0$ and $t \in (-R^2, R^2)$. Then

$$\frac{u(0,t)}{u(x,t)} = e^{-\frac{\xi^2-(x+\xi)^2}{4(t+2R^2)}} = e^{-\frac{-x^2-2x\xi}{4(t+2R^2)}} = e^{\frac{x^2+2x\xi}{4(t+2R^2)}} \to 0$$

as $\xi \operatorname{sign} x \to -\infty$.

4.6. **Hölder continuity.** By iterating (weak) Harnack's inequality we may prove the local Hölder continuity of weak solutions.

Theorem 4.28. *Let u be a positive weak solution to $\frac{\partial u}{\partial t} - \Delta u = 0$. Then there exists $\gamma \in (0,1)$ and a representative such that*

$$|u(x,t) - u(y,s)| \le C(|x-y| + |t-s|^{1/2})^\gamma$$

locally.

Proof. We take the weak Harnack for $s = 1$ for [1]granted and using that for a weak (super)solutions $u - \operatorname{ess\,inf}_{\tilde{Q}} u$ and $\operatorname{ess\,sup}_{\tilde{Q}} u - u$, we have

$$\begin{aligned} &\fint_{\tilde{Q}} u\,dx - \operatorname*{ess\,inf}_{\tilde{Q}} u \leq C\Big(\operatorname*{ess\,inf}_{Q^+} u - \operatorname*{ess\,inf}_{\tilde{Q}} u\Big) \\ &\operatorname*{ess\,sup}_{\tilde{Q}} u - \fint_{\tilde{Q}} u\,dx \leq C\Big(\operatorname*{ess\,sup}_{\tilde{Q}} u - \operatorname*{ess\,sup}_{Q^+} u\Big). \end{aligned}$$

Summing up yields

$$\operatorname{osc}_{\tilde{Q}} u \leq C\Big(\operatorname{osc}_{\tilde{Q}} u - \operatorname{osc}_{Q^+} u\Big),$$

where we denoted

$$\operatorname{osc}_{\tilde{Q}} u := \operatorname*{ess\,sup}_{\tilde{Q}} u - \operatorname*{ess\,inf}_{\tilde{Q}} u.$$

Rearranging the terms, we have

$$\operatorname{osc}_{Q^+} u \leq \Big(1 - \frac{1}{C}\Big) \operatorname{osc}_{\tilde{Q}} u.$$

Thus by setting $\theta := 1 - 1/C \in (0, 1)$ we obtain

$$\operatorname{osc}_{Q^+} \leq \theta \operatorname{osc}_{\tilde{Q}} u. \tag{4.51}$$

The proof of (weak) Harnack would also work in the geometry

$$\begin{aligned} \tilde{Q} &:= B(0, R) \times (-R^2, R^2) \\ Q^+ &:= B(0, R/2) \times (R^2/2 - (R/2)^2, R^2/2 + (R/2)^2). \end{aligned}$$

Using this and denoting, with a slight abuse of notation,

$$Q_k := B(0, R/2^k) \times \Big(t_k - (R/2^k)^2, t_k + (R/2^k)^2\Big)$$

for a suitable t_k we obtain $\operatorname{osc}_{Q_1} u \leq \theta \operatorname{osc}_{Q_0} u$. Repeating the argument, we deduce

$$\operatorname{osc}_{Q_k} u \leq \theta^k \operatorname{osc}_{Q_0} u.$$

Then fix $\rho < R$ and k such that $2^k < R/\rho \leq 2^{k+1}$, $k = 0, 1, 2, \ldots$ so that $2^{-k}R > \rho \geq 2^{-(k+1)}R$ and

$$k \leq \log(R/\rho)/\log(2) \leq k + 1$$

i.e.

$$\log(R/(2\rho))/\log(2) \leq k$$

[1]Or use (strong) Harnack to have $\fint_{Q^-} u\,dx \leq C \operatorname{ess\,inf}_{Q^+} u$. Then applying this to $u - \operatorname{ess\,inf}_{\tilde{Q}} u$ and $\operatorname{ess\,sup}_{\tilde{Q}} u - u$ gives the same oscillation estimate.

Thus

$$\begin{aligned}
\operatorname{osc}_{Q_\rho} u &\le \theta^{\log(R/(2\rho))/\log(2)} \operatorname{osc}_{Q_0} u \\
&= \theta^{\log(R/(2\rho))\log(\theta)/(\log(\theta)\log(2))} \operatorname{osc}_{Q_0} u \\
&= \left(\frac{\rho}{2R}\right)^{\log(\theta)/\log(2)} \operatorname{osc}_{Q_0} u \\
&= C\left(\frac{\rho}{R}\right)^{\underbrace{-\log(\theta)/\log(2)}_{=:\gamma}} \operatorname{osc}_{Q_0} u.
\end{aligned}$$

Since ρ can be chosen arbitrarily small and u is locally bounded, this implies Hölder-continuity. □

4.7. **Remarks.** Also a similar regularity theory that we established for the elliptic equations can be developed for $\frac{\partial u}{\partial t} + Lu = f$ if the coefficients are smooth enough.

