Solutions for Demonstration 3

Problem 1. For each $k = 1, \dots, 4$, let $X^{(k)}$ be $\mathbb{F}$-adapted with $X^{(k)} = 0$. Define

$$\tau_k := \inf\{t > 0 : X_t^{(k)} > 1\}.$$

First, observe that for any $k = 1, \dots, 4$, the process $X^{(k)}$ is left-continuous or right-continuous, then for all $t > 0$ one has

$$\{\tau_k < t\} = \bigcup_{s<t}\{X_s^{(k)} > 1\} = \bigcup_{s<t, s\in\mathbb{Q}}\{X_s^{(k)} > 1\} \in \mathcal{F}_t.$$

Hence τ_k is an *optional time* w.r.t. $\mathbb{F}$.

We now check whether τ_k is a stopping time. For $t > 0$, since $\{\tau_k \leqslant t\} = \{\tau_k < t\} \cup \{\tau_k = t\}$ and $\{\tau_k < t\} \in \mathcal{F}_t$, it is equivalent to check whether $\{\tau_k = t\} \in \mathcal{F}_t$.

(a) The construction of paths of $X^{(1)}$ implies that τ_1 is the second jump time of $X^{(1)}$, which means

$$\{\tau_1 = t\} = \{X_t^{(1)} > 1\} = \{X_t^{(1)} = 2\} \in \mathcal{F}_t.$$

Hence τ_1 is a *stopping time* w.r.t $\mathbb{F}$.

(b) $X_t^{(2)} = \lim_{s\uparrow t} X_s^{(1)}$, $t > 0$.
Let ε be an random variable with $\mathbb{P}(\varepsilon = 1) = \mathbb{P}(\varepsilon = 0) = \frac{1}{2}$. Define

$$X_t^{(2)}(\omega) = \begin{cases} 0 & \text{if } t \in [0,1] \\ 1 & \text{if } t \in (1,2] \\ \mathbb{1}_{\{\varepsilon=0\}}(\omega) + 2\mathbb{1}_{\{\varepsilon=1\}}(\omega) & \text{if } t > 2. \end{cases}$$

Let $\mathcal{F}_t^{(2)} = \sigma\{X_s^{(2)} : s \leqslant t\}$. Then we have $\mathcal{F}_2^{(2)} = \{\emptyset, \Omega\}$, however

$$\{\tau_2 = 2\} = \{\varepsilon = 1\} \notin \mathcal{F}_2^{(2)}.$$

Hence τ_2 is *not* a stopping time.

(c) Consider the random variable ε as in (b). Define the following continuous process

$$X_t^{(3)}(\omega) = \begin{cases} t & \text{if } t \in [0,1] \\ \mathbb{1}_{\{\varepsilon=0\}}(\omega) + t\mathbb{1}_{\{\varepsilon=1\}}(\omega) & \text{if } t > 1. \end{cases}$$

Let $\mathcal{F}_t^{(3)} = \sigma\{X_s^{(3)} : s \leqslant t\}$. Then we have $\mathcal{F}_1^{(3)} = \{\emptyset, \Omega\}$, however

$$\{\tau_3 = 1\} = \{\varepsilon = 1\} \notin \mathcal{F}_1^{(3)}.$$

Hence τ_3 is *not* a stopping time.

