EXERCISE SET 3
PARTIAL DIFFERENTIAL EQUATIONS 2
2023

Return the exercises to jarkko.siltakoski@jyu.fi or to my box in the ware room (opposite to MaD356).

1. Consider the elliptic equation
$$Lu = -\sum_{i,j=1}^{n} D_i(a_{ij}D_j u) + cu = 0 \tag{1}$$
with $c \geq 0$. Does the weak maximum principle hold for weak subsolutions of (1)? And for supersolutions? Provide an explanation or a counterexample in each case.

2. Formulate and prove the weak minimum principle for weak solutions to (1).

3. Combine the weak maximum principle and the weak minimum principle to show that
$$\operatorname*{ess\,sup}_{\Omega} |u| \leq \sup_{\partial\Omega} |u|$$
for every weak solution $u \in W^{1,2}(\Omega)$ to (1). Does this inequality hold for subsolutions or supersolutions?

4. Give a counterexample to the weak maximum principle for (1) if we drop the assumption $c \geq 0$.

5. Give an example showing that Hopf's lemma does not hold if we drop the interior ball condition. (Hint: consider the harmonic function $u(x, y)$ defined as the real part of the complex function $f(z) = z^{\alpha}$ (c.f. Remark 3.41) and choose suitable $\alpha > 0$ and $\Omega \subset \mathbb{R}^2$.)

6. Show that the test function space $C_0^\infty(\Omega_T)$ in Definition 4.8 can be extended to
$$\left\{\varphi \in L^2(0, T; W_0^{1,2}(\Omega)) : \partial_t\varphi \in L^2(\Omega_T) \quad \text{and} \quad \varphi(x, 0) = 0 = \varphi(x, T)\right\}.$$