(1) We check the properties of an algebra:

a) $\Omega, \emptyset \in \mathcal{B}(\xi_1, \dots, \xi_n)$ for all $n \geq 1$, so that (of course)

$$\Omega, \emptyset \in \bigcup_{n=1}^{\infty} \mathcal{B}(\xi_1, \dots, \xi_n).$$

b) $A \in \bigcup_{n=1}^{\infty} \mathcal{B}(\xi_1, \dots, \xi_n) \implies \exists\, n_0 \geq 1 \quad A \in \mathcal{B}(\xi_1, \dots, \xi_{n_0})$

$$\implies A^c \in \mathcal{B}(\xi_1, \dots, \xi_{n_0})$$

$$\implies A^c \in \bigcup_{n=1}^{\infty} \mathcal{B}(\xi_1, \dots, \xi_n)$$

c) $A, B \in \bigcup_{n=1}^{\infty} \mathcal{B}(\xi_1, \dots, \xi_n) \implies \exists \begin{array}{ll} n_A \geq 1 & A \in \mathcal{B}(\xi_1, \dots, \xi_{n_A}) \\ n_B \geq 1 & B \in \mathcal{B}(\xi_1, \dots, \xi_{n_B}) \end{array}$

$$\implies A, B \in \mathcal{B}(\xi_1, \dots, \xi_{n_0}) \text{ with } n_0 := \max\{n_A, n_B\}$$

$$\implies A \cup B \in \mathcal{B}(\xi_1, \dots, \xi_{n_0})$$

$$\implies A \cup B \in \bigcup_{n=1}^{\infty} \mathcal{B}(\xi_1, \dots, \xi_n)$$

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(2) a) $A_1 \in \mathcal{F}^\infty$ because, for all $N \geq 1$,

$$\{\omega \in \Omega : \xi_n(\omega) > 1 \text{ for infinitely many } n \text{ with } n \geq 1\}$$
$$= \{\omega \in \Omega : \xi_n(\omega) > 1 \text{ for infinitely many } n \text{ with } n \geq N\}$$

b) A_1 is not always in $\mathcal{F}^\infty$. Here is an example:

$\varepsilon_1 \equiv 0$, $\varepsilon_2 : \Omega \to \{-1, 1\}$ with $\mathbb{P}(\varepsilon_2 = -1) = \mathbb{P}(\varepsilon_2 = 1) = \frac{1}{2}$,

$\varepsilon_3 = \varepsilon_4 = \dots \equiv -1$. Then

$$A_2 = \{\omega \in \Omega : \varepsilon_2(\omega) = 1\} \text{ and } \mathbb{P}(A_2) = \frac{1}{2}.$$

Hence $A_2 \in \mathcal{F}^\infty$ is not possible by Kolmogorov's 0-1 law.