(1) **an algebra**
Let $(\xi_n)_{n=1}^\infty$, $\xi_n : \Omega \to \mathbb{R}$, be random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Show that

$$\bigcup_{n=1}^{\infty} \sigma(\xi_1, ..., \xi_n)$$

is an algebra on Ω.

(2) **the tail** $sigma$**-algebra** $\mathcal{F}^\infty$
Assume a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and independent random variables $\xi_n : \Omega \to \mathbb{R}$, $n = 1, 2, 3...$ and let

$$\mathcal{F}^\infty = \bigcap_{N=1}^{\infty} \sigma(\xi_N, \xi_{N+1}, ...).$$

Decide which of the following sets belongs to $\mathcal{F}^\infty$:
 (a) $A_1 := \{\omega \in \Omega : \xi_n(\omega) > 1 \text{ for infinitely many } n\}$
 (b) $A_2 := \{\omega \in \Omega : \exists n \geq 1 : \xi_{n+1}(\omega) > \xi_n(\omega)\}$
 (c) $A_3 := \{\omega \in \Omega : \exists c > 0 \forall n \geq 1 |\xi_n(\omega)| \leq cn\}$
 (d) $A_4 := \{\omega \in \Omega : \exists c > 0 \forall n \geq 1 |\xi_n(\omega)| \geq cn\}$

(3) **a Three-Series-Theorem application**
Let $(S_n)_{n=1}^\infty$ be a sequence of independent random variables where each S_n is exponentially distributed with parameter $\lambda_n > 0$. Set

$$T_n := \begin{cases} S_n, & S_n \leq 1 \\ 1, & S_n > 1. \end{cases}$$

 (a) Show that $\mathbb{E}T_n = \frac{1-e^{-\lambda_n}}{\lambda_n}$, $\mathbb{E}(T_n - \mathbb{E}T_n)^2 = \frac{1-2\lambda_n e^{-\lambda_n} - e^{-2\lambda_n}}{\lambda_n^2}$ and $\mathbb{P}(S_n \geq 1) = e^{-\lambda_n}$.
 (b) Show:

$$\sum_{n=1}^{\infty} S_n < \infty \quad a.s. \iff \sum_{n=1}^{\infty} \lambda_n^{-1} < \infty.$$

(4) **0-1-law application**
Let $\xi_1, \xi_2, ... : \Omega \to [0, 1]$ be a sequence of independent random variables and $c \in [0, 1]$.
 (a) Use the 0-1-law of Kolomogorov to show that

$$p_c = \mathbb{P}(\limsup_n \xi_n = c) \in \{0, 1\}.$$

 (b*) Is it possible to choose $c \in [0, 1]$ such that $p_c = 1$?