EXERCISE SET 3
PARTIAL DIFFERENTIAL EQUATIONS 2
2023

Return the exercises to jarkko.siltakoski@jyu.fi or to my box in the ware room (opposite to MaD356).

1. Let $a_{ij}, b_i, c \in L^\infty(\Omega)$, $f \in L^2(\Omega)$ and $u \in W^{1,2}(\Omega)$ be a weak solution to $Lu = f$ in Ω. Show that for any $\Omega' \Subset \Omega$ we have
$$\int_{\Omega'} |Du|^2 \, dx \leq C \int_\Omega (u^2 + f^2) \, dx,$$
where C is independent of u. (Use $v := \eta^2 u$ as a test function, where η is a suitable cut-off function).

2. Let $f \in L^2(\Omega)$, $g \in W^{1,2}(\Omega)$ and $u \in W^{1,2}(\Omega)$ be a weak solution to $\Delta u = f$ such that $u - g \in W_0^{1,2}(\Omega)$. Show that
$$\|u\|_{W^{1,2}(\Omega)} \leq C(\|g\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)}).$$

3. Let $u \in W_{loc}^{1,2}(\Omega)$ be a weak *subsolution* to $\Delta u = 0$, i.e.
$$\int_\Omega Du \cdot D\varphi \, dx \leq 0 \quad \text{for all non-negative } \varphi \in C_0^\infty(\Omega). \tag{1}$$
Let $0 < r < R < \infty$ be such that $B(x_0, R) \subset \Omega$ and consider a cut-off function $\eta \in C_0^\infty(B(x_0, R))$ such that $0 \leq \eta \leq 1$, $\eta = 1$ in $B(x_0, r)$ and $|D\eta| \leq C/(R - r)$. Show that
$$\int_{B(x_0,R)} \eta^2 \left|D(u - k)_+\right|^2 \, dx \leq \left(\frac{C}{R - r}\right)^2 \int_{B(x_0,R)} \left|(u - k)_+\right|^2 \, dx,$$
where $k \in \mathbb{R}$ and $u_+ = \max\{u, 0\}$.

4. Does the ess-sup estimate
$$\operatorname*{ess\,sup}_{B(x_0,r/2)} u \leq k_0 + C\Big(\fint_{B(x_0,r)} \left|(u - k_0)_+\right|^2 \, dx\Big)^{1/2}$$
hold for a subsolution to $\Delta u = 0$?

5. Let $n > 2$, $u : \mathbb{R}^n \to (0, \infty]$ be the function defined by $u(x) = |x|^{2-n}$ and $k > 0$. Show that $\min\{u, k\}$ is a weak *supersolution* to $\Delta u = 0$, i.e. that (1) holds with "$\geq$".

6. Show by a counter example that the ess-sup estimate in Exercise 4 does not hold in the case of a supersolution.