Solutions for Demonstration 4

Problem 1. Let $C[0,\infty)$ be the family of continuous functions $f\colon [0,\infty) \to \mathbb{R}$. Define

$$\mathcal{F}_t := \sigma\{\{g \in C[0,\infty) : g(s) \in B\} : s \in [0,t], B \in \mathcal{B}(\mathbb{R})\}.$$

Fix $t > 0$. We show the strict inclusion $\mathcal{F}_t \subsetneq \mathcal{F}_{t+}$. Define

$$A_t := \{g \in C[0,\infty) : g \text{ attains a local maximum at } t\}.$$

Since each element of A_t is a continuous function, it implies for all $N \in \mathbb{N}$ that

$$A_t := \bigcup_{n=N}^{\infty} \bigcap_{s \in (t-\frac{1}{n}, t+\frac{1}{n}) \cap \mathbb{Q}} \{g \in C[0,\infty) : g(t) \geqslant g(s)\} \in \mathcal{F}_{t+\frac{1}{N}}.$$

Hence

$$A_t \in \cap_{N=1}^{\infty} \mathcal{F}_{t+\frac{1}{N}} = \mathcal{F}_{t+}.$$

In order to show $A_t \notin \mathcal{F}_t$, we use the Fact: "*Let $f_1, f_2 \in C[0,\infty)$ such that $f_1(s) = f_2(s)$ for all $s \in [0,t]$. If $E \in \mathcal{F}_t$ and $f_1 \in E$, then $f_2 \in E$*".

Consider $g_1(s) = s$ and

$$g_2(s) = \begin{cases} s & \text{if } s \leqslant t \\ 2t - s & \text{if } s > t. \end{cases}$$

It is clear that $g_2 \in A_t$ and $g_1(s) = g_2(s)$ for all $s \in [0,t]$. If $A_t \in \mathcal{F}_t$, then it follows from the Fact above that $g_1 \in A_t$, which leads to the contradiction. Therefore, $A_t \notin \mathcal{F}_t$.

Prove the Fact: Define

$$\mathcal{E}_t := \{E \in \mathcal{F}_t : \text{if } f \in E \text{ and } \hat{f} \in C[0,\infty) \text{ with } f(s) = \hat{f}(s)\ \forall s \in [0,t], \text{ then } \hat{f} \in E\}.$$

We prove $\mathcal{E}_t = \mathcal{F}_t$ by showing that $\mathcal{E}_t$ is a σ-algebra and

$$\{g \in C[0,\infty) : g(s) \in B\} \in \mathcal{E}_t, \quad \forall s \in [0,t], B \in \mathcal{B}(\mathbb{R}).$$

- It is clear that $C[0,\infty) \in \mathcal{E}_t$.
- Let $E \in \mathcal{E}_t$. Consider $f \in C[0,\infty)\backslash E$ and $\hat{f} \in C[0,\infty)$ with $f(s) = \hat{f}(s)$ for all $s \in [0,t]$. If $\hat{f} \in E$, then the definition of $\mathcal{E}_t$ implies that $f \in E$, which is the contradiction. Hence $\hat{f} \in C[0,\infty)\backslash E$, which means $C[0,\infty)\backslash E \in \mathcal{E}_t$.
- Let $E_n \in \mathcal{E}_t$, $n \geqslant 1$. Let $f \in \cup_{n\geqslant 1} E_n$ and $\hat{f} \in C[0,\infty)$ with $f(s) = \hat{f}(s)$ for all $s \in [0,t]$. Then there exists n_0 such that $f \in E_{n_0}$. Since $E_{n_0} \in \mathcal{E}_t$, it asserts that $\hat{f} \in E_{n_0}$. Hence $\cup_{n\geqslant 1} E_n \in \mathcal{E}_t$.
- Let $s \in [0,t]$ and $B \in \mathcal{B}(\mathbb{R})$. Set $G := \{g \in C[0,\infty) : g(s) \in B\}$. If $f \in G$ and $\hat{f} \in C[0,\infty)$ such that $f(u) = \hat{f}(u)$ for all $u \in [0,t]$, then we have $\hat{f}(s) = f(s) \in B$, which means $\hat{f} \in G$. Thus $G \in \mathcal{E}_t$. □

