# Solutions for Demonstration 2

**Problem 1.** Let $E = \{0, 1, 2, \dots\}$ and $\mathcal{E} = 2^E$. We define for $h \geqslant 0, s > 0, k \in E, B \in \mathcal{E}$ that

$$P_h(k, B) := \mathbb{P}(N_{s+h} \in B | N_s = k). \tag{1}$$

(a) We show that $(P_h(k, B))$ is a transition function.

- By properties of conditional probability, $B \mapsto P_h(k, B)$ is a probability measure on $\mathcal{E}$ for each $h \geqslant 0, k \in E$.
- Since $\mathcal{E} = 2^E$, it is obvious to verify the measurability of $E \ni k \mapsto P_h(k, B)$.
- $P_0(k, B) = \mathbb{P}(N_s \in B | N_s = k) = \delta_k(B)$.
- Check the Chapman-Kolmogorov condition: let $u, v \geqslant 0$, one has

$$\begin{aligned} P_{u+v}(k, B) &= \mathbb{P}(N_{s+u+v} \in B | N_s = k) = \mathbb{P}(N_{s+u+v} - N_s + N_s \in B | N_s = k) \\ &\overset{(N_{s+u+v} - N_s) \perp N_s}{=} \mathbb{P}(N_{s+u+v} - N_s + k \in B). \end{aligned}$$

We also have

$$\begin{aligned} \int_E P_u(y, B) P_v(k, dy) &= \sum_{m=0}^{\infty} P_u(m, B) P_v(k, \{m\}) \\ &= \sum_{m=0}^{\infty} \mathbb{P}(N_{s+u+v} \in B | N_{s+v} = m) \mathbb{P}(N_{s+v} = m | N_s = k) \\ &= \sum_{m=0}^{\infty} \mathbb{P}(N_{s+u+v} - N_{s+v} + m \in B) \mathbb{P}(N_{s+v} - N_s + k = m) \\ &= \sum_{m=0}^{\infty} \mathbb{P}(N_{s+u+v} - N_{s+v} + m \in B, N_{s+v} - N_s + k = m) \\ &= \sum_{m=0}^{\infty} \mathbb{P}(N_{s+u+v} - N_s + k \in B, N_{s+v} - N_s + k = m) \\ &= \mathbb{P}(N_{s+u+v} - N_s + k \in B). \end{aligned}$$

Hence $\int_E P_u(y, B) P_v(k, dy) = P_{u+v}(k, B)$, which asserts the Chapman-Kolmgorov condition.

(b) Let $\mathbb{F}$ be the natural filtration of $(N_t)_{t \geqslant 0}$.
Let $f : (E, \mathcal{E}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be bounded. For $s \leqslant t$, we have

$$\mathbb{E}(f(N_t) | \mathcal{F}_s) = \mathbb{E}(f(N_t - N_s + N_s) | \mathcal{F}_s) = \mathbb{E} f(N_t - N_s + m)|_{m = N_s},$$

and

$$\begin{aligned} \int_E f(y) P_{t-s}(N_s, dy) &= \left[ \sum_{k=0}^{\infty} f(k) P_{t-s}(m, \{k\}) \right]_{m = N_s} = \left[ \sum_{k=0}^{\infty} f(k) \mathbb{P}(N_t = k | N_s = m) \right]_{m = N_s} \\ &= \left[ \sum_{k=0}^{\infty} f(k) \mathbb{P}(N_t - N_s + m = k) \right]_{m = N_s} = \mathbb{E} f(N_t - N_s + m)|_{m = N_s}. \end{aligned}$$

Thus

$$\mathbb{E}(f(N_t) | \mathcal{F}_s) = \int_E f(y) P_{t-s}(N_s, dy),$$

which implies that $(N_t)$ is a Markov process w.r.t. $\mathbb{F}$ with the transition function above. □

**Problem 2.** Let $M_t := \frac{\mathrm{e}^{\mathrm{i}aW_t}}{\mathbb{E}\,\mathrm{e}^{\mathrm{i}aW_t}}$, where $W$ is a standard Brownian motion. We show that $M$ is a martingale w.r.t. $\mathbb{F}^W = (\mathcal{F}^W_t)_{t\geqslant 0}$.

- It is clear that $M$ is adapted;
- Since $\mathbb{E}\,\mathrm{e}^{\mathrm{i}aW_t} = \mathrm{e}^{-\frac{a^2t}{2}} > 0$ for any $a \in \mathbb{R}, t \geqslant 0$, one has $\mathbb{E}|M_t| = \frac{1}{|\mathbb{E}\,\mathrm{e}^{iaW_t}|} < \infty$;
- For $0 \leqslant s < t$, we have

$$\mathbb{E}(M_t|\mathcal{F}^W_s) = \mathbb{E}\Big(\frac{\mathrm{e}^{\mathrm{i}a(W_t-W_s)}\,\mathrm{e}^{\mathrm{i}aW_s}}{\mathbb{E}\,\mathrm{e}^{\mathrm{i}a(W_t-W_s)}\,\mathrm{e}^{\mathrm{i}aW_s}}\;\Big|\;\mathcal{F}^W_s\Big) = \frac{\mathrm{e}^{\mathrm{i}aW_s}}{\mathbb{E}\,\mathrm{e}^{\mathrm{i}aW_s}} = M_s \quad a.s.$$

Hence $M$ is a martingale w.r.t. $\mathbb{F}^W$. □

**Problem 3.** Let $(\mathcal{F})_{t\geqslant 0}$ be a filtration and define $\mathcal{G}_t := \mathcal{F}_{t+} = \cap_{u>t}\mathcal{F}_u$. We have

$$\mathcal{G}_{t+} = \cap_{u>t}\mathcal{G}_u = \cap_{u>t}\cap_{s>u}\mathcal{F}_s = \cap_{s>t}\mathcal{F}_s = \mathcal{F}_{t+} = \mathcal{G}_t,$$

which means that $\mathcal{G}$ is right continuous. □

**Problem 4.** Let $X = (X_t)_{t\geqslant 0}$ be a process such that $X_t - X_s$ is independent of $\mathcal{F}^X_s$ for all $0 \leqslant s \leqslant t$.

