(1) **2 or 3 Series Theorem?**
Let $(X_k)_{k=1}^{\infty}$ be a sequence of independent random variables such that $\mathbb{E}X_k = 0$ for all $k \geq 1$. Moreover, assume that

$$\sum_{k=1}^{\infty} \mathbb{E}\psi(X_k) < \infty,$$

where

$$\psi(x) := x^2 \mathbb{1}_{\{|x| \leq 1\}} + |x| \mathbb{1}_{\{|x|>1\}}.$$

Is it true that

$$\sum_{k=1}^{\infty} X_k \quad \text{converges (to a finite number) a.s. ?}$$

(2) **Conditional expectation**
We use the probability space $([0,4), \mathcal{B}([0,4)), \frac{1}{4}\lambda)$. Let $\mathcal{G} = \sigma([0,1),[1,3)) \subseteq \mathcal{B}([0,4))$ be a sub-σ-algebra on $[0,4)$. Find out $\mathbb{E}[f_k|\mathcal{G}]$, for $k = 1,2,3$, where

- $f_1(x) = x$,
- $f_2(x) = \mathbb{1}_{[1,4)}(x)$,
- $f_3(x) = e^x$.

(3) **Closable martingale?**
Let $(\varepsilon_k)_{k=1}^{\infty}$ be i.i.d. with $\mathbb{P}(\varepsilon_k = \pm 1) = 1/2$. Define $M_0 := 0$ and $M_n := \varepsilon_1 + ... + \varepsilon_n$ for $n \in \mathbb{N}^*$. We have shown that the process $N = (N_n)_{n=0}^{\infty}$ given by $N_0 = 1$ and

$$N_n = \left(\frac{2}{e + e^{-1}}\right)^n e^{M_n}$$

is a martingale. Does there exist an $Z \in \mathcal{L}_1(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}|N_n - Z| \to 0$ as $n \to \infty$?

(4) **Radon-Nikodym Theorem**
Let $\Omega \neq \emptyset$. Assume a filtration $(\mathcal{F}_n)_{n=0}^{\infty}$ on Ω such that

- $\mathcal{F}_n = \sigma\left(A_1^{(n)}, ..., A_{L_n}^{(n)}\right)$,
- the $A_1^{(n)}, ..., A_{L_n}^{(n)}$ are pair-wise disjoint and $\bigcup_{l=1}^{L_n} A_l^{(n)} = \Omega$,
- every $A_l^{(n)}$ is a union of elements from $\left\{A_1^{(n+1)}, ..., A_{L_{n+1}}^{(n+1)}\right\}$,
- $\mathcal{F} = \sigma\left(A_l^{(n)} : n = 0,1,... \text{ and } l = 1,...,L_n\right)$.

Assume probability measures $\mathbb{P}$ and μ on $(\Omega, \mathcal{F})$ such that $\mathbb{P}(A) = 0$ implies $\mu(0) = 0$ (in other words, μ is absolutely continues with respect to $\mathbb{P}$. We define the random variables $M_n : \Omega \to \mathbb{R}$ by

$$M_n(\omega) := \begin{cases} \frac{\mu(A_l^{(n)})}{\mathbb{P}(A_l^{(n)})} & : \ \mathbb{P}(A_l^{(n)}) > 0 \\ 1 & : \ \mathbb{P}(A_l^{(n)}) = 0 \end{cases}$$

whenever $\omega \in A_l^{(n)}$.

(a) Show that $M = (M_n)_{n=0}^{\infty}$ is a martingale with respect to the filtration $(\mathcal{F}_n)_{n=0}^{\infty}$.
(b) Show that $\mu(A) = \int_A M_n d\mathbb{P}$ for $A \in \mathcal{F}_n$.
(c) Show that $(M_n)_{n=0}^{\infty}$ is uniformly integrable.
Hint: Here you can use the following fact: Assume a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a finite measure μ on $(\Omega, \mathcal{F})$ such that μ is absolutely continuous with respect to $\mathbb{P}$. Then, given $\varepsilon > 0$ there is some $\delta \in (0,1)$ such that $\mathbb{P}(A) \leq \delta$ implies that $\mu(A) \leq \varepsilon$.
($d*$) What is the meaning of the limit random variable $M_\infty = \lim_n M_n$ that exists according to Proposition 3.8.6?