(1) We check the properties of an algebra:

a) $\Omega, \emptyset \in \sigma(\xi_1, \dots, \xi_n)$ for all $n \geq 1$, so that (of course)

$$\Omega, \emptyset \in \bigcup_{n=1}^{\infty} \sigma(\xi_1, \dots, \xi_n).$$

b) $$A \in \bigcup_{n=1}^{\infty} \sigma(\xi_1, \dots, \xi_n) \implies \exists n_0 \geq 1 \quad A \in \sigma(\xi_1, \dots, \xi_{n_0})$$

$$\implies A^c \in \sigma(\xi_1, \dots, \xi_{n_0})$$

$$\implies A^c \in \bigcup_{n=1}^{\infty} \sigma(\xi_1, \dots, \xi_n)$$

c) $$A, B \in \bigcup_{n=1}^{\infty} \sigma(\xi_1, \dots, \xi_n) \implies \exists \begin{matrix} n_A \geq 1 & A \in \sigma(\xi_1, \dots, \xi_{n_A}) \\ n_B \geq 1 & B \in \sigma(\xi_1, \dots, \xi_{n_B}) \end{matrix}$$

$\implies A, B \in \sigma(\xi_1, \dots, \xi_{n_0})$ with $n_0 := \max\{n_A, n_B\}$

$$\implies A \cup B \in \sigma(\xi_1, \dots, \xi_{n_0})$$

$$\implies A \cup B \in \bigcup_{n=1}^{\infty} \sigma(\xi_1, \dots, \xi_n)$$

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(2) a) $A_1 \in \mathcal{F}^\infty$ because, for all $N \geq 1$,

$$\{\omega \in \Omega : \xi_n(\omega) > 1 \text{ for infinitely many } n \text{ with } n \geq 1\}$$
$$= \{\omega \in \Omega : \xi_n(\omega) > 1 \text{ for infinitely many } n \text{ with } n \geq N\}$$

b) A_2 is not always in $\mathcal{F}^\infty$. Here is an example:

$\varepsilon_1 \equiv 0$, $\varepsilon_2 : \Omega \to \{-1, 1\}$ with $\mathbb{P}(\varepsilon_2 = -1) = \mathbb{P}(\varepsilon_2 = 1) = \frac{1}{2}$,

$\varepsilon_3 = \varepsilon_4 = \dots \equiv -1$. Then

$$A_2 = \{\omega \in \Omega : \varepsilon_2(\omega) = 1\} \text{ and } \mathbb{P}(A_2) = \frac{1}{2}.$$

Hence $A_2 \in \mathcal{F}^\infty$ is not possible by Kolmogorov's 0-1 law.

c) $A_3 \in \mathcal{F}^\infty$. Assume $N \geq 1$ and

$$|\xi_n(\omega)| \leq C_N(\omega)\, n \quad \text{for } n \geq N.$$

Then we choose $d_N(\omega) > 0$ such that

$$|\xi_1(\omega)| \leq d_N(\omega), \dots, |\xi_{N-1}(\omega)| \leq d_N(\omega).$$

This implies

$$|\xi_n(\omega)| \leq \max\{C_N(\omega), d_N(\omega)\}\, n \quad \text{for } n \geq 1$$

d) A_4 is not always in $\mathcal{F}^\infty$. Here is an example:

$$\mathbb{P}(\xi_1 = 0) = \mathbb{P}(\xi_1 = 1) = \frac{1}{2}, \quad \xi_2 = \xi_3 = \dots \equiv 1$$

Then

$$A_4 = \{\omega \in \Omega : \xi_1(\omega) = 1\}$$

and $\mathbb{P}(A_4) = \frac{1}{2}$. As in b) $A_4 \notin \mathcal{F}^\infty$.