Let $0 \leqslant s \leqslant t_0 < t_1 < \cdots < t_n$. We show that the vector $Y_n := (X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}})$ is independent of $\mathcal{F}^X_s$. Take $D \in \mathcal{F}^X_s$ arbitrarily.
The characteristic function of $(X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}, \mathbb{1}_D)$ is

$$\begin{aligned}
\varphi(x_1,\ldots,x_n,y) &:= \mathbb{E}\,\mathrm{e}^{\mathrm{i}\left[\sum_{k=1}^n x_k(X_{t_k}-X_{t_{k-1}})+y\mathbb{1}_D\right]}\\
&= \mathbb{E}\,\mathrm{e}^{\mathrm{i}x_n(X_{t_n}-X_{t_{n-1}})+\mathrm{i}\left[\sum_{k=1}^{n-1} x_n(X_{t_k}-X_{t_{k-1}})+y\mathbb{1}_D\right]}\\
&\overset{X_{t_n}-X_{t_{n-1}}\perp\mathcal{F}^X_{t_{n-1}}}{=} \mathbb{E}\,\mathrm{e}^{\mathrm{i}x_n(X_{t_n}-X_{t_{n-1}})}\,\mathbb{E}\,\mathrm{e}^{\mathrm{i}\left[\sum_{k=1}^{n-1} x_n(X_{t_k}-X_{t_{k-1}})+y\mathbb{1}_D\right]}\\
&= \mathbb{E}\,\mathrm{e}^{\mathrm{i}x_n(X_{t_n}-X_{t_{n-1}})}\cdots\mathbb{E}\,\mathrm{e}^{\mathrm{i}x_2(X_{t_2}-X_{t_1})}\,\mathbb{E}\,\mathrm{e}^{\mathrm{i}x_1(X_{t_1}-X_{t_0})}\,\mathbb{E}\,\mathrm{e}^{\mathrm{i}y\mathbb{1}_D}\\
&= \mathbb{E}\,\mathrm{e}^{\mathrm{i}x_n(X_{t_n}-X_{t_{n-1}})}\cdots\mathbb{E}\,\mathrm{e}^{\mathrm{i}x_2(X_{t_2}-X_{t_1})+\mathrm{i}x_1(X_{t_1}-X_{t_0})}\,\mathbb{E}\,\mathrm{e}^{\mathrm{i}y\mathbb{1}_D}\\
&= \mathbb{E}\,\mathrm{e}^{\mathrm{i}\left[\sum_{k=1}^n x_k(X_{t_k}-X_{t_{k-1}})\right]}\,\mathbb{E}\,\mathrm{e}^{\mathrm{i}y\mathbb{1}_D}\\
&= \varphi_{Y_n}(x_1,\ldots,x_n)\varphi_{\mathbb{1}_D}(y),
\end{aligned}$$

where $\varphi_{Y_n}$ and $\varphi_{\mathbb{1}_D}$ is the characteristic function of $Y_n$ and $\mathbb{1}_D$ respectively. Thus we conclude that $Y_n$ and $\mathcal{F}^X_s$ are independent. □

**Problem 5.** Let $X$ have independent increments with $X_0 \equiv 0$. We prove that $X_t - X_s$ is independent of $\mathcal{F}^X_s$.

- We first observe that

$$\begin{aligned}
\mathcal{F}^X_s &= \sigma\{X_u : 0 \leqslant u \leqslant s\} = \sigma\{X_u - X_v : 0 \leqslant v < u \leqslant s\}\\
&= \sigma\Big\{(X_{u_n} - X_{u_{n-1}}, \ldots, X_{u_1} - X_{u_0}) : 0 \leqslant u_0 < u_1 < \cdots < u_n \leqslant s, \text{ for all } n\Big\}.
\end{aligned}$$

Define

$$\mathcal{A}_s := \bigcup_{n=1}^{\infty}\;\bigcup_{0\leqslant u_0<u_1<\cdots<u_n\leqslant s}\sigma\{X_{u_n} - X_{u_{n-1}}, \ldots, X_{u_1} - X_{u_0}\}.$$

It is clear that $\mathcal{A}$ is an algebra and $\mathcal{F}^X_s = \sigma(\mathcal{A}_s)$.

Since $X$ has independent increment, $X_t - X_s$ is independent of $\mathcal{A}_s$. Denote

$$\mathcal{G}_s = \{G \in \mathcal{F}^X_s : \mathbb{P}((X_t - X_s \in B) \cap G) = \mathbb{P}(X_t - X_s \in B)\mathbb{P}(G) \text{ for all } B \in \mathcal{B}(\mathbb{R})\}.$$

We have $\mathcal{A}_s \subset \mathcal{G}_s \subset \mathcal{F}^X_s$. It is easy to show that $\mathcal{G}_s$ is a monotone class (or a $\lambda$-class). Using the monotone class theorem (or $\pi$-$\lambda$ Dynkin's theorem, resp.) we conclude that $\mathcal{F}^X_s = \sigma(\mathcal{A}_s) \subset \mathcal{G}_s$. Therefore, $\mathcal{G}_s = \mathcal{F}^X_s$, and we conclude that $X_t - X_s$ is independent of $\mathcal{F}^X_s$.