Problem 2. Let $\bar{W}$ be a *continuous* modification of W. For any bounded Borel f, we have

$$\int_{\mathbb{R}} f(y)P_t(x,dy) = \int_{\mathbb{R}} f(y)\mathbb{P}(W_t + x \in dy) = \int_{\mathbb{R}} f(y)\mathbb{P}_{W_t+x}(dy) = \mathbb{E}f(W_t + x),$$

where $\mathbb{P}_{W_t+x}$ denote the image measure of $\mathbb{P}$ via $W_t + x$. Since $\bar{W}$ is a modification of W, we get $\bar{W}_t = W_t$ in distribution, and hence

$$\int_{\mathbb{R}} f(y)P_t(x,dy) = \mathbb{E}f(\bar{W}_t + x).$$

(a) For $f = \sin$, we have that, for all $x \in \mathbb{R}$,

$$\lim_{t\downarrow 0}\int_{\mathbb{R}} \sin(y)P_t(x,dy) = \lim_{t\downarrow 0}\mathbb{E}\sin(\bar{W}_t + x) = \mathbb{E}\lim_{t\downarrow 0}\sin(\bar{W}_t + x) = \sin x = f(x),$$

where one applies the dominated convergence theorem to get the second equality.

(b) For $f = \mathbb{1}_{(-\infty,0]}$, by choosing $x = 0$ we have

$$\lim_{t\downarrow 0}\int_{\mathbb{R}} \mathbb{1}_{(-\infty,0]}(y)P_t(0,dy) = \lim_{t\downarrow 0}\mathbb{E}\mathbb{1}_{\{W_t\leqslant 0\}} = \lim_{t\downarrow 0}\mathbb{P}(W_t \leqslant 0) = \frac{1}{2},$$

while $f(0) = 1$. Hence $\lim_{t\downarrow 0}\int_{\mathbb{R}} \mathbb{1}_{(-\infty,0]}(y)P_t(0,dy) \neq f(0)$. □

Problem 3. Recall that $\mathcal{F}_t^{\mathbb{P}} = \mathcal{F}_t^X \vee \mathcal{N}^{\mathbb{P}}$. Define

$$\mathcal{G}_t := \{G \subseteq \Omega : \exists H \in \mathcal{F}_t^X : H\Delta G \in \mathcal{N}^{\mathbb{P}}\}.$$

First, we show that $\mathcal{G}_t$ is a σ-algebra. Indeed,

- $\Omega \in \mathcal{G}_t$: clear;
- Let $G \in \mathcal{G}_t$. Then there is an $H \in \mathcal{F}_t^X$ such that $H\Delta G \in \mathcal{N}^{\mathbb{P}}$. Noticing that $H\Delta G = (\Omega\backslash G)\Delta(\Omega\backslash H)$ and $\Omega\backslash H \in \mathcal{F}_t^X$, we obtain that $\Omega\backslash G \in \mathcal{G}_t$.
- Let $(G_n)_{n\geqslant 1} \subset \mathcal{G}_t$. There exists the corresponding $(H_n)_{n\geqslant 1} \subset \mathcal{F}_t^X$ such that $G_n\Delta H_n \in \mathcal{N}^{\mathbb{P}}$. Now,

  $$\Big(\cup_{n\geqslant 1} G_n\Big)\Delta\Big(\cup_{n\geqslant 1} H_n\Big) \subseteq \cup_{n\geqslant 1}(G_n\Delta H_n) \in \mathcal{N}^{\mathbb{P}}.$$

  By the definition of $\mathcal{N}^{\mathbb{P}}$, we get that $\Big(\cup_{n\geqslant 1} G_n\Big)\Delta\Big(\cup_{n\geqslant 1} H_n\Big) \in \mathcal{N}^{\mathbb{P}}$. Since $\cup_{n\geqslant 1}H_n \in \mathcal{F}_t^X$, it implies $\cup_{n\geqslant 1}G_n \in \mathcal{G}_t$.

Next, we prove that $\mathcal{F}_t^{\mathbb{P}} = \mathcal{G}_t$.

"$\subseteq$": This direction is clear because $\mathcal{F}_t^X \subseteq \mathcal{G}_t$ and $\mathcal{N}^{\mathbb{P}} \subseteq \mathcal{G}_t$.