Intuition: We consider formally the heat equation

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = f & \text{in } \mathbb{R}^n \times (0,T] \\ u = g & \text{on } \mathbb{R}^n \times \{0\} \end{cases}$$

and u decays fast enough at infinity. Then integration by parts gives

$$\begin{aligned}
\int_{\mathbb{R}^n} f^2\,dx &= \int_{\mathbb{R}^n} (\frac{\partial u}{\partial t} - \Delta u)^2\,dx \\
&= \int_{\mathbb{R}^n} \frac{\partial u}{\partial t}^2 - 2\frac{\partial u}{\partial t}\Delta u + \Delta u^2\,dx \\
&= \int_{\mathbb{R}^n} \frac{\partial u}{\partial t}^2 + 2\frac{\partial Du}{\partial t}\cdot Du + \Delta u^2\,dx
\end{aligned}$$

Then we calculate

$$\begin{aligned}
\int_0^t \int_{\mathbb{R}^n} \frac{\partial Du}{\partial t}\cdot Du\,dx\,ds &= \int_0^t \int_{\mathbb{R}^n} \frac{\partial |Du|^2}{\partial t}\,dx\,ds \\
&\overset{\text{init cond}}{=} \int_{\mathbb{R}^n} |Du(x,t)|^2\,dx - \int_{\mathbb{R}^n} |Dg|^2\,dx.
\end{aligned}$$

Moreover, similarly as with the elliptic equations

$$\begin{aligned}\int_{\mathbb{R}^n} (\Delta u)^2\,dx &= \int_{\mathbb{R}^n} \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} \sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2}\,dx \\ &= \sum_{i,j=1}^n \int_{\mathbb{R}^n} \frac{\partial^2 u}{\partial x_i^2}\frac{\partial^2 u}{\partial x_j^2}\,dx \\ &\overset{\text{int by parts}}{=} -\sum_{i,j=1}^n \int_{\mathbb{R}^n} \frac{\partial^3 u}{\partial x_i^2 \partial x_j}\frac{\partial u}{\partial x_j}\,dx \\ &\overset{\text{int by parts}}{=} \sum_{i,j=1}^n \int_{\mathbb{R}^n} \frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j}\,dx \\ &= \int_{\mathbb{R}^n} \left|D^2 u\right|^2 dx.\end{aligned}$$

Choosing t so that $\int_{\mathbb{R}^n} |Du|^2(x,t)\,dx \geq \frac{1}{2}\sup_{t\in(0,T)} \int_{\mathbb{R}^n} |Du(x,t)|^2\,dx$ and combining the estimates, we end up with

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^n} \left|\frac{\partial u}{\partial t}\right|^2 + \left|D^2 u\right|^2 dx\,dt + \sup_{t\in(0,T)} \int_{\mathbb{R}^n} |Du(x,t)|^2\,dx \\ \leq C\int_0^T \int_{\mathbb{R}^n} |f|^2\,dx\,dt + C\int_{\mathbb{R}^n} |Dg|^2\,dx.\end{aligned}$$

Continuing in this way (cf. elliptic), we would obtain higher regularity estimates as well. The solution has two more space derivatives than f etc. To make above conclusions rigorous, we could again utilize difference quotients both in space and time.

5. Schauder estimates

We finish the course by briefly returning to the elliptic theory, and sketching the Schauder theory because this is needed to finish the story with Hilbert's 19th problem.

Recall Hölder continuity

Definition 5.1. *Let $u : \Omega \to \mathbb{R}$. For $\alpha \in (0,1)$, we denote the semi-norm*

$$|u|_{C^\alpha(\Omega)} = \sup_{x,y\in\Omega, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha},$$

and a set of all functions satisfying $|u|_{C^\alpha(\Omega)} < \infty$ by $C^\alpha(\overline{\Omega})$. This space can be equipped with the norm

$$||f||_{C^\alpha(\Omega)} = ||f||_{L^\infty(\Omega)} + \sup_{x,y\in\Omega, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

Similarly

$$C^{k,\alpha}(\overline{\Omega}) = \{u \,:\, D^\beta u \in C^\alpha(\overline{\Omega}) \text{ for } |\beta| \leq k\},$$

where β is a multi-index.

The main result of this section is

Theorem 5.2. *Let u be a weak solution to $-\Delta u = f$ in $B(0,2R)$ with $f \in C^\alpha(\overline{B}(0,2R))$. Then $u \in C^{2,\alpha}(\overline{B}(0,R/4))$ with an explicit estimate.*

Remark 5.3.
- *Theorem 5.2 actually comes with estimate, see Theorem 5.13.*
- *The result can be extended to $Lu = f$ with*

  $$||a_{ij}||_{C^\alpha(B(0,2R))}, ||b_i||_{C^\alpha(B(0,2R)}, ||c||_{C^\alpha(B(0,2R)} \leq M,$$

  uniform ellipticity, and $a_{ij} = a_{ji}$ by the freezing of coefficients technique.
- *In regular domains, there is also a corresponding global result.*

First, we look at the important step i.e. how to pass from integral estimates (natural from the point of view what we have done so far) to the pointwise Hölder norms. For this, we use a theory of Campanato spaces. Denote

$$u_{x,\rho} = \frac{1}{|\Omega \cap B(x,\rho)|} \int_{\Omega\cap B(x,\rho)} u(y)\, dy.$$

where Ω is a regular domain, for example $\Omega = B(0,R)$.

