(1)a) By definition $\mathbb{E}(f|\mathcal{G})$ and $\mathbb{E}(g|\mathcal{G})$ are $\mathcal{G}$-measurable, so that h is $\mathcal{G}$-measurable. Moreover,

$$\int_\Omega |h|\,d\mathbb{P} = \int_\Omega |\lambda\,\mathbb{E}(f|\mathcal{G}) + \mu\,\mathbb{E}(g|\mathcal{G})|\,d\mathbb{P}$$

$$\leq |\lambda| \int_\Omega |\mathbb{E}(f|\mathcal{G})|\,d\mathbb{P} + |\mu| \int_\Omega |\mathbb{E}(g|\mathcal{G})|\,d\mathbb{P}$$

$$\leq |\lambda| \int_\Omega \mathbb{E}(|f|\,|\mathcal{G})\,d\mathbb{P} + |\mu| \int_\Omega \mathbb{E}(|g|\,|\mathcal{G})\,d\mathbb{P}$$

$$= |\lambda| \int_\Omega |f|\,d\mathbb{P} + |\mu| \int_\Omega |g|\,d\mathbb{P} < \infty.$$

Finally, for $A \in \mathcal{G}$, we get

$$\int_A h\,d\mathbb{P} = \lambda \int_A \mathbb{E}(f|\mathcal{G})\,d\mathbb{P} + \mu \int_A \mathbb{E}(g|\mathcal{G})\,d\mathbb{P}$$

$$= \lambda \int_A f\,d\mathbb{P} + \mu \int_A g\,d\mathbb{P}$$

$$= \int_A [\lambda f + \mu g]\,d\mathbb{P}.$$

b) $\mathcal{G} = \{\emptyset, \Omega\}$. A random variable h is $\mathcal{G}$-measurable $\Leftrightarrow$ h is constant on Ω.

Ansatz: $h(\omega) := \mathbb{E} f$ for all $\omega \in \Omega$. Then for

$B = \Omega$,

$$\int_B h\,d\mathbb{P} = \int_B \mathbb{E}f\,d\mathbb{P} = \mathbb{E}f = \int_B f\,d\mathbb{P},$$

$B = \emptyset$,

$$\int_B h\,d\mathbb{P} = 0 = \int_B f\,d\mathbb{P}$$

[4] Let $B \in \mathcal{B}(\mathbb{R})$. Then

$$X_\tau^{-1}(B) = \{\omega \in \Omega : X_{\tau(\omega)}(\omega) \in B\}$$

$$= \bigcup_{n \geq 0} \{\omega \in \Omega : \tau(\omega) = n,\ X_n(\omega) \in B\}$$

$$= \bigcup_{n \geq 0} \{\tau = n\} \cap \{X_n \in B\}$$

and, for $m \geq 0$,

$$\{\tau = m\} \cap \{X_\tau^{-1}(B)\} = \underbrace{\{\tau = m\}}_{\in \mathcal{F}_m} \cap \underbrace{\{X_m \in B\}}_{\in \mathcal{F}_m} \in \mathcal{F}_m$$

[3] a) For $n \geq 0$ one has

$$\{\sigma + \tau = n\} = \bigcup_{k=0}^{n} \{\sigma = k,\ \tau = n-k\}$$

$$= \bigcup_{k=0}^{n} \Big[\underbrace{\{\sigma = k\}}_{\in \mathcal{F}_k \subseteq \mathcal{F}_n} \cap \underbrace{\{\tau = n-k\}}_{\in \mathcal{F}_{n-k} \subseteq \mathcal{F}_n} \Big] \in \mathcal{F}_n.$$

b) $A \in \mathcal{F}_\sigma \iff A \cap \{\sigma = n\} \in \mathcal{F}_n$ for $n \geq 0$.

Then
$$A \cap \{\tau = m\} = A \cap \{\sigma \leq m\} \cap \{\tau = m\}$$

$$= \bigcup_{k=0}^{m} \underbrace{A \cap \{\sigma = k\}}_{\in \mathcal{F}_k \subseteq \mathcal{F}_m} \cap \underbrace{\{\tau = m\}}_{\in \mathcal{F}_m}$$

and $A \cap \{\tau = m\} \in \mathcal{F}_m$.

(5) From the definition we know that

a) $|M_n(\omega)| \leq \dfrac{e^n}{\alpha^n} < \infty$,

b) M_n depends only on $\varepsilon_1, \dots, \varepsilon_n$, so that

M_n is $\mathcal{F}_n$-measurable

c) From a) and b) we also get

$\mathbb{E}|M_n| < \infty$.

We only need to check

$$\mathbb{E}(M_{n+1} | \mathcal{F}_n) = M_n.$$

This is equivalent to

$$\mathbb{E}\left(e^{\sum_{k=1}^{n+1} \varepsilon_k} / \alpha^{n+1} \,\middle|\, \mathcal{F}_n \right) = e^{\sum_{k=1}^{n} \varepsilon_k} / \alpha^n,$$

$$\mathbb{E}\left(e^{\varepsilon_1} / \alpha \,\middle|\, \mathcal{F}_0 \right) = 1.$$

