EXERCISE SET 2
PARTIAL DIFFERENTIAL EQUATIONS 2
2023

Return the exercises to jarkko.siltakoski@jyu.fi or to my box in the ware room (opposite to MaD356).

1. Let $g \in \hat{W}^{1,p}(\Omega)$. Show that
$$F(v) := \int_\Omega Dg \cdot Dv \, dx$$
is a bounded linear functional in $\hat{W}_0^{1,2}(\Omega)$.

2. Let $g \in W^{1,2}(\Omega)$ and $f \in L^2(\Omega)$. Explain how you would prove the existence of a weak solution to
$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{1}$$

3. Let $u_k \rightharpoonup u$ converge weakly in $L^2(\Omega)$ as $k \to \infty$. Show that
$$\|u\|_{L^2(\Omega)} \le \liminf_k \|u_k\|_{L^2(\Omega)} .$$

4. Let $p > 2$. Formulate and prove the Dirichlet principle for the variational integral
$$I(v) = \frac{1}{p} \int_\Omega |Dv|^p \, dx.$$

5. Prove the uniqueness of weak solutions to
$$-\Delta u(x) + \sum_{i=1}^n b_i(x) D_i u(x) + c(x)u(x) = f(x)$$
with standard assumptions on the data and the coefficients and for large enough c_0 for which $c \ge c_0$. (Hint: Choose large enough c_0 such that $c \ge c_0$ in order to absorb the bad terms)

6. Prove that for a weak solution $u \in W^{1,2}(\Omega)$ to $-\Delta u = f$, where $f \in L^2(\Omega)$, it holds that $u \in W^{2,2}_{loc}(\Omega)$. Instruction: We could of course use the more general theorem from the lectures, but here you should work out the proof by hand.