Definition 5.4 (Campanato space)**.** *Let $\mu \geq 0$ and $u \in L^2(\Omega)$. Then the functions satisfying*

$$|u|_{\mathcal{L}^{2,\mu}(\Omega)} = \sup_{x\in\Omega, 0<\rho<\operatorname{diam}(\Omega)} \left(\frac{1}{\rho^\mu}\int_{\Omega\cap B(x,\rho)} |u(y)-u_{x,\rho}|^2\, dy\right)^{1/2} < \infty$$

belong to the Campanato space $\mathcal{L}^{2,\mu}(\Omega)$. *We use the norm*

$$||u||_{\mathcal{L}^{2,\mu}(\Omega)} = |u|_{\mathcal{L}^{2,\mu}(\Omega)} + ||u||_{L^2(\Omega)}.$$

Lemma 5.5 (Mean value lemma)**.** *Let $u \in \mathcal{L}^{2,\mu}(\Omega)$, $x \in \overline{\Omega}$ and $0 < \rho < R < \operatorname{diam}(\Omega)$. Then*

$$|u_{x,R} - u_{x,\rho}| \leq C\rho^{-n/2} R^{\mu/2} |u|_{\mathcal{L}^{2,\mu}(\Omega)}.$$

Proof. Let $y \in B(x,\rho) \cap \Omega$ and write

$$|u_{x,R} - u_{x,\rho}|^2 \le C(|u_{x,R} - u(y)|^2 + |u(y) - u_{x,\rho}|^2,$$

integrate over $B(x,\rho) \cap \Omega \subset B(x,R) \cap \Omega$ to have

$$\begin{aligned}
|u_{x,R} - u_{x,\rho}|^2 &= \fint_{B(x,\rho)\cap\Omega} |u_{x,R} - u_{x,\rho}|^2 \, dy \\
&\le C \fint_{B(x,\rho)\cap\Omega} |u_{x,R} - u(y)|^2 + |u(y) - u_{x,\rho}|^2 \, dy \\
&\le C \fint_{B(x,\rho)\cap\Omega} |u_{x,R} - u(y)|^2 + |u(y) - u_{x,\rho}|^2 \, dy \\
&\le C\Big(\frac{R^\mu}{\rho^n} + \frac{\rho^\mu}{\rho^n}\Big)|u|^2_{\mathcal{L}^{2,\mu}(\Omega)} \\
&\le 2C\frac{R^\mu}{\rho^n}|u|^2_{\mathcal{L}^{2,\mu}(\Omega)}.
\end{aligned}$$

□

In the proof of the key result, we need integral characterization of Hölder continuous functions i.e. Campanato estimate.

Lemma 5.6 (Integral characterization of Hölder continuous functions)**.** *Let $n < \mu \le n+2$. Then $\mathcal{L}^{2,\mu}(\Omega) = C^\alpha(\overline{\Omega})$ and*

$$C^{-1}|u|_{C^\alpha(\Omega)} \le |u|_{\mathcal{L}^{2,\mu}(\Omega)} \le C^2|u|_{C^\alpha(\Omega)}$$

with $\alpha = (\mu - n)/2$ and $C = C(n,\mu)$.

Interpretation: $C^\alpha(\overline{\Omega}) \subset \mathcal{L}^{2,\mu}(\Omega)$ and each $u \in \mathcal{L}^{2,\mu}(\Omega)$ has a presentative $\tilde{u}$ in $C^\alpha(\overline{\Omega})$.

Proof. The second inequality: Let $u \in C^\alpha(\overline{\Omega})$, $x \in \Omega$ and $0 < \rho < \operatorname{diam}(\Omega)$ and $y \in \Omega \cap B(x,\rho)$. We have

$$\begin{aligned}
|u(y) - u_{x,\rho}| &= \Big|\fint_{B(x,\rho)\cap\Omega} u(y) - u(z) \, dz\Big| \\
&\le \fint_{B(x,\rho)\cap\Omega} |u(y) - u(z)| \, dz \\
&\le |u|_{C^\alpha(\Omega)} \fint_{B(x,\rho)\cap\Omega} |y - z|^\alpha \, dz \\
&\le \frac{C|u|_{C^\alpha(\Omega)}}{\rho^n} \int_{B(x,\rho)\cap\Omega} |y - z|^\alpha \, dz = *
\end{aligned}$$

since $|\Omega \cap B(x,\rho)|C \geq \rho^n$. Moreover, since $y-x \in B(0,\rho)$ it follows that $B(y-x,\rho)\cap\Omega \subset B(0,2\rho)\cap\Omega$ and by the change of variables that

$$\begin{aligned}
* &\leq \frac{C|u|_{C^\alpha(\Omega)}}{\rho^n}\int_{B(0,2\rho)\cap\Omega} |z|^\alpha\, dz \\
&\leq \frac{C|u|_{C^\alpha(\Omega)}}{\rho^n}\int_{B(0,2\rho)} |z|^\alpha\, dz \\
&\leq \frac{C|u|_{C^\alpha(\Omega)}}{\rho^n}\int_0^{2\rho} r^{n-1+\alpha}\, dz \\
&\leq C|u|_{C^\alpha(\Omega)}\rho^\alpha.
\end{aligned} \tag{5.52}$$

