(1) **Conditional expectations**
  (a) Prove Proposition 3.1.8(i).
  Hint: Define $h := \lambda \mathbb{E}(f|\mathcal{G}) + \mu \mathbb{E}(g|\mathcal{G})$ and check that h is $\mathcal{G}$-measurable and that

$$\int_A h d\mathbb{P} = \int_A (\lambda f + \mu g) d\mathbb{P} \quad \text{for} \quad A \in \mathcal{G}.$$

  (b) Prove Proposition 3.1.8(viii) by the definition of the conditional expectation.

(2) **Stopping times?**
  Let $M_0 := 0$ and $M_n := \varepsilon_1 + \cdots + \varepsilon_n$, $n \geq 1$, where $\varepsilon_1, \varepsilon_2, ... : \Omega \to \{-1, 1\}$ are independent Bernoulli random variables. Let $\mathcal{F}_0 := \{\emptyset, \Omega\}$ and $\mathcal{F}_n := \sigma(\varepsilon_1, ..., \varepsilon_n)$. Which of the following maps are stopping times ($\inf \emptyset := \infty$)?
  (a) $\sigma(\omega) := \inf \{n \geq 0 : M_n(\omega) \in (10, 12)\}$
  (b) $\sigma(\omega) := \inf \{n \geq 0 : M_n(\omega) \in (10, 12)\} - 1$
  (c) $\sigma(\omega) := \inf \{n \geq 0 : M_n(\omega) \in (10, 12)\} + 1$
  (d) $\sigma(\omega) := \inf \{n \geq 0 : M_{n+1}(\omega) \in (10, 12)\}$
  (e) $\sigma(\omega) := \inf \{n \geq 0 : M_{n+1}(\omega) \in (10, 11)\}$
  (f) $\sigma(\omega) := \inf \{n \geq 1 : M_{n-1}(\omega) = 10\}$
  (g) $\sigma(\omega) := \inf \{n \geq 1 : M_{n-1}(\omega) = 10\} - 1$

(3) **Stopping times and their σ-algebras**
  Let $(\mathcal{F}_n)_{n=0}^\infty$ be a filtration and $\sigma, \tau : \Omega \to \mathbb{N} \cup \{\infty\}$ stopping times.
  (a) Show that $\sigma + \tau$ is a stopping time.
  (b) Show that $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$ if $0 \leq \sigma \leq \tau$.

(4) **Stopped processes**
  Let $(\Omega, \mathcal{F})$ be a measurable space equipped with a filtration $(\mathcal{F}_n)_{n=0}^\infty$. Assume that $(X_n)_{n=0}^\infty$ is an adapted process, i.e. $X_n : \Omega \to \mathbb{R}$ is $\mathcal{F}_n$-measurable for all $n \geq 0$. Assume a stopping time $\tau : \Omega \to \{0, 1, 2, \ldots\}$. Prove that that the map

$$X_\tau : \omega \mapsto X_{\tau(\omega)}(\omega)$$

  is an $\mathcal{F}_\tau$-measurable map from Ω into $\mathbb{R}$.

(5) **Martingales**
  Let $\varepsilon_1, \varepsilon_2, ... : \Omega \to \{-1, 1\}$ be independent Bernoulli random variables, i.e. $\mathbb{P}(\varepsilon_n = 1) = \mathbb{P}(\varepsilon_n = -1) = \frac{1}{2}$. Define the natural filtration $\mathcal{F}_0 := \{\Omega, \emptyset\}$ and $\mathcal{F}_n := \sigma(\varepsilon_1, ..., \varepsilon_n)$ for $n \geq 1$. Let $M_0 := 1$ and

$$M_n := \frac{e^{\sum_{k=1}^n \varepsilon_k}}{\alpha^n}, \quad n \geq 1,$$

  where $\alpha > 0$. For which $\alpha > 0$ the process $(M_n)_{n=0}^\infty$ is a martingale?
  (*) What happens if one considers the complex valued process

$$M_n := \frac{e^{i \sum_{k=1}^n \varepsilon_k}}{\alpha^n}, \quad n \geq 1?$$