Demo 1

(1) For $p = q = \frac{1}{2}$ we get

$$r_x = \frac{1}{2} r_{x+1} + \frac{1}{2} r_{x-1}$$

for $0 < x < B$. So the function on $\{x-1, x, x+1\}$ is linear, so the function

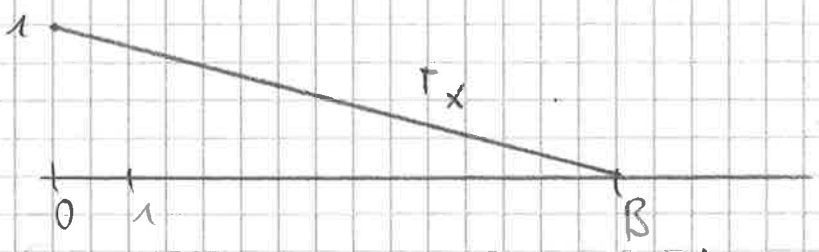
$$r_x : \{0, \dots, B\} \longrightarrow \mathbb{R}$$

is <u>linear</u> with

$$r_0 = 1 \quad \text{and} \quad r_B = 0$$

so that

$$r_x = \underline{(B-x)/B} = 1 - \frac{x}{B}.$$



By the same argument we get $\omega_x = \frac{x}{B}$.

---

(2) a) $\Omega \in \mathcal{B}(f)$: $B = \mathbb{R} \in \mathcal{B}(\mathbb{R})$

$$\Rightarrow f^{-1}(B) = f^{-1}(\mathbb{R}) = \Omega$$

$$A \in \mathcal{B}(f) \Longrightarrow \exists \; B \in \mathcal{B}(\mathbb{R}) \quad f^{-1}(B) = A$$

$$\Rightarrow B^c \in \mathcal{B}(\mathbb{R}) \text{ and}$$

$$f^{-1}(B^c) = \left(f^{-1}(B)\right)^c = A^c$$

$$\Rightarrow A^c \in \mathcal{B}(f)$$

$$A_1, A_2, \dots \in \mathcal{B}(f) \Rightarrow \exists \; B_n \in \mathcal{B}(\mathbb{R}) \quad f^{-1}(B_n) = A_n$$

$$\Rightarrow \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} f^{-1}(B_n) = f^{-1}\Big(\underbrace{\bigcup_{n=1}^{\infty} B_n}_{\in \mathcal{B}(\mathbb{R})}\Big)$$

$$\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{B}(f)$$

b) f is a random variable $\iff$

$$f^{-1}(B) \in \mathcal{F} \quad \forall \quad B \in \mathcal{B}(\mathbb{R}) \iff$$

$$\{f^{-1}(B) : B \in \mathcal{B}(\mathbb{R})\} \subseteq \mathcal{F} \iff$$

$$\sigma(f) \subseteq \mathcal{F}.$$

---

(3) a) $x \in A \Delta B \iff x \in A \setminus B$ or $x \in B \setminus A$

$\iff x \in A \cap B^c$ or $x \in B \cap A^c$

$\iff x \in (A \cap B^c) \cup (B \cap A^c)$

b) $A^c \Delta B^c = (A^c \setminus (B^c)) \cup (B^c \setminus (A^c))$

$= (x \in A^c, x \notin B^c)$ or $(x \in B^c, x \notin A^c)$

$= (x \notin A, x \in B)$ or $(x \notin B, x \in A)$

$= A \Delta B$

c) $A \Delta B = A \Delta C \Rightarrow A \setminus B \subseteq (A \setminus C) \cup (C \setminus A)$

$\Rightarrow A \setminus B \subseteq A \setminus C$

In the same way we get

$A \setminus C \subseteq A \setminus B$ and therefore

$\boxed{A \setminus C = A \setminus B}$

Also $B \setminus A \subseteq (A \setminus C) \cup (A \setminus C)$ implies

$B \setminus A \subseteq A \setminus C.$

In the same way

$C \setminus A \subseteq B \setminus A$ and therefore

$\boxed{C \setminus A = B \setminus A} \longrightarrow A \cap C^c = A \cap B^c$

$A = (A \setminus C) \cup (A \cap C^c) = (A \setminus B) \cup (A \cap B^c) = B$

d) $x \in (B \cup C) \Delta A \rightarrow x \in (B \cup C)$ and $x \notin A$

or $x \notin (B \cup C)$ and $x \in A$

$\Rightarrow$

$x \in A$ | $x \notin A$
$\downarrow$ | $\downarrow$
$x \notin (B \cup C)$ | $x \in B \cup C$
 | $x \notin A$
$\downarrow$ | $\downarrow$
$x \in A,\ x \notin C$ | $x \in (B \setminus A)$
$\downarrow$ |
$x \in C \Delta A$ |