Hence

$$\begin{aligned}
\frac{1}{\rho^\mu}\int_{B(x,\rho)\cap\Omega} |u(y)-u_{x,\rho}|^2\, dy &\leq C|u|^2_{C^\alpha(\Omega)}\rho^{2\alpha-\mu}|B(x,\rho)\cap\Omega| \\
&\leq C|u|^2_{C^\alpha(\Omega)}\rho^{\underbrace{2\alpha-\mu+n}_{0}}
\end{aligned}$$

and thus the second inequality follows.

The proof of the first inequality is in three steps: construction of the representative $\tilde{u}$, showing that $\tilde{u}=u$ a.e., and showing that $\tilde{u}\in C^\alpha(\Omega)$ **Step1(construction of the representative $\tilde{u}$):** Let $x\in\overline{\Omega}$, $0<R<$ diam(Ω) and $R_i = 2^{-i}R,\ i=0,1,\ldots$. Then by Lemma 5.5

$$\begin{aligned}
\left|u_{x,R_j}-u_{x,R_{j+1}}\right| &\leq CR_{j+1}^{-n/2}R_j^{\mu/2}|u|_{\mathcal{L}^{2,\mu}(\Omega)} \\
&= C2^{j(n-\mu)/2+n/2}R^{(\mu-n)/2}|u|_{\mathcal{L}^{2,\mu}(\Omega)}.
\end{aligned}$$

Thus for $0\leq j<i$

$$\begin{aligned}
&\left|u_{x,R_j}-u_{x,R_{j+1}}+u_{x,R_{j+1}}-\ldots+u_{x,R_{i-1}}-u_{x,R_i}\right| \\
&\leq CR^{(\mu-n)/2}|u|_{\mathcal{L}^{2,\mu}(\Omega)}\sum_{k=j}^{i-1}2^{k(n-\mu)/2+n/2} \\
&= CR^{(\mu-n)/2}|u|_{\mathcal{L}^{2,\mu}(\Omega)}2^{j(n-\mu)/2+n/2}\sum_{k=0}^{i-j-1}2^{k(n-\mu)/2+n/2} \\
&= CR^{(\mu-n)/2}|u|_{\mathcal{L}^{2,\mu}(\Omega)}2^{j(n-\mu)/2+n/2}\frac{1-2^{(i-j)(n-\mu)/2+n/2}}{1-2^{(n-\mu)/2+n/2}} \\
&= CR_j^{(\mu-n)/2}|u|_{\mathcal{L}^{2,\mu}(\Omega)},
\end{aligned}$$

where $C=C(n,\mu)$. We have derived the estimate

$$\left|u_{x,R_j}-u_{x,R_i}\right| \leq CR_j^{(\mu-n)/2}|u|_{\mathcal{L}^{2,\mu}(\Omega)}, \tag{5.53}$$

It follows that $u_{x,R_i},\ i = 0,1,2,\dots$ is a Cauchy sequence. Hence we may define

$$\tilde{u}(x) = \lim_{i\to\infty} u_{x,R_i}, \quad x \in \overline{\Omega}.$$

It also holds that the limit does not depend on the particular choice of R. To see this, take $0 < r < R$ and let $r_i = 2^{-i}r,\ i = 0,1,\dots$. Then again by Lemma 5.5

$$\begin{aligned}
|u_{x,R_i} - u_{x,r_i}| &\le C r_i^{-n/2} R_i^{\mu/2} |u|_{\mathcal{L}^{2,\mu}(\Omega)} \\
&\le C\left(\frac{R_i}{r_i}\right)^{n/2} R_i^{(\mu-n)/2} |u|_{\mathcal{L}^{2,\mu}(\Omega)} \\
&\le C\left(\frac{R}{r}\right)^{n/2} R_i^{(\mu-n)/2} |u|_{\mathcal{L}^{2,\mu}(\Omega)} \to 0
\end{aligned}$$

as $i \to \infty$, since $\mu > n$. Thus $\tilde{u}_R(x) = \tilde{u}_r(x)$. Moreover, by (5.53) setting $j = 0$ and letting $i \to \infty$

$$|u_{x,R} - \tilde{u}(x)| \le C R^{(\mu-n)/2} |u|_{\mathcal{L}^{2,\mu}(\Omega)} \tag{5.54}$$

so that $\tilde{u}(x) = \lim_{R\to 0} u_{x,R}$.
Step2 ($\tilde{u} = u$ a.e): By Lebesgue's theorem

$$\tilde{u}(x) = \lim_{R\to 0} u_{x,R} \overset{\text{Leb.}}{=} u(x) \quad \text{a.e. in } \Omega.$$

