EXERCISE SET 4
PARTIAL DIFFERENTIAL EQUATIONS 2
2023

Return the exercises to jarkko.siltakoski@jyu.fi or to my box in the ware room (opposite to MaD356).

1. Provide the definition of weak sub- and supersolutions to $\partial_t u + Lu = f$ in Ω_T.
2. Show that the step function
$$u(x,t) = \begin{cases} 0, & t \le \frac{1}{2}, \\ 1, & t > \frac{1}{2}, \end{cases}$$
is a weak supersolution to $\partial_t u - \Delta u \ge 0$ in $\Omega \times (0,1)$. Observe that u has no time derivative in the weak sense.
3. Let $u, g \in C([0,T]; L^2(\Omega))$. Show that the following three initial conditions are equivalent
$$\frac{1}{h} \int_{\Omega_h} |u(x,t) - g(x,t)|^2 \, dx \, dt \to 0 \quad \text{as } h \to 0,$$
$$\int_\Omega |u(x,t) - g(x,0)|^2 \, dx \to 0 \quad \text{as } t \to 0,$$
and
$$\int_\Omega u(x,0)\varphi(x) \, dx = \int_\Omega g(x,0)\varphi(x) \, dx \quad \text{for all } \varphi \in C_0^\infty(\Omega).$$
4. Let u be a weak solution with zero boundary condition and an initial data g to the equation $\partial_t u - \Delta u = f$, where $f \in L^2(\Omega_T)$ and $g \in W_0^{1,2}(\Omega)$. Prove the following energy estimate
$$\sup_{t \in [0,T]} \int_\Omega u^2 \, dx + \int_{\Omega_T} |Du|^2 \, dx \, dt \le C \|g\|^2_{L^2(\Omega)} + C \|f\|^2_{L^2(\Omega_T)} .$$
5. Under the assumptions of the previous exercise, prove the estimate
$$\|\partial_t u\|_{L^2(\Omega)} \le C \|Dg\|_{L^2(\Omega)} + C \|f\|_{L^2(\Omega_T)} .$$
(Hint: uniqueness + existence proof in section 4.1)