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(4) For all $N \geq 1$ one has

$$\limsup_{n \geq 1} \xi_n(\omega) = \limsup_{n \geq N} \xi_n(\omega)$$

so that

$$\{\omega \in \Omega : \limsup_n \xi_n(\omega) = c\} \in \bigcap_{N=1}^\infty \sigma(\xi_N, \xi_{N+1}, \dots)$$

and $\mathbb{P}(\limsup_n \xi_n = c) \in \{0,1\}$ by Kolmogorov's 0-1-law. Exactly the same argument yields to

$$\mathbb{P}(\limsup_n \xi_n \leq c) \in \{0,1\}. \qquad \text{(Eq 1)}$$

We let $f : \Omega \to \mathbb{R}$ be defined by

$$f(\omega) := \limsup_n \xi_n(\omega).$$

Then (Eq 1) reads as

$$F_f(c) = \mathbb{P}(f \leq c) \in \{0, 1\},$$

where F_f is the distribution function of f. The distribution function is right-continuous so that there must be a $c_0 \in \mathbb{R}$ with

$$F_f(c) = 0 \quad \text{for} \quad c < c_0$$
$$F_f(c) = 1 \quad \text{for} \quad c \geq c_0.$$

For this c_0 we obtain

$$\mathbb{P}(f = c_0) = \mathbb{P}(f \leq c_0) - \mathbb{P}(f < c_0)$$
$$= \underbrace{\mathbb{P}(f \leq c_0)}_{1} - \lim_{c \uparrow c_0} \underbrace{\mathbb{P}(f \leq c)}_{0}$$
$$= 1.$$

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(3) In this solution we do not need to compute

$$\mathbb{E}(T_n - \mathbb{E}T_n)^2$$

so we leave out this and you will not be asked later for this.

(a) $$\mathbb{P}(S_n \geq 1) = \lambda_n \int_1^\infty e^{-\lambda_n t} dt = -e^{-\lambda_n t} \Big|_1^\infty$$
$$= e^{-\lambda_n}$$

$$\mathbb{E}\, T_n = \lambda_n \int_0^1 t e^{-\lambda_n t} dt + \int_1^\infty \lambda_n e^{-\lambda_n t} dt$$

$$= \lambda_n \left[-t e^{-\lambda_n t} \frac{1}{\lambda_n} \Big|_0^1 + \int_0^1 \frac{1}{\lambda_n} e^{-\lambda_n t} dt \right]$$

$$+ e^{-\lambda_n}$$

$$= -e^{-\lambda_n} + -\frac{1}{\lambda_n} e^{-\lambda_n t} \Big|_0^1 + e^{\lambda_n}$$

$$= \frac{1 - e^{-\lambda_n}}{\lambda_n}$$

(b) "$\Rightarrow$" From the 3-series theorem we get that

$$\sum_{n=1}^{\infty} \mathbb{E}\, T_n = \sum_{n=1}^{\infty} \frac{1 - e^{-\lambda_n}}{\lambda_n} < \infty$$

This implies that

$$\frac{1 - e^{-\lambda_n}}{\lambda_n} \to 0$$

Which is only possible if $\lambda_n \to \infty$:

$$\lambda \mapsto \frac{1 - e^{-\lambda}}{\lambda} \text{ is continuous on } (0, \infty)$$

and $$\lim_{\lambda \downarrow 0} \frac{1 - e^{-\lambda}}{\lambda} \underset{\text{L'Hospital}}{=} \lim_{\lambda \downarrow 0} \frac{e^{-\lambda}}{1} = 1$$

Therefore

$$\sum_{n=1}^{\infty} \frac{1 - e^{-\lambda_n}}{\lambda_n} < \infty \Leftrightarrow \sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty$$

because $\frac{1}{2} \leq 1 - e^{-\lambda_n} \leq 1$ for $\lambda_n \geq C$.

$\Leftarrow$ This can be done with the 3-series theorem of Kolmogorov or more easily in this special case:

We have by monotone convergence that

$$\mathbb{E}\left(\sum_{n=1}^{\infty} S_n\right) = \sum_{n=1}^{\infty} \mathbb{E} S_n = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty$$

which implies that

$$\mathbb{P}\left(\left\{\omega \in \Omega : \sum_{n=1}^{\infty} S_n(\omega) < \infty\right\}\right) = 1.$$