Step3 ($\tilde{u} \in C^{\alpha}(\Omega)$):
Let $x, y \in \overline{\Omega},\ x \ne y$ and set $R := |x-y|$. By (5.54)

$$\begin{aligned}
|\tilde{u}(x) - \tilde{u}(y)| &\le |\tilde{u}(x) - u_{x,R}| + |u_{x,R} - u_{y,R}| + |u_{y,R} - \tilde{u}(y)| \\
&\le C R^{(\mu-n)/2} |u|_{\mathcal{L}^{2,\mu}(\Omega)} + |u_{x,R} - u_{y,R}|.
\end{aligned}$$

Set $G = \Omega \cap B(x,2R) \cap B(y,2R)$. Observe that $G \subset \Omega \cap B(x,2R)$ and $G \subset \Omega \cap B(y,2R))$, and $C|G| \ge R^n$ because Ω is smooth. Estimate the second term on the RHS as

$$\begin{aligned}
&|u_{x,R} - u_{y,R}| \\
&= \fint_G |u_{x,R} - u_{y,R}|\, dz \\
&\le \frac{|\Omega \cap B(x,2R)|^{1-1/2}}{|G|} \Big(\int_{\Omega\cap B(x,2R)} |u_{x,R} - u(z)|^2 \, dz \Big)^{1/2} \\
&\quad + \frac{|\Omega \cap B(y,2R)|^{1-1/2}}{|G|} \Big(\int_{\Omega\cap B(y,2R)} |u(z) - u_{y,R}|^2 \, dz \Big)^{1/2} \\
&\le C R^{(\mu-n)/2} |u|_{\mathcal{L}^{2,\mu}(\Omega)}.
\end{aligned}$$

Combining the estimates, we have

$$|\tilde{u}(x) - \tilde{u}(y)| \leq CR^{(\mu-n)/2}|u|_{\mathcal{L}^{2,\mu}(\Omega)}$$

so that $\tilde{u} \in C^\alpha(\Omega)$ with $\alpha = (\mu - n)/2$, and

$$|u|_{C^\alpha(\Omega)} \leq C|u|_{\mathcal{L}^{2,\mu}(\Omega)}. \qquad \square$$

Calculation (5.52) gives us a useful estimate, that we state separately as a lemma

Lemma 5.7. *Let $u \in C^\alpha(\overline{\Omega})$, $x \in \overline{\Omega}$. Then*

$$\int_{B(x,\rho)\cap\Omega} |u(y) - u_{x,\rho}|^2 \, dy \leq C|u|^2_{C^\alpha(\Omega)}\rho^{n+2\alpha}.$$

Without loss of generality, as long as we have the uniqueness, we may derive the Schauder estimates assuming smoothness, by using smooth approximations, and passing to the limit at the end.

Lemma 5.8. *Let u be a weak solution $\Delta u = 0$ in $B(0, 2R)$. Then for any $0 < \rho \leq R$, it holds*

$$\int_{B(0,\rho)} u^2 \, dx \leq C \left(\frac{\rho}{R}\right)^n \int_{B(0,R)} u^2 \, dx$$

$$\int_{B(0,\rho)} (u - u_\rho)^2 \, dx \leq C \left(\frac{\rho}{R}\right)^{n+2} \int_{B(0,R)} (u - u_R)^2 \, dx,$$

with $C = C(n)$.

Proof. The first estimate: By the elliptic counterpart of the (ess)sup-estimate (cf. Lemma 4.17, and ex 13 in set 3), we have for $0 < \rho < R/2$

$$\int_{B(0,\rho)} u^2 \, dx \leq C\rho^n \sup_{B(0,\rho)} u^2 \leq C \left(\frac{\rho}{R}\right)^n \int_{B(0,R)} u^2 \, dx.$$

For $R/2 \leq \rho \leq R$ the result immediately follows

$$\int_{B(0,\rho)} u^2 \, dx \leq C\Big(\underbrace{\frac{\rho}{R}}_{\geq C}\Big)^n \int_{B(0,R)} u^2 \, dx.$$

The second estimate: The second follows from the first one by observing that $w = D_i u$ is also a solution to the Laplace equation, and thus by the first estimate

$$\int_{B(0,\rho)} (D_i u)^2 \, dx \leq C \left(\frac{\rho}{R}\right)^n \int_{B(0,R)} (D_i u)^2 \, dx \qquad (5.55)$$

Summing over i, assuming $0 < \rho < R/2$, and using Poincaré's inequality

$$\begin{aligned}\int_{B(0,\rho)} (u-u_\rho)^2\,dx &\overset{\text{Poinc.}}{\leq} C\rho^2 \int_{B(0,\rho)} |Du|^2\,dx \\ &\leq C\rho^2 \left(\frac{\rho}{R}\right)^n \int_{B(0,R/2)} |Du|^2\,dx\end{aligned}$$

By Caccioppoli's inequality (ex)

$$\int_{B(0,R/2)} |Du|^2\,dx \leq \frac{C}{R^2} \int_{B(0,R)} (u-u_R)^2\,dx.$$

Combining the previous two estimates, we have

$$\int_{B(0,\rho)} (u-u_\rho)^2\,dx \leq C\left(\frac{\rho}{R}\right)^{n+2} \int_{B(0,R)} (u-u_R)^2\,dx.$$

