(1) For $0 < p < q < \infty$ we have

$$(\mathbb{E}|X|^p)^{1/p} \le (\mathbb{E}|X|^q)^{1/q}$$

for any random variable $X : \Omega \to \mathbb{R}$. Hence

$$\mathbb{E}|f| \le (\mathbb{E}|f|^3)^{1/3} < \infty$$

$$(\mathbb{E}|f|^2)^{1/2} \le (\mathbb{E}|f|^3)^{1/3} < \infty$$

(a) $\mathbb{E}[f|\mathcal{G}] = \mathbb{E}[f|\sigma(g)] \underset{f \text{ and } g \text{ independent}}{=} \underline{\mathbb{E}[f]}$

(b) $\mathbb{E}[f|\mathcal{H}] \overset{\Rightarrow f \text{ and } \sigma(g) \perp}{=} \mathbb{E}[f|\sigma(f,g)]$

$= \underline{f}$

because f is $\sigma(f,g)$-measurable.

(c) $\mathbb{E}|f+g|^3 \le \mathbb{E}|f|^3 + 3\mathbb{E}|f|^2|g| + 3\mathbb{E}|f||g|^2$

$+ \mathbb{E}|g|^3$

$\underset{|g| \le c}{\le} \mathbb{E}|f|^3 + 3c\,\mathbb{E}|f|^2 + 3c^2\mathbb{E}|f| + c^3 < \infty$

$\mathbb{E}[(f+g)^3|\mathcal{G}] \underset{\text{linearity of cond. exp}}{=} \mathbb{E}[f^3|\mathcal{G}] + 3\,\mathbb{E}[f^2 g|\mathcal{G}]$

$+ 3\,\mathbb{E}[f g^2|\mathcal{G}] +$

$+ \mathbb{E}[g^3|\mathcal{G}]$

$=$

[f indep. from $\mathcal{G}$
→ f^3 indep. from $\mathcal{G}$
f^2 indep. from $\mathcal{G}$

[g is $\mathcal{G}$-measurable

$$\boxed{\begin{aligned}&\mathbb{E} f^3 + 3g\,\mathbb{E} f^2 \\ &+ 3g^2\,\mathbb{E} f \\ &+ g^3\end{aligned}}$$

(2)

(a) $\Omega_n \in \mathcal{G}$, g is a simple random variable constant of the $\Omega_1, \dots, \Omega_N$. Hence g is $\mathcal{G}$ measurable.

(b) g takes only finitely many values. Hence $\sup_{\omega \in \Omega} |g(\omega)| < \infty$ and $\mathbb{E}|g| < \infty$.

(c) Assume $B \in \mathcal{G}$. Then there is a subset $I \subseteq \{1, \dots, N\}$ such that $B = \bigcup_{n \in I} \Omega_n$

Then we get

$$
\begin{aligned}
\int_B g \, d\mathbb{P} &= \sum_{n \in I} \int_{\Omega_n} g \, d\mathbb{P} \\
&= \sum_{n \in I} \int_{\Omega_n} \frac{\mathbb{E}[f 1_{\Omega_n}]}{\mathbb{P}(\Omega_n)} \, d\mathbb{P} \\
&= \sum_{n \in I} \mathbb{E}[f 1_{\Omega_n}] \frac{\mathbb{P}(\Omega_n)}{\mathbb{P}(\Omega_n)} \\
&= \sum_{n \in I} \int_{\Omega_n} f \, d\mathbb{P} \\
&= \int_B f \, d\mathbb{P}.
\end{aligned}
$$

(3) (a) X_i - $\mathcal{F}_i$ measurable, hence $\mathcal{F}_n$-measurable for $n \geq i$

$\Rightarrow$ $X_1 + \cdots + X_n$ is $\mathcal{F}_n$-measurable

$\Rightarrow$ Y_n is $\mathcal{F}_n$-measurable

(b) $|Y_n| \leq (nc)^2$ where $c > 0$ is a constant with $|X_i(\omega)| \leq c$ for all $\omega \in \Omega$.

Hence $\mathbb{E}|Y_n| < \infty$.

(c) $\mathbb{E}(Y_1 | \mathcal{F}_0) = \mathbb{E}(X_1^2 | \mathcal{F}_0) \geq 0 = Y_0$ a.s.

For $n \geq 1$ we get, a.s.,

$$\mathbb{E}(Y_{n+1} | \mathcal{F}_n) = \mathbb{E}\left(\left([X_1 + \cdots + X_n] + X_{n+1}\right)^2 \middle| \mathcal{F}_n\right)$$

$$= \mathbb{E}\left([X_1 + \cdots + X_n]^2 \middle| \mathcal{F}_n\right) + 2\,\mathbb{E}\left([X_1 + \cdots + X_n] X_{n+1} \middle| \mathcal{F}_n\right) + \mathbb{E}\left(X_{n+1}^2 \middle| \mathcal{F}_n\right)$$

$$= Y_n + 2[X_1 + \cdots + X_n]\, \mathbb{E}[X_{n+1} | \mathcal{F}_n] + \mathbb{E}[X_{n+1}^2 | \mathcal{F}_n]$$

$$= Y_n + 2[X_1 + \cdots + X_n] \underbrace{\mathbb{E}[X_{n+1}]}_{=0} + \mathbb{E}[X_{n+1}^2]$$

X_{n+1} and X_{n+1}^2 independent from $\mathcal{F}_n$

$$= Y_n + \mathbb{E}\, X_1^2$$

$X_{n+1} \stackrel{d}{=} X_1$

Hence, for $C := \mathbb{E}X_1^2 \geq 0$,

$$\mathbb{E}(Y_{n+1} | \mathcal{F}_n) = Y_n + C \geq Y_n \quad \text{a.s.}$$

and

$$\mathbb{E}(\underbrace{Y_{n+1} - C(n+1)}_{M_{n+1}} | \mathcal{F}_n) = \underbrace{Y_n - Cn}_{M_n}$$

14) see solutions Demo 2.