Solutions for Demonstration 5

Problem 1. Assume that $Y = (Y_1, \ldots, Y_d)$ and

$$\mathbb{E}\left(e^{i(x_1Y_1+\cdots+x_dY_d)}\,\mathbb{1}_G\right) = \mathbb{P}(G)\mathbb{E}\,e^{i(x_1Y_1+\cdots+x_dY_d)}, \quad \forall x \in \mathbb{R}^d, G \in \mathcal{G}.$$

We prove that Y and $\mathcal{G}$ are independent. Indeed, for any $G \in \mathcal{G}$ and any $x_i, y \in \mathbb{R}$ we have

$$\begin{aligned}
\mathbb{E}\,e^{i(x_1Y_1+\cdots+x_dY_d+y\mathbb{1}_G)} &= \mathbb{E}\left(\mathbb{1}_G\,e^{i(x_1Y_1+\cdots+x_dY_d+y\mathbb{1}_G)}\right) + \mathbb{E}\left(\mathbb{1}_{\Omega\setminus G}\,e^{i(x_1Y_1+\cdots+x_dY_d+y\mathbb{1}_G)}\right)\\
&= \mathbb{E}\left(\mathbb{1}_G\,e^{i(x_1Y_1+\cdots+x_dY_d+y)}\right) + \mathbb{E}\left(\mathbb{1}_{\Omega\setminus G}\,e^{i(x_1Y_1+\cdots+x_dY_d)}\right)\\
&= e^{iy}\,\mathbb{P}(G)\mathbb{E}\,e^{i(x_1Y_1+\cdots+x_dY_d)} + \mathbb{P}(\Omega\setminus G)\mathbb{E}\,e^{i(x_1Y_1+\cdots+x_dY_d)}\\
&= \mathbb{E}\,e^{i(x_1Y_1+\cdots+x_dY_d)}\,\mathbb{E}\,e^{iy\mathbb{1}_G},
\end{aligned}$$

which implies that Y and $\mathcal{G}$ are independent.

Alternative proof: We use Taylor's expansion

$$\mathbb{E}\left(e^{i(x_1Y_1+\cdots+x_dY_d}\,e^{iy\mathbb{1}_G}\right) = \mathbb{E}\left(e^{i(x_1Y_1+\cdots+x_dY_d)}\sum_{n=0}^{\infty}\frac{(iy\mathbb{1}_G)^n}{n!}\right)$$

$$\overset{DCT}{=} \mathbb{E}\left(e^{i(x_1Y_1+\cdots+x_dY_d)}\right) + \mathbb{E}\left(e^{i(x_1Y_1+\cdots+x_dY_d)}\,\mathbb{1}_G\right)\sum_{n=1}^{\infty}\frac{(iy)^n}{n!} = \mathbb{E}\,e^{iy\mathbb{1}_G}\,\mathbb{E}\,e^{i(x_1Y_1+\cdots+x_dY_d)}.$$

Problem 2. Let $E := \mathbb{N}$ and $2^{\mathbb{N}} := \mathcal{B}(E)$. Assume that $(N_t)_{t\geqslant 0}$ is a Poisson process with parameter $\lambda > 0$ and

$$T(t)f(x) = \mathbb{E}f(N_t + x), \quad f \in B_b(E),$$

where $B_b(E)$ denotes the family of bounded Borel measurable functions on E.

(a) We show that the infinitesimal generator A of $(T(t))_{t\geqslant 0}$ is

$$Af(x) = \lambda(f(x+1) - f(x)) \quad \text{for } f \in D(A) = B_b(E).$$

First, observe that $Af \in B_b(E)$ whenever $f \in B_b(E)$. Next, we need to prove

$$\lim_{t\downarrow 0}\left\|\frac{T(t)f - f}{t} - Af\right\|_{B_b(E)} = \lim_{t\downarrow 0}\sup_{x\in E}\left|\frac{T(t)f(x) - f(x)}{t} - Af(x)\right| = 0. \tag{1}$$