The case $R/2 \leq \rho \leq R$ is again easier:

$$\begin{aligned}&\int_{B(0,\rho)} (u-u_\rho)^2\,dx \\ &\leq \int_{B(0,\rho)} (u-u_R+u_R-u_\rho)^2\,dx \\ &\leq C\int_{B(0,\rho)} (u-u_R)^2\,dx + C\int_{B(0,\rho)} \fint_{B(0,\rho)} (u_R-u)^2\,dx\,dx \\ &\leq C\int_{B(0,R)} (u-u_R)^2\,dx \\ &\leq C\Big(\underbrace{\frac{\rho}{R}}_{\geq C}\Big)^{n+2} \int_{B(0,R)} (u-u_R)^2\,dx.\end{aligned} \tag{5.56}$$

□

Lemma 5.9. *Let u be a solution to $\Delta u = f$ in $B(0,2R)$, and let $w = D_i u$, $f \in C^\alpha(\overline{B}(0,2R))$. Then for $0 < \rho \leq R$*

$$\begin{aligned}&\frac{1}{\rho^{n+2\alpha}} \int_{B(0,\rho)} |Dw-(Dw)_\rho|^2\,dx \\ &\qquad\leq \frac{C}{R^{n+2\alpha}} \int_{B(0,R)} |Dw-(Dw)_R|^2\,dx + C|f|^2_{C^\alpha(B(0,R))}.\end{aligned}$$

Proof. Decompose $w = w_1 + w_2$, where

$$\begin{cases} -\Delta w_1 = 0 & \text{in } B(0,R) \\ w_1 = w & \text{on } \partial B(0,R) \end{cases}$$

and

$$\begin{cases} -\Delta w_2 = D_i f = D_i(f - f_R) & \text{in } B(0,R) \\ w_2 = 0 & \text{on } \partial B(0,R) \end{cases}$$

in the weak sense. Then use Lemma 5.8 for $D_i w_1$ (this is also a solution to Laplace eq) to have

$$\int_{B(0,\rho)} (D_i w_1 - (D_i w_1)_\rho)^2 \, dx \le C \left(\frac{\rho}{R}\right)^{n+2} \int_{B(0,R)} (D_i w_1 - (D_i w_1)_R)^2 \, dx$$

Summing over i

$$\int_{B(0,\rho)} (D w_1 - (D w_1)_\rho)^2 \, dx \le C \left(\frac{\rho}{R}\right)^{n+2} \int_{B(0,R)} (D w_1 - (D w_1)_R)^2 \, dx$$

and further (change of radius as before in (5.56))

$$\begin{aligned}
&\int_{B(0,\rho)} (Dw - (Dw)_\rho)^2 \, dx \\
&\le C \int_{B(0,\rho)} (Dw_1 - (Dw_1)_\rho)^2 \, dx + C \int_{B(0,\rho)} (Dw_2 - (Dw_2)_\rho)^2 \, dx \\
&\le C \left(\frac{\rho}{R}\right)^{n+2} \int_{B(0,R)} (Dw_1 - (Dw_1)_R)^2 \, dx + C \int_{B(0,R)} (Dw_2 - (Dw_2)_R)^2 \, dx \\
&\le C \left(\frac{\rho}{R}\right)^{n+2} \int_{B(0,R)} (Dw - (Dw)_R)^2 \, dx + C \int_{B(0,R)} (Dw_2 - (Dw_2)_R)^2 \, dx \\
&\le C \left(\frac{\rho}{R}\right)^{n+2} \int_{B(0,R)} (Dw - (Dw)_R)^2 \, dx + C \int_{B(0,R)} |Dw_2|^2 \, dx,
\end{aligned}$$

where we wrote $Dw_1 = D(w_1 + w_2) - Dw_2$ etc.

By using $\varphi = w_2$ as a test function in $\int Dw_2 \cdot D\varphi \, dx = -\int f D_i \varphi \, dx$ we get (recall zero bdr values)

$$\begin{aligned}
\int_{B(0,R)} |Dw_2|^2 \, dx &= -\int_{B(0,R)} (f - f_R) D_i w_2 \, dx \\
&\le \frac{1}{2} \int_{B(0,R)} (f - f_R)^2 \, dx + \frac{1}{2} \int_{B(0,R)} |Dw_2|^2 \, dx.
\end{aligned}$$

Thus

$$\begin{aligned}
\int_{B(0,R)} |Dw_2|^2 \, dx &\le C \int_{B(0,R)} (f - f_R)^2 \, dx \\
&\le C R^{n+2\alpha} |f|^2_{C^\alpha(B(0,R))}
\end{aligned}$$

where we also used estimate similar to Lemma 5.7.

Combining the estimates

$$\int_{B(0,\rho)} (Dw - (Dw)_\rho)^2 \, dx \\ \le C \left(\frac{\rho}{R}\right)^{n+2} \int_{B(0,R)} (Dw - (Dw)_R)^2 \, dx + CR^{n+2\alpha} |f|^2_{C^\alpha(B(0,R))}.$$

Then by Lemma 5.10

$$\int_{B(0,\rho)} (Dw - (Dw)_\rho)^2 \, dx \\ \le C \left(\frac{\rho}{R}\right)^{n+2\alpha} \Big(\int_{B(0,R)} (Dw - (Dw)_R)^2 \, dx + R^{n+2\alpha} |f|^2_{C^\alpha(B(0,R))} \Big). \quad \square$$

In the previous proof, we used the following iteration lemma for

$$G(r) = \int_{B(0,r)} (Dw - (Dw)_r)^2 \, dx$$

where $r \in [0, \, R]$.