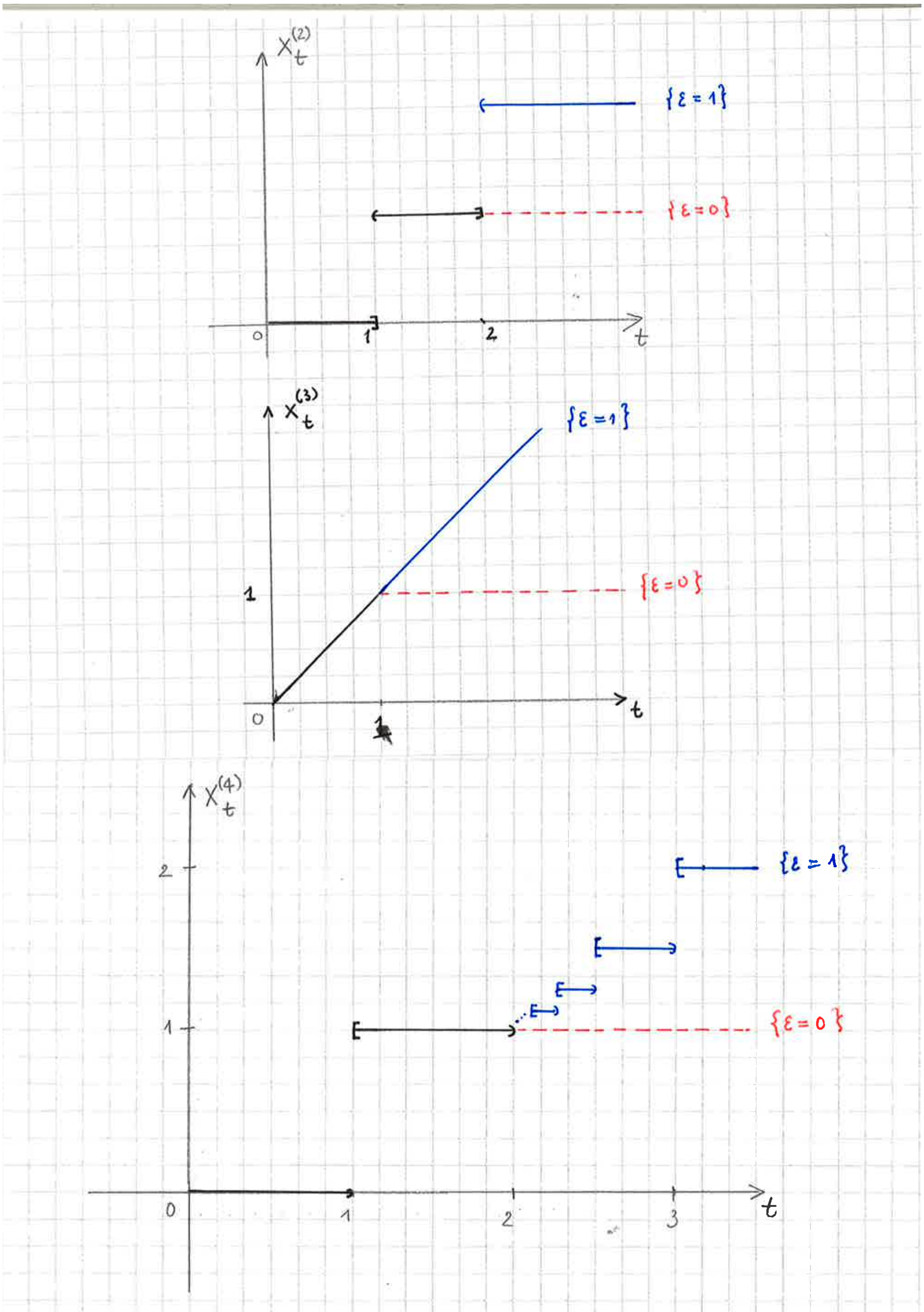
(d) Consider the random variable ε as in (b). Define

$$X_t^{(4)}(\omega) = \begin{cases} 0 & \text{if } t \in [0,1) \\ 1 & \text{if } t \in [1,2] \\ \mathbb{1}_{\{\varepsilon=0\}}(\omega) + \mathbb{1}_{\{\varepsilon=1\}}(\omega)\left[1 + \sum_{n=0}^{\infty} \frac{1}{2^{n+1}}\chi_{[2+\frac{1}{2^{n+1}}, 2+\frac{1}{2^n})}(t) + \chi_{[3,\infty)}(t)\right] & \text{if } t > 2. \end{cases}$$

Notice that $X_t^{(4)}$ is continuous at $t = 2$.
Let $\mathcal{F}_t^{(4)} = \sigma\{X_s^{(4)} : s \leqslant t\}$. Then we have $\mathcal{F}_2^{(4)} = \{\emptyset, \Omega\}$, however

$$\{\tau_4 = 2\} = \{\varepsilon = 1\} \notin \mathcal{F}_2^{(4)}.$$

Hence τ_4 is *not* a stopping time.