For any $f \in B_b(E)$ and $x \in E$,

$$\begin{aligned}
&\left|\frac{T(t)f(x) - f(x)}{t} - Af(x)\right| = \left|\frac{\mathbb{E}f(N_t + x) - f(x)}{t} - \lambda(f(x+1) - f(x))\right|\\
&= \frac{1}{t}\left|\mathbb{E}\left[f(N_t + x) - f(x)\right] - \lambda t(f(x+1) - f(x))\right|\\
&= \frac{1}{t}\left|\left(e^{-\lambda t}\sum_{n=0}^{\infty}\left[f(n+x) - f(x)\right]\frac{(\lambda t)^n}{n!}\right) - \lambda t(f(x+1) - f(x))\right|\\
&= \frac{1}{t}\left|\lambda t(e^{-\lambda t} - 1)(f(x+1) - f(x)) + e^{-\lambda t}\sum_{n=2}^{\infty}\left[f(n+x) - f(x)\right]\frac{(\lambda t)^n}{n!}\right|\\
&\leqslant \lambda|e^{-\lambda t} - 1||f(x+1) - f(x)| + \left|e^{-\lambda t}\sum_{n=2}^{\infty}\left[f(n+x) - f(x)\right]\frac{\lambda^n t^{n-1}}{n!}\right|
\end{aligned}$$

$$\leqslant 2\|f\|_{B_b(E)}\lambda|\mathrm{e}^{-\lambda t}-1| + 2\|f\|_{B_b(E)}\mathrm{e}^{-\lambda t}\sum_{n=2}^{\infty}\frac{\lambda^n t^{n-1}}{n!}$$
$$\leqslant 2\|f\|_{B_b(E)}\lambda|\mathrm{e}^{-\lambda t}-1| + 2\|f\|_{B_b(E)}\lambda^2 t.$$

Then

$$\limsup_{t\downarrow 0}\sup_{x\in E}\left|\frac{T(t)f(x)-f(x)}{t} - Af(x)\right| \leqslant \lim_{t\downarrow 0}\left(\|f\|_{B_b(E)}\lambda|\mathrm{e}^{-\lambda t}-1| + 2\|f\|_{B_b(E)}\lambda^2 t\right) = 0$$

which verifies (1).

(b) In the case of compound Poisson process $(X_t)_{t\geqslant 0}$ where $X_t = \sum_{k=1}^{N_t} Y_k$ and the $\mathbb{R}^d$-valued iid (Y_k) is independent from (N_t).
Let $E := \mathbb{R}^d$. We define the measure

$$\nu := \lambda \mathbb{P}_{Y_1},$$

then we can prove by using the same arguments as above that the infinitesimal generator of $(T(t))_{t\geqslant 0}$ associated with $(X_t)_{t\geqslant 0}$, $T(t)f(x) = \mathbb{E}f(X_t + x)$, $x \in \mathbb{R}^d$, $f \in B_b(\mathbb{R}^d)$, is

$$Af(x) = \int_{\mathbb{R}^d}(f(x+y) - f(x))\nu(dy), \quad x \in \mathbb{R}^d, f \in B_b(\mathbb{R}^d).$$

Indeed, we have for $B \in \mathcal{B}(\mathbb{R}^d)$ that

$$\mathbb{P}_{X_t}(B) = \mathbb{P}(X_t \in B) = \mathrm{e}^{-\lambda t}\delta_0(B) + \sum_{k=1}^{\infty}\mathbb{P}\left(\sum_{i=1}^{k} Y_i \in B\right)\mathbb{P}(N_t = k)$$
$$= \mathrm{e}^{-t\nu(\mathbb{R}^d)}\left(\delta_0(B) + \sum_{k=1}^{\infty}\mathbb{P}_{Y_1}^{*k}(B)\frac{(\lambda t)^k}{k!}\right) = \mathrm{e}^{-t\nu(\mathbb{R}^d)}\sum_{k=0}^{\infty}\frac{t^k}{k!}\nu^{*k}(B).$$

We now show that $Af(x) = \int_{\mathbb{R}^d}(f(y+x) - f(x))\nu(dy)$ for $f \in B_b(\mathbb{R}^d)$. Indeed, since

$$T(t)f(x) = \mathbb{E}f(X_t + x) = \int_{\mathbb{R}^d} f(y+x)\mathbb{P}_{X_t}(dy) = \mathrm{e}^{-t\nu(\mathbb{R}^d)}\sum_{k=0}^{\infty}\frac{t^k}{k!}\int_{\mathbb{R}^d} f(y+x)\nu^{*k}(dy)$$

one has

$$\frac{T(t)f(x)-f(x)}{t} = \frac{1}{t}\mathrm{e}^{-t\nu(\mathbb{R}^d)}\sum_{k=0}^{\infty}\frac{t^k}{k!}\int_{\mathbb{R}^d}(f(y+x) - f(x))\nu^{*k}(dy)$$
$$= \mathrm{e}^{-t\nu(\mathbb{R}^d)}\int_{\mathbb{R}^d}(f(y+x) - f(x))\nu(dy) + \frac{1}{t}\mathrm{e}^{-t\nu(\mathbb{R}^d)}\sum_{n=2}^{\infty}\frac{t^k}{k!}\int_{\mathbb{R}^d}(f(y+x) - f(x))\nu^{*k}(dy).$$