"$\supseteq$": Let $G \in \mathcal{G}_t$. Then there is an $H \in \mathcal{F}_t^X$ such that $F := G\Delta H \in \mathcal{N}^{\mathbb{P}}$. Since $G = H\Delta F$, we conclude that $G \in \mathcal{F}_t^{\mathbb{P}}$. Hence $\mathcal{G}_t \subseteq \mathcal{F}_t^{\mathbb{P}}$. □

Problem 4. Let $X = (X_t)_{t\geqslant 0}$ be a d-dimensional Lévy process in law. Define

$$f_t(u) := \mathbb{E}\,\mathrm{e}^{\mathrm{i}\langle u,X_t\rangle}, \quad u \in \mathbb{R}^d, t \geqslant 0.$$

(a) For all $u \in \mathbb{R}^d$, $f_0(u) = \mathbb{E}\,\mathrm{e}^{\mathrm{i}\langle u,X_0\rangle} = 1$ a.s.

(b) For all $u \in \mathbb{R}^d$, $s,t \geqslant 0$,

$$\begin{aligned} f_{t+s}(u) &= \mathbb{E}\,\mathrm{e}^{\mathrm{i}\langle u,X_{t+s}\rangle} = \mathbb{E}\,\mathrm{e}^{\mathrm{i}\langle u,X_{t+s}-X_t\rangle+\mathrm{i}\langle u,X_t\rangle} \\ &\overset{X_{t+s}-X_t\perp X_t}{=} \mathbb{E}\,\mathrm{e}^{\mathrm{i}\langle u,X_{t+s}-X_t\rangle}\,\mathbb{E}\,\mathrm{e}^{\mathrm{i}\langle u,X_t\rangle} \overset{X_{t+s}-X_t\sim X_s}{=} \mathbb{E}\,\mathrm{e}^{\mathrm{i}\langle u,X_s\rangle}\,\mathbb{E}\,\mathrm{e}^{\mathrm{i}\langle u,X_t\rangle} \\ &= f_t(u)f_s(u). \end{aligned}$$

(c) Let $u \in \mathbb{R}^d$ and $t \geqslant 0$. For any $n \in \mathbb{N}$, applying (b) we get

$$f_t(u) = \left(f_{\frac{t}{n}}(u)\right)^n.$$

If $f_t(u) = 0$, then $f_{t/n}(u) = 0$ for all n. Since $X_{t/n} \to X_0 = 0$ as $n \to \infty$ in probability, the dominated convergence theorem implies that

$$\lim_{n\to\infty} f_{\frac{t}{n}}(u) = \lim_{n\to\infty} \mathbb{E}\,\mathrm{e}^{\mathrm{i}\langle u, X_{t/n}\rangle} = \mathbb{E} \lim_{n\to\infty} \mathrm{e}^{\mathrm{i}\langle u, X_{t/n}\rangle} = f_0(u) = 1,$$

which leads to a contradiction. Thus $f_t(u) \neq 0$ for all u, t. □

Problem 5. Let X be a d-dimensional Lévy process in law. Fix $\theta \in \mathbb{R}^d$. Define

$$Z_t := \frac{\mathrm{e}^{\mathrm{i}\langle \theta, X_t\rangle}}{\mathbb{E}\,\mathrm{e}^{\mathrm{i}\langle \theta, X_t\rangle}}.$$

We show that $(Z_t)_{t\geqslant 0}$ is a martingale w.r.t. $(\mathcal{F}_t^X)_{t\geqslant 0}$.

- Z_t is $\mathcal{F}_t^X$-measurable: clear.
- For $t \geqslant 0$, due to problem 4(c) we have

$$\mathbb{E}|Z_t| = \mathbb{E}\left|\frac{1}{\mathbb{E}\,\mathrm{e}^{\mathrm{i}\langle \theta, X_t\rangle}}\right| = \frac{1}{|f_t(\theta)|} < \infty.$$

- For $0 \leqslant s \leqslant t$,

$$\mathbb{E}[Z_t|\mathcal{F}_s^X] = \mathbb{E}\left[\frac{\mathrm{e}^{\mathrm{i}\langle \theta, X_t - X_s\rangle}\,\mathrm{e}^{\mathrm{i}\langle \theta, X_s\rangle}}{\mathbb{E}\,\mathrm{e}^{\mathrm{i}\langle \theta, X_t - X_s\rangle}\,\mathbb{E}\,\mathrm{e}^{\mathrm{i}\langle \theta, X_s\rangle}}\middle|\mathcal{F}_s^X\right] = \frac{\mathrm{e}^{\mathrm{i}\langle \theta, X_s\rangle}}{\mathbb{E}\,\mathrm{e}^{\mathrm{i}\langle \theta, X_s\rangle}} = Z_s \quad a.s.$$