Lemma 5.10 (another iteration lemma)**.** *If*

$$G(\rho) \le A \left(\frac{\rho}{R}\right)^\gamma G(R) + BR^\beta, \quad 0 < \rho < R \le R_0$$

where $0 < \beta < \gamma$*, then there is* $C = C(A, \gamma, \beta)$ *such that*

$$G(\rho) \le C \left(\frac{\rho}{R}\right)^\beta (G(R) + BR^\beta), \quad 0 < \rho < R \le R_0.$$

Proof. Ex. $\square$

We also need a Caccioppoli type estimate.

Lemma 5.11. *Let* u *be a solution to* $-\Delta u = f$ *in* $B(0, 2R)$*,* $f \in C^\alpha(\overline{B}(0, 2R))$*. Then there is* $C = C(n)$ *such that*

$$\int_{B(0,R/2)} \left|D^2 u\right|^2 dx \\ \le C\Big(\frac{1}{R^4} \int_{B(0,R)} u^2 \, dx + R^n \, ||f||^2_{L^\infty(\Omega)} + R^{n+2\alpha} |f|^2_{C^\alpha(B(0,R))}\Big).$$

Proof. Since $u \in W^{2,2}_{\text{loc}}(B(0, 2R))$ by our earlier regulatity results, we may test with $\varphi = D_i \phi$ with a smooth function ϕ. Integrating by parts

$$\int f D_i \phi \, dx = \int Du \cdot DD_i \phi \, dx \\ \overset{\text{int by parts}}{=} - \int DD_i u \cdot D\phi \, dx.$$

Thus in the weak sense $w = D_i u$

$$-\Delta w = D_i f = D_i(f - f_R).$$

Testing this with $\varphi = \eta^2 w$, where $\eta \in C_0^\infty(B(0,R)),\ 0 \le \eta \le 1,\ \eta = 1$ in $B(0,R/2)$ and $|D\eta|^2 \le C/R^2$, we have (ex)

$$\begin{aligned}\int_{B(0,R)} |Dw|^2 \eta^2 \,dx &\le C \int_{B(0,R)} w^2 |D\eta|^2 \,dx + C \int_{B(0,R)} \eta^2 |f - f_R|^2 \,dx \\ &\le \frac{C}{R^2} \int_{B(0,R)} w^2 \,dx + C R^{n+2\alpha} |f|_{C^\alpha(B(0,R))},\end{aligned}$$

where at the last step we also used Lemma 5.7. Further testing the weak formulation of $-\Delta u = f$ by $\varphi = \eta^2 u$ where $\eta \in C_0^\infty(B(0,3R/2)),\ 0 \le \eta \le 1,\ \eta = 1$ in $B(0,R)$ and $|D\eta|^2 \le C/R^2$, we have

$$\begin{aligned}\int_{B(0,R)} w^2 \,dx &\le \int_{B(0,3R/2)} |Du|^2 \eta^2 \,dx \\ &\le C \int_{B(0,3R/2)} u^2 |D\eta|^2 \,dx + C R^2 \int_{B(0,3R/2)} \eta^2 |f|^2 \,dx \\ &\le \frac{C}{R^2} \int_{B(0,3R/2)} u^2 \,dx + C R^{n+2} \left|\left|f\right|\right|^2_{L^\infty(B(0,3R/2))},\end{aligned}$$

where we estimated for example $\int_{B(0,3R/2)} \frac{R}{R} \eta^2 u f \,dx \le C R^2 \int_{B(0,3R/2)} \eta^2 |f|^2 \,dx + \frac{C}{R^2} \int_{B(0,3R/2)} \eta^2 u^2 \,dx$. Combining the previous two estimates we have

$$\begin{aligned}&\int_{B(0,R/2)} \left|D^2 u\right|^2 dx \\ &\le \frac{C}{R^4} \int_{B(0,3R/2)} u^2 \,dx + C R^n \left|\left|f\right|\right|^2_{L^\infty(B(0,3R/2))} + C R^{n+2\alpha} |f|_{C^\alpha(B(0,R))}.\end{aligned}$$

□

Lemma 5.12. *Let u be a solution to $-\Delta u = f$ in $B(0,2R)$, and let $w = D_i u$, $f \in C^\alpha(\overline{B}(0,2R))$. Then for $0 < \rho \le R/2$ there is $C = C(n)$ st*

$$\int_{B(0,\rho)} |Dw - (Dw)_\rho|^2 \,dx \le \rho^{n+2\alpha} M_R C$$

where

$$M_R = \frac{1}{R^{4+2\alpha}} \left|\left|u\right|\right|^2_{L^\infty(B(0,R))} + \frac{1}{R^{2\alpha}} \left|\left|f\right|\right|^2_{L^\infty(B(0,R))} + |f|^2_{C^\alpha(B(0,R))}.$$