$X_t^{(2)}$
{ε = 1}
{ε = 0}
0
1
2
t
$X_t^{(3)}$
{ε = 1}
1
{ε = 0}
0
1
t
$X_t^{(4)}$
2
{ε = 1}
1
{ε = 0}
0
1
2
3
t

Problem 2. We show that

$$\{A \in \mathcal{F} : A \cap \{\tau \leqslant t\} \in \mathcal{F}_{t+} \quad \forall t \in [0,\infty)\} = \{A \in \mathcal{F} : A \cap \{\tau < t\} \in \mathcal{F}_t \quad \forall t \in [0,\infty)\}.$$

"$\subset$": Let $A \in \mathcal{F}$ be such that $A \cap \{\tau \leqslant t\} \in \mathcal{F}_{t+}$ for all $t \in [0,\infty)$. Then $A \cap \{\tau < 0\} = \emptyset \in \mathcal{F}_0$ and for $t \in (0,\infty)$,

$$A \cap \{\tau < t\} = \cup_{n>\frac{1}{t}}^{\infty} \underbrace{\left(A \cap \{\tau \leqslant t - \tfrac{1}{n}\}\right)}_{\in \mathcal{F}_{(t-\frac{1}{n})+} \subset \mathcal{F}_t} \in \mathcal{F}_t.$$

"$\supset$": Let $A \in \mathcal{F}$ be such that $A \cap \{\tau < t\} \in \mathcal{F}_t$, $\forall t \in [0,\infty)$. Then we have

$$A \cap \{\tau \leqslant t\} = \cup_{n=N}^{\infty} \left(A \cap \{\tau < t + \tfrac{1}{n}\}\right) \in \mathcal{F}_{t+\frac{1}{N}}, \quad \forall N \in \mathbb{N}.$$

Hence $A \cap \{\tau \leqslant t\} \in \cap_{N=1}^{\infty} \mathcal{F}_{t+\frac{1}{N}} = \mathcal{F}_{t+}$. □

Problem 3. We have

$$\bigcap_{s>t} \mathcal{F}_s = \bigcap_{n=1}^{\infty} \bigcap_{s \geqslant t+\frac{1}{n}} \mathcal{F}_s = \bigcap_{n=1}^{\infty} \mathcal{F}_{t+\frac{1}{n}}.$$

Problem 4. Let τ and η be stopping times. Then $\tau \wedge \eta$ is a stopping times and we prove that $\mathcal{F}_\tau \cap \mathcal{F}_\eta = \mathcal{F}_{\tau\wedge\eta}$.

"$\subset$": Let $A \in \mathcal{F}_\tau \cap \mathcal{F}_\eta$, which means $A \cap \{\tau \leqslant t\} \in \mathcal{F}_t$ and $A \cap \{\eta \leqslant t\} \in \mathcal{F}_t$ for all t. Then

$$A \cap \{\tau \wedge \eta \leqslant t\} = A \cap (\{\tau \leqslant t\} \cup \{\eta \leqslant t\}) = (A \cap \{\tau \leqslant t\}) \cup (A \cap \{\eta \leqslant t\}) \in \mathcal{F}_t.$$

"$\supset$": Let $A \cap \{\tau \wedge \eta \leqslant t\} \in \mathcal{F}_t$ for all t. We have

$$A \cap \{\tau \leqslant t\} = A \cap (\{\tau \wedge \eta \leqslant t\} \cap \{\tau \leqslant t\}) = (A \cap \{\tau \wedge \eta \leqslant t\}) \cap \{\tau \leqslant t\} \in \mathcal{F}_t.$$

Similarly, $A \cap \{\eta \leqslant t\} \in \mathcal{F}_t$ for all t. Hence $A \in \mathcal{F}_\tau \cap \mathcal{F}_\eta$. □