To recall

- σ - algebra
- Borel σ - algebra $\mathcal{B}(\mathbb{R})$
- probability space
- properties of $\mathbb{P}$ like continuity from below
- properties of measurable maps (random variables)

you can use the Lecture Notes 'An Introduction to Probability Theory' which is available in KOPPA.

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(1) **Gambler's ruin with a fair coin**
Prove Proposition 1.1.1 from the lecture notes for the case $p = q = \frac{1}{2}$.

(2) **σ-algebra generated by $f : \Omega \to \mathbb{R}$**

(a) Let $f : \Omega \to \mathbb{R}$ and define

$$\sigma(f) := \{A = f^{-1}(B) \subseteq \Omega : B \in \mathcal{B}(\mathbb{R})\}.$$

Prove that $\sigma(f)$ is a σ-algebra on Ω.

(b) If $(\Omega, \mathcal{F})$ is a measurable space (this means $\Omega \neq \emptyset$ and $\mathcal{F}$ is a σ-algebra on Ω or in other words, a probability space without measure), show that f is a random variable if and only if

$$\sigma(f) \subseteq \mathcal{F}.$$

(3) **symmetric difference**
Let Ω be a non-empty set and $A, B, C \subseteq \Omega$. For the symmetric difference defined by $A \Delta B := (A \cup B) \setminus (A \cap B)$ show the following relations:

(a) $A \Delta B = (A \cap B^c) \cup (A^c \cap B)$
(b) $A^c \Delta B^c = A \Delta B$
(c) $A \Delta B = A \Delta C \implies B = C$.
(d) $(B \cup C) \Delta A \subseteq (B \setminus A) \cup (C \Delta A)$.

(4) **a monotone class theorem**
Let Ω be a non-empty set and 2^Ω the set of all subsets (the power set).
A system of subsets $\mathcal{G} \subseteq 2^\Omega$ is called a *monotone class* if $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ with $A_k \in \mathcal{G}$ implies that $\bigcup_k A_k \in \mathcal{G}$.
Show that the following holds:
If $\mathcal{G} \subseteq 2^\Omega$ a monotone class and an algebra, then it is a σ-algebra.
Hint: In fact, one only has to show that also for arbitrary $(B_k)_{k=1}^\infty \subseteq \mathcal{G}$ it follows $\bigcup_{k=1}^\infty B_k \in \mathcal{G}$.

(5*) **an algebra is 'quite close' to its generated σ-algebra: Lemma 2.1.7**
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{A}$ an algebra such that $\sigma(\mathcal{A}) = \mathcal{F}$. Check that for

$$\mathcal{G} := \{B \in \mathcal{F} : \forall \varepsilon > 0\, \exists A \in \mathcal{A} : \mathbb{P}(A \Delta B) < \varepsilon\}$$

the assertions are true:

(a) $\mathcal{A} \subseteq \mathcal{G}$.
(b) $\mathcal{G}$ is a σ-algebra.
Hint: Use (3) and continuity of $\mathbb{P}$ from below and properties of Δ.
(c) $\mathcal{G} = \mathcal{F}$.