(1) **conditional expectation**
Let f, g be indendent random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G} = \sigma(g)$, the smallest σ-algebra, such that g is measurable, and $\mathcal{H} = \sigma(f, g)$. Assume that $\mathbb{E}|f|^3 < \infty$. Use Proposition 3.1.8 to find out
(a) $\mathbb{E}[f|\mathcal{G}]$,
(b) $\mathbb{E}[f|\mathcal{H}]$.
(c) $\mathbb{E}[(f+g)^3|\mathcal{G}]$, if g is bounded.

(2) **conditional expectation: the discrete case**
Assume a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a partition $\Omega_1, ..., \Omega_N$ of Ω, (i.e. $\Omega = \bigcup_{n=1}^N \Omega_n$ and $\Omega_k \cap \Omega_l = \emptyset$ for $k \neq l$). Assume moreover that for $n = 1, ..., N$ it holds $\Omega_n \in \mathcal{F}$ and $\mathbb{P}(\Omega_n) > 0$. Put $\mathcal{G} := \sigma(\Omega_1, \ldots, \Omega_N)$ and assume $f \in \mathcal{L}_1(\Omega, \mathcal{F}, \mathbb{P})$. Define

$$g(\omega) := \sum_{n=1}^N \frac{\mathbb{E}(f \mathbb{1}_{\Omega_n})}{\mathbb{P}(\Omega_n)} \mathbb{1}_{\Omega_n}(\omega).$$

Show that $g = \mathbb{E}[f|\mathcal{G}]$ a.s.

(3) **sub-martingales and martingales**
Let $X_1, X_2, ...$ be bounded, i.i.d. random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}X_1 = 0$. We will use the natural filtration given by $\mathcal{F}_0 := \{\Omega, \emptyset\}$ and $\mathcal{F}_n := \sigma(X_1, ..., X_n)$ $(n \geq 1)$. Let $(Y_n)_{n=0}^\infty$ be given by $Y_0 := 0$ and $Y_n := (X_1 + ... + X_n)^2$ for $n \geq 1$.
(a) Show that $(Y_n)_{n=0}^\infty$ is a sub-martingale.
(b) Does there exist a constant $c > 0$ such that the process $(M_n)_{n=0}^\infty$ given by $M_n := Y_n - cn$ is a martingale?

(4) **Problem 3 of Demo 2 and Problem 2 of Demo 2**
We interpret
(a) $S_1, S_2, \ldots$ as waiting times with parameters $\lambda_1, \lambda_2, \ldots$,
(b) $T_1, T_2, \ldots$ as claim arrival times,
(c) X_t as the claim amount process, i.e. the number of claims at time t.

Explosion means that the waiting are such that one has, up to some *finite* time, *infinitely many* claims. In the demo we already checked with $\mathbb{E}S_n = 1/\lambda_n$ (we assumed this to be known) that, by monotone convergence,

$$\sum_{n=1}^\infty \frac{1}{\lambda_n} = \sum_{n=1}^\infty \mathbb{E}S_n = \mathbb{E}\left[\sum_{n=1}^\infty S_n\right] \in [0, \infty].$$

This gives that $\sum_{n=1}^\infty \frac{1}{\lambda_n} < \infty$ implies $\sum_{n=1}^\infty S_n < \infty$ almost surely, i.e.

$$\mathbb{P}\left(\left\{\omega \in \Omega : \sum_{n=1}^\infty S_n(\omega) < \infty\right\}\right) = 1.$$

This means, that if the waiting times become too short, then one has an explosion almost surely. The following problem is left:

Problem: Deduce by the 3-series Theorem of Kolmogorov that

$$\mathbb{P}\left(\left\{\omega \in \Omega : \sum_{n=1}^\infty S_n(\omega) < \infty\right\}\right) = 1$$

implies $\sum_{n=1}^\infty \frac{1}{\lambda_n} < \infty$.

Hint: Use and verify part (a) of Problem 3 of Demo 2.