EXERCISE SET 1
PARTIAL DIFFERENTIAL EQUATIONS 2
2023

Return the exercises to jarkko.siltakoski@jyu.fi or to my box in the ware room (opposite to MaD356).

1. Consider the equation
$$\Delta u(x) + \operatorname{div}(b(x)u(x)) = f(x) \quad \text{in } B_1(0),$$
where $f : B(0,1) \to \mathbb{R}$ and $b : B(0,1) \to \mathbb{R}^n$ are given. Give a physical justification for each of the terms.

2. Let $u(x) = |x|^{\alpha}$, $x \in \mathbb{R}^n \setminus \{0\}$, where $\alpha \in \mathbb{R}$. Calculate $\Delta u(x) := \operatorname{div}(Du(x))$ and find α so that $\Delta u(x) = 0$ in the classical sense.

3. Prove the ε-version of Young's inequality (as stated in the lecture note on page 6).

4. Find a weak derivative (and show that it is a weak derivative) of
$$u : (-1,1) \to \mathbb{R}, \quad u(x) = \begin{cases} 0, & -1 < x \leq 0, \\ \sqrt{x}, & 0 < x < 1. \end{cases}$$

5. Let $p > 0$. Show that there exists a constant $C \geq 1$, depending only on p, such that
$$\frac{1}{C}(a^p + b^p) \leq (a+b)^p \leq C(a^p + b^p)$$
for all $a, b \geq 0$. Use this to prove that $\|u\|_{W^{k,p}(\Omega)}$ is equivalent with the norm
$$\sum_{|\alpha| \leq k} \left(\int_\Omega |D^\alpha u|^p \, dx \right)^{1/p} \quad \text{if } 1 < p < \infty.$$

6. Let $\Omega = (0,2)$, $f = 1$, $b = 0 = c$,
$$a(x) = \begin{cases} 1, & x \in (0,1], \\ 2, & x \in (1,2). \end{cases}$$
Consider the problem
$$\begin{cases} Lu = f, & x \in \Omega, \\ u(0) = 0 = u(2). \end{cases}$$
Show that
$$u(x) = \begin{cases} -\frac{x^2}{2} + \frac{5}{6}x, & x \in [0,1], \\ -\frac{x^2}{4} + \frac{5}{12}x + \frac{1}{6}, & x \in (1,2], \end{cases}$$
is a weak solution to the above problem.

7. Consider $\Omega = (0,2)$, $c = 0 = b$, $f = 1$ and

$$a(x) = \begin{cases} x, & x \in (0,1], \\ 1, & x \in (1,2), \end{cases}$$

and a problem

$$\begin{cases} Lu = f, & x \in \Omega, \\ u(0) = 0 = u(2). \end{cases}$$

Show that

$$u(x) = \begin{cases} -x, & x \in (0,1], \\ -\frac{1}{2}x^2 + 2.5x - 3, & x \in (1,2), \end{cases}$$

is not a weak solution. What is the problem with this PDE?

8. Same data as in the previous exercise except

$$a(x) = \begin{cases} 1+x, & x \in (0,1], \\ 2, & x \in (1,2). \end{cases}$$

Find a weak solution and show that your function is a weak solution.

9. Let $\alpha \in (0,1)$ and

$$\mathcal{A} = \begin{bmatrix} \frac{x_1^2+\alpha^2 x_2^2}{|x|^2} & (1-\alpha^2)\frac{x_1 x_2}{|x|^2} \\ (1-\alpha^2)\frac{x_1 x_2}{|x|^2} & \frac{\alpha^2 x_1^2 + x_2^2}{|x|^2} \end{bmatrix}.$$

Show that

$$\alpha^2 |\xi|^2 \le \sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j \le |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^2.$$

10. Let $\alpha \in (0,1)$. Show that

$$u : B(0,1) \to \mathbb{R}, \quad u(x) = |x|^{\alpha-1} x_1$$

is in $W^{1,2}(B(0,1))$

11. Show that u as defined in the exercise 10 is a weak solution to

$$\operatorname{div}(\mathcal{A}(x)Du) = 0 \quad \text{in } B(0,1),$$

where $\mathcal{A}$ is as in exercise 9. You may take for granted that $\operatorname{div}(\mathcal{A}(x)Du) = 0$ pointwise in the set where u is classically differentiable.