$x \in (B \setminus A) \cup (C \Delta A)$

---

(4) we only have to show $B_1, B_2, \ldots \in \mathcal{G}$

$$\Rightarrow \bigcup_{n=1}^{\infty} B_n \in \mathcal{G}$$

$A_1 := B_1,\quad A_2 := B_1 \cup B_2, \ldots, A_n := B_1 \cup \ldots \cup B_n$

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} \underbrace{A_n}_{\in \mathcal{G}} \in \mathcal{G}$$

monotone class

$\in \mathcal{G}$ as $\mathcal{G}$ is an algebra

15*)

(a) Assume that $B \in \mathcal{A}$. Then we choose

$A := B$ so that $A \Delta B = \emptyset$ and

$\mathbb{P}(A \Delta B) = 0$. Hence $B \in \mathcal{G}$ and $\mathcal{A} \subseteq \mathcal{G}$.

(b) • $\Omega \in \mathcal{G}$ is true because $\Omega \in \mathcal{A}$ and a)

• $B \in \mathcal{G}$, $\varepsilon > 0$. Find $A \in \mathcal{A}$ with

$$\mathbb{P}(A \Delta B) < \varepsilon$$

By (3b): $\mathbb{P}(A^c \Delta B^c) = \mathbb{P}(A \Delta B) < \varepsilon$

and because $A^c \in \mathcal{A}$ we get $B^c \in \mathcal{G}$.

• $B_1, B_2 \in \mathcal{G}$, $\varepsilon > 0$. $\exists$ $A_1, A_2 \in \mathcal{A}$

$$\mathbb{P}(A_1 \Delta B_1) < \varepsilon/2,$$

$$\mathbb{P}(A_2 \Delta B_2) < \varepsilon/2.$$

Hence

$$\mathbb{P}(\underbrace{(A_1 \cup A_2)}_{\in \mathcal{A}} \Delta (B_1 \cup B_2))$$

$$\underset{(3d)}{\leq} \mathbb{P}\Big([B_1 \setminus (A_1 \cup A_2)] \cup [B_2 \Delta (A_1 \cup A_2)]\Big)$$

$$\leq \mathbb{P}((B_1 \setminus A_1) \cup [B_2 \setminus (A_1 \cup A_2)])$$

$$\leq \mathbb{P}((B_1 \setminus A_1) \cup (B_2 \setminus A_2))$$

$$\leq \mathbb{P}(B_1 \setminus A_1) + \mathbb{P}(B_2 \setminus A_2) < \varepsilon.$$

Therefore, $\mathcal{G}$ is an algebra and a monotone class. Let

$$B_1, B_2, \ldots \in \mathcal{G} \text{ with } B_1 \subseteq B_2 \subseteq \ldots$$

and $\varepsilon > 0$. Find $A_\ell \in \mathcal{A}$ with

$$\mathbb{P}(A_\ell \Delta B_\ell) < \varepsilon/2^\ell .$$

Then

$$\mathbb{P}\Big(\Big(\bigcup_1^\infty B_\ell\Big) \Delta \underbrace{\Big(\bigcup_1^L A_\ell\Big)}_{\in \mathcal{A}} \Big)$$

$$= \mathbb{P}\Big(\Big[\bigcup_1^L B_\ell \cup \Big(\bigcup_1^\infty B_\ell \setminus \bigcup_1^L B_\ell\Big)\Big] \Delta \Big(\bigcup_1^L A_\ell\Big)\Big)$$

$$\underset{(3d)}{\leq} \mathbb{P}\Big(\Big(\bigcup_1^L B_\ell\Big) \Delta \Big(\bigcup_1^L A_\ell\Big) \Big) + \mathbb{P}\Big(\Big(\bigcup_1^\infty B_\ell \setminus \bigcup_1^L B_\ell\Big) \Delta \Big(\bigcup_1^L A_\ell\Big)\Big)$$

$$\underset{(3d)}{\leq} \sum_{\ell=1}^L \underbrace{\mathbb{P}(B_\ell \Delta A_\ell)}_{< \varepsilon/2^\ell} + \underbrace{\mathbb{P}\Big(\bigcup_1^\infty B_\ell \setminus \bigcup_1^L B_\ell \Big)}_{\downarrow\, L \to \infty \atop 0}$$

$L \to \infty \quad \leq \varepsilon$

By (4) $\mathcal{G}$ is a σ-algebra.

(c) By (a) and (c): $\mathcal{F} = \sigma(\mathcal{A}) \subseteq \mathcal{G} \subseteq \mathcal{F}$

and $\mathcal{G} = \mathcal{F}$