Proof. By Lemma 5.9

$$\begin{aligned}
&\int_{B(0,\rho)} |Dw - (Dw)_\rho|^2 \, dx \\
&\quad \leq C\rho^{n+2\alpha}\Big(\frac{1}{R^{n+2\alpha}} \int_{B(0,R/2)} |Dw - (Dw)_R|^2 \, dx + |f|^2_{C^\alpha(B(0,R/2))}\Big) \\
&\quad \leq C\rho^{n+2\alpha}\Big(\frac{1}{R^{n+2\alpha}} \int_{B(0,R/2)} |Dw|^2 \, dx + |f|^2_{C^\alpha(B(0,R))}\Big).
\end{aligned}$$

Then by the Caccioppoli type estimate Lemma 5.11 we have

$$\begin{aligned}
&\int_{B(0,\rho)} |Dw - (Dw)_\rho|^2 \, dx \\
&\qquad \leq C\rho^{n+2\alpha}\Big(\frac{1}{R^{n+2\alpha}} \int_{B(0,R/2)} |Dw|^2 \, dx + |f|^2_{C^\alpha(B(0,R))}\Big) \\
&\qquad \leq C\rho^{n+2\alpha}\Bigg(\frac{1}{R^{n+2\alpha}}\Big(\frac{1}{R^4} \int_{B(0,R)} u^2 \, dx \\
&\qquad\quad + R^n \, ||f||^2_{L^\infty(B(0,R))}\Big) + |f|^2_{C^\alpha(B(0,R))}\Bigg).
\end{aligned}$$

□

Theorem 5.13. *Let u be a solution to $-\Delta u = f$ in $B(0,2R)$ with $f \in C^\alpha(B(0,2R))$. Then*

$$\begin{aligned}
&\left|D^2 u\right|_{C^\alpha(B(0,R/4))} \\
&\quad \leq C\Big(\frac{1}{R^{2+\alpha}} \, ||u||_{L^\infty(B(0,R))} + \frac{1}{R^\alpha} \, ||f||_{L^\infty(B(0,R))} + |f|_{C^\alpha(B(0,R))}\Big).
\end{aligned}$$

Proof. What we have in Lemma 5.12 looks very much like the Campanato seminorm. Indeed, this is exactly the idea of the proof. To be more precise, by Lemma 5.6 it suffices to bound the Campanato seminorm.

$$|u|_{\mathcal{L}^{2,\mu}(B(0,R/4))} = \sup_{x\in\Omega, 0<\rho<\operatorname{diam}(B(0,R/4))} \Big(\frac{1}{\rho^\mu} \int_{\Omega_p(x)} |u(y) - u_{x,\rho}|^2 \, dx\Big)^{1/2}.$$

To this end, let $x \in B(0,R/4)$ and $0 < \rho \leq R/2$ and observe that similarly as before in (5.56)

$$\begin{aligned}
&\int_{B(x,\rho)\cap B(0,R/4)} \left|D^2 u(y) - (D^2 u)_{B(x,\rho)\cap B(0,R/4)}\right|^2 dx \\
&\qquad\qquad \leq C \int_{B(x,\rho)} \left|D^2 u(y) - (D^2 u)_{B(x,\rho)}\right|^2 dx.
\end{aligned}$$

Then by Lemma 5.12,

$$\int_{B(x,\rho)} \left|D^2u(y) - (D^2u)_{B(x,\rho)}\right|^2 dx$$

$$\leq C\rho^{n+2\alpha}\left(\frac{1}{R^{4+2\alpha}}\,||u||^2_{L^\infty(B(0,R)))} + \frac{1}{R^{2\alpha}}\,||f||^2_{L^\infty(B(0,R))} + |f|^2_{C^\alpha(B(0,R))}\right).$$

We combine the estimates, divide on both sides by $\rho^{n+2\alpha}$, take $\sup_{x\in\Omega, 0<\rho<\operatorname{diam}(B(0,R/4))}$, and power $\frac{1}{2}$ to obtain the result. □

The previous theorem immediately implies Theorem 5.2.

By differentiating the Euler-Lagrange equation related to a minimizer, using the Hölder-continuity result, then Schauder estimates and iterating using so called bootstrapping argument, Hilbert's 19th problem was settled.

6. Notes and comments

I would like to thank Juha Kinnunen for providing his lecture notes at my disposal when designing this course. Other material includes "Elliptic & Parabolic Equations" (Wu, Yin and Wang, 2006, World Scientific), "Partial Differential Equations" (Evans, 1998, American Mathematical Society), "Elliptic Partial Differetential Equations of Second Order" (Gilbarg, Trudinger, 1977, Springer), "Second Order Elliptic Equations and Elliptic Systems" (Chen, Wu, 1998, American Mathematical Society), "Direct Methods in the Calculus of Variations" (Giusti, 2003, World Scientific), "Partial Differential Equations" (Jost, 2002, Springer), and "Partial Differential Equations" (DiBenedetto, 2010, Birkhäuser).

Department of Mathematics and Statistics, University of Jyväskylä, PO Box 35, FI-40014 Jyväskylä, Finland

E-mail address: Please email typos/errors to:mikko.j.parviainen@jyu.fi