Demo 6

(1) From our assumption we know that

$$\sum_{k=1}^{\infty} \mathbb{E}\, |X_k|^2 \mathbb{1}_{\{|X_k| \leq 1\}} < \infty \quad \text{and}$$

$$\sum_{k=1}^{\infty} \mathbb{E}\, |X_k| \mathbb{1}_{\{|X_k| > 1\}} < \infty .$$

From this we get that

a) $$\sum_{k=1}^{\infty} \mathbb{P}(|X_k| > 1) \leq \sum_{k=1}^{\infty} \mathbb{E}\, |X_k| \mathbb{1}_{\{|X_k| > 1\}} < \infty ,$$

b) $$\sum_{k=1}^{\infty} \mathbb{E}\, |X_k^1 - \mathbb{E} X_k^1|^2 \leq \sum_{k=1}^{\infty} \mathbb{E}\, |X_k^1|^2$$

$$\leq \sum_{k=1}^{\infty} \mathbb{E}\left[\mathbb{1}_{\{|X_k| > 1\}} + |X_k|^2 \mathbb{1}_{\{|X_k| \leq 1\}}\right]$$

$$\leq \sum_{k=1}^{\infty} \mathbb{E}\, \varphi(X_k) < \infty ,$$

c) $$\sum_{k=1}^{\infty} |\mathbb{E}\, X_k^1| = \sum_{k=1}^{\infty} \left| \mathbb{E}\left[X_k - X_k^1\right] \right| \qquad (\mathbb{E} X_k = 0)$$

$$\leq \sum_{k=1}^{\infty} \mathbb{E}\, |X_k| \mathbb{1}_{|X_k| > 1} < \infty$$

Hence we can apply the 3-series theorem and the answer is YES.

12) $\mathcal{G} = \sigma([0,1), [1,3))$. For

$A_1 := [0,1)$, $A_2 := [1,3)$, $A_3 := [3,4)$ all

elements of $\mathcal{G}$ are unions of A_1, A_2, A_3. Hence

$$\mathbb{E}[f|\mathcal{G}] = \sum_{k=1}^{3} 1_{A_k} \int_{A_k} f(x)\, d\mathbb{P}(x) \frac{1}{\mathbb{P}(A_k)}$$

for $d\mathbb{P}(x) = \frac{1}{4} d\lambda(x)$. Hence

$$\mathbb{E}[f|\mathcal{G}] = 1_{A_1} \int_{A_1} f(x) \frac{dx}{4} \frac{1}{1/4}$$

$$+ 1_{A_2} \int_{A_2} f(x) \frac{dx}{4} \frac{1}{1/2}$$

$$+ 1_{A_3} \int_{A_3} f(x) \frac{dx}{4} \frac{1}{1/4}$$

$$= 1_{[0,1)} \int_0^1 f(x)\, dx + \frac{1}{2} 1_{[1,3)} \int_1^3 f(x)\, dx$$

$$+ 1_{[3,4)} \int_3^4 f(x)\, dx$$

f_1: $\mathbb{E}[f_1|\mathcal{G}] = \frac{1}{2} 1_{[0,1)} + 2\, 1_{[1,3)} + \frac{7}{2} 1_{[3,4)}$

f_2 is $\mathcal{G}$-measurable, so that

$$\mathbb{E}[f_2|\mathcal{G}] = f_2$$

f_3: $\mathbb{E}[f_3|\mathcal{G}] = 1_{A_1}(e-1) + 1_{A_2} \frac{1}{2}(e^3 - e) + 1_{A_3} (e^4 - e^3)$

(3) If $N_n \to Z$ in $\mathcal{L}_1$, then $(N_n)_{n=1}^{\infty}$ would be a Cauchy sequence in $\mathcal{L}_1$. Hence it would need to hold that

$$\mathbb{E}\,|N_{n+1} - N_n| \xrightarrow[n\to\infty]{} 0.$$

But we get

$$\mathbb{E}\left|N_{n+1} - N_n\right| = \mathbb{E}\,|N_n|\left|\frac{N_{n+1}}{N_n} - 1\right|$$

$$= \mathbb{E}\left|N_n\left[e^{\varepsilon_{n+1}}\frac{2}{e+e^{-1}} - 1\right]\right|$$

$$= \underbrace{\mathbb{E}\,N_n}_{=1}\ \underbrace{\mathbb{E}\left|e^{\varepsilon_{n+1}}\frac{2}{e+e^{-1}} - 1\right|}_{\underbrace{\mathbb{E}\left|e^{\varepsilon_1}\frac{2}{e+e^{-1}} - 1\right|}_{=\varepsilon>0}}$$

Hence $\mathbb{E}\,|N_{n+1} - N_n| \longrightarrow 0$ is not true.

Hence $N_n \to Z$ in $\mathcal{L}_1$ is not satisfied.