The first equality is equivalent to

$$e^{\sum_{k=1}^{n} \varepsilon_k} / \alpha^n \, \mathbb{E}\left(e^{\varepsilon_{n+1}} / \alpha \,\middle|\, \mathcal{F}_n \right) = e^{\sum_{k=1}^{n} \varepsilon_k} / \alpha^n$$

and

$$\mathbb{E}\left(\underbrace{e^{\varepsilon_{n+1}} / \alpha}_{\text{independent from } \mathcal{F}_n} \,\middle|\, \mathcal{F}_n \right) = 1,$$

or

$$\mathbb{E}\left(\frac{e^{\varepsilon_{n+1}}}{\alpha} \right) = 1$$

or $\mathbb{E}\, e^{\varepsilon_{n+1}} = \frac{1}{2}\left(e + \frac{1}{e}\right) = \alpha$. Because

$\mathbb{E}(e^{\varepsilon_1} | \mathcal{F}_0) = \mathbb{E}(e^{\varepsilon_2} | \mathcal{F}_1) = \dots = \alpha$, we get that

$(M_n)_{n=0}^{\infty}$ is a martingale $\Leftrightarrow \alpha = \frac{1}{2}\left(e + \frac{1}{e}\right)$

(1) part: The computation is exactly the same, but we get

$$\alpha = \mathbb{E}\left(e^{i\varepsilon_1}\right) = \frac{1}{2}\left(e^{i} + e^{-i}\right)$$

$$\underset{\text{Euler formula}}{=} \frac{1}{2}\big(\cos 1 + i\sin 1 + \cos(-1) + i\sin(-1)\big)$$

$$= \cos 1$$

(2) $M_n = \varepsilon_1 + \dots + \varepsilon_n$ depends only on $\varepsilon_1, \dots, \varepsilon_n$ and is therefore $\mathcal{F}_n = \sigma(\varepsilon_1, \dots, \varepsilon_n)$ measurable for $n \geq 1$. M_0 is constant and therefore $\mathcal{F}_0$-measurable. Hence, $(M_n)_{n=0}^{\infty}$ is adapted.

a) $$\{\tau = n\} = \{\omega \in \Omega : M_0^{(\omega)} \notin (10,12), \dots, M_{n-1}^{(\omega)} \notin (10,12), M_n \in (10,12)\}$$
$n > 10$

$$= \underbrace{\{M_0 \notin (10,12)\}}_{\in \mathcal{F}_0} \cap \dots \cap \underbrace{\{M_{n-1} \notin (10,12)\}}_{\in \mathcal{F}_{n-1}} \cap \underbrace{\{M_n \in (10,12)\}}_{\in \mathcal{F}_n} \in \mathcal{F}_n.$$

For $0 \leq n \leq 10$ we have
$$\{\tau = n\} = \emptyset \in \mathcal{F}_n$$

Answer: Yes.

c) Let us denote τ from part a) by τ_a. Then

$$\{\tau = n\} = \{\tau_a + 1 = n\} = \{\tau_a = n-1\}$$

$$= \begin{cases} \emptyset & n = 0 \\ \mathcal{F}_{n-1} \text{ measurable if } & n \geq 1 \end{cases} \in \mathcal{F}_n$$

Answer: Yes.

e) $\tau(\omega) = \infty$ because the condition $M_{n+1}(\omega) \in (10, 11)$ can never be satisfied. Hence τ is a stopping time

Answer: Yes.

f) $$\{\tau = n\} = \underbrace{\{M_0 \neq 10\}}_{\in \mathcal{F}_0} \cap \dots \cap \underbrace{\{M_{n-2} \neq 10\}}_{\in \mathcal{F}_{n-2}} \cap \underbrace{\{M_{n-1} = 10\}}_{\in \mathcal{F}_{n-1}}$$

$(\{\tau = n\} = \emptyset \in \mathcal{F}_{n-1} \text{ for } n \leq 10)$ $\in \mathcal{F}_{n-1} \subseteq \mathcal{F}_n$

Answer: Yes

g) Let τ_f be the stopping time from f). Then

$$\{\tau = n\} = \{\tau_f - 1 = n\} = \underbrace{\{\tau_f = n+1\}}$$

$\in \mathcal{F}_n$ because of the computation in f).

Answer: Yes

d) $$\{\tau = n\} = \{M_0 \neq 11\} \cap \dots \cap \{M_n \neq 11\} \cap \{M_{n+1} = 11\}$$

$$= (\{M_n = 9\} \cap \{\varepsilon_{n+1} = 1\}) \cup (\{M_n = 11\} \cap \{\varepsilon_{n+1} = -1\})$$

If $\{\tau = n\} \in \mathcal{F}_n$ this would imply

$$\underbrace{1_{\{\tau=n\}}}_{\in\{0,1\}} = \mathbb{E}\left(1_{\{\tau=n\}} \middle| \mathcal{F}_n\right)$$

$$= 1_{\{M_n=9\}} \mathbb{E}\left(1_{\{\varepsilon_{n+1}=1\}} \middle| \mathcal{F}_n\right)$$

$$+ 1_{\{M_n=11\}} \mathbb{E}\left(1_{\{\varepsilon_{n+1}=-1\}} \middle| \mathcal{F}_n\right)$$

$$= 1_{\{M_n=9\}} \mathbb{E}\, 1_{\{\varepsilon_{n+1}=1\}}$$

$$+ 1_{\{M_n=11\}} \mathbb{E}\, 1_{\{\varepsilon_{n+1}=-1\}}$$

$$= \frac{1}{2}\left(1_{\{M_n=9\}} + 1_{\{M_n=11\}}\right) \in \{0, \tfrac{1}{2}\}$$

this is a contradiction.

Answer: NO

b) $\{\tau = n\} = \{M_0 \neq 11\} \cap \cdots \cap \{M_n \neq 11\} \cap \{M_{n+1} = 11\}$

which is the same as a).

Answer: NO