EXERCISE SET 4
PARTIAL DIFFERENTIAL EQUATIONS 2
2023

Return the exercises to jarkko.siltakoski@jyu.fi or to my box in the ware room (opposite to MaD356).

1. Let $k \in \mathbb{R}$ and define $A(k,r) = B(x_0,r) \cap \{x \in \Omega : u(x) > k\}$ and $k_0 = (m(2r) + M(2r))/r$. The measure decay lemma in the De Giorgi's method states that if u is a weak solution to $\Delta u = 0$ and

$$|A(k_0,r)| \leq \gamma \, |B(x_0,r)| \,, \quad 0 < \gamma < 1,$$

then

$$\lim_{k \uparrow M(2r)} |A(k,r)| = 0.$$

Is this still true if we just assume u to be a weak subsolution?

2. Let $n > 2$, $u : \mathbb{R}^n \to (0, \infty]$ be the function defined by $u(x) = |x|^{2-n}$ and $k > 0$. Show that $\min\{u, k\}$ does not satisfy the measure decay lemma.

3. Fill the gap in the proof of Hölder continuity: Let $\operatorname{osc}_{B(x_0,2r)} u = M(2r) - m(2r)$. Assume that u satisfies

$$\operatorname{osc}_{B(x_0,r/4)} u \leq \gamma \operatorname{osc}_{B(x_0,r)} u$$

for some $\gamma \in (0,1)$. Show that there exists a constant $C > 0$ and $\alpha \in (0,1)$ such that

$$\operatorname{osc}_{B(x_0,r/4)} u \leq C \left(\frac{r}{R}\right)^{\alpha} \operatorname{osc}_{B(x_0,R)} u$$

for every $R \geq r$. (Hint: Choose $j \in \mathbb{N}$ such that $4^{j-1} r \leq R < 4^j r$ and iterate)

4. Does Hölder continuity hold for subsolutions of $\Delta u = 0$?

5. Formulate and prove a weak max principle for $u \in C^2(\Omega) \cap C(\overline{\Omega})$,

$$Lu = -\sum_{i=1}^{n} a_{ii}(x) D_{ii} u(x) + \sum_{i=1}^{n} b_i(x) D_i u(x) \leq 0,$$

with $a_{ii}, b_i \in C(\overline{\Omega})$ and uniform ellipticity. Tips: Suppose that there is a point violating the weak max principle. What do we know about Du and $D^2 u$ at that point? Modify the solution to get strict inequalities and arrive at a contradiction.