Thus, with $\lambda = \nu(\mathbb{R}^d)$ we obtain

$$\sup_{x\in\mathbb{R}^d}\left|\frac{T(t)f(x)-f(x)}{t} - \int_{\mathbb{R}^d}(f(y+x) - f(x))\nu(dy)\right|$$
$$\leqslant 2\lambda|\mathrm{e}^{-\lambda t}-1|\|f\|_{B_b(\mathbb{R}^d)} + 2\|f\|_{B_b(\mathbb{R}^d)}\left|\frac{1}{t}\sum_{k=2}^{\infty}\mathrm{e}^{-\lambda t}\frac{(\lambda t)^k}{k!}\right|$$
$$\leqslant 2\lambda|\mathrm{e}^{-\lambda t}-1|\|f\|_{B_b(\mathbb{R}^d)} + 2\|f\|_{B_b(\mathbb{R}^d)}\mathrm{e}^{-\lambda t}\lambda^2 t\sum_{k=0}^{\infty}\frac{(\lambda t)^k}{(k+2)!}$$
$$\leqslant 2\lambda|\mathrm{e}^{-\lambda t}-1|\|f\|_{B_b(\mathbb{R}^d)} + 2\|f\|_{B_b(\mathbb{R}^d)}\lambda^2 t$$
$$\to 0 \text{ as } t \downarrow 0.$$

Therefore $Af(x) = \int_{\mathbb{R}^d}(f(y+x) - f(x))\nu(dy)$ and $D(A) \supseteq B_b(\mathbb{R}^d)$. □

Problem 3. (a) $(N_t)_{t\geqslant 0}$: Poisson process with intensity $\lambda > 0$. The characteristic function of N_t is

$$\varphi_{N_t}(u) = \mathbb{E}\,\mathrm{e}^{\mathrm{i}uN_t} = \mathrm{e}^{\lambda t(\mathrm{e}^{\mathrm{i}u}-1)}, \quad u \in \mathbb{R}.$$

Let $T(t)f(x) = \mathbb{E}f(N_t + x)$ and $f_u(x) = \mathrm{e}^{iux}$ for $u, x \in \mathbb{R}$. We have

$$\frac{T(t)f_u(x) - f_u(x)}{t} = \frac{\mathbb{E}f_u(N_t + x) - f_u(x)}{t} = \frac{\mathbb{E}\,\mathrm{e}^{\mathrm{i}u(N_t+x)} - \mathrm{e}^{\mathrm{i}ux}}{t} = \frac{\mathrm{e}^{\mathrm{i}ux}\,\mathrm{e}^{\lambda t(\mathrm{e}^{\mathrm{i}u}-1)}}{t}$$
$$\to \lambda\,\mathrm{e}^{\mathrm{i}ux}(\mathrm{e}^{\mathrm{i}u}-1) = \lambda(\mathrm{e}^{\mathrm{i}u(x+1)} - \mathrm{e}^{\mathrm{i}ux}) = \lambda(f_u(x+1) - f_u(x)) \text{ as } t \downarrow 0.$$

Moreover, by the same arguments as in Problem 2, one can show that

$$\lim_{t\downarrow 0} \sup_{x\in\mathbb{R}} \left| \frac{T(t)f_u(x) - f_u(x)}{t} - \lambda(f_u(x+1) - f_u(x)) \right| = 0$$

Hence,

$$\lim_{t\downarrow 0} \frac{T(t)f_u - f_u}{t} = Af_u.$$

(b) Let W be a standard Brownian motion and $f_u(x) = \mathrm{e}^{\mathrm{i}ux}$ for $x, u \in \mathbb{R}$. We have

$$\frac{T(t)f_u(x) - f_u(x)}{t} = \frac{\mathbb{E}f_u(W_t + x) - f(x)}{t} = \frac{(\mathbb{E}\,\mathrm{e}^{\mathrm{i}uW_t}-1)\,\mathrm{e}^{\mathrm{i}ux}}{t} = \frac{\mathrm{e}^{\mathrm{i}ux}(\mathrm{e}^{-\frac{tu^2}{2}}-1)}{t}.$$

Hence

$$\lim_{t\downarrow 0} \sup_{x\in\mathbb{R}} \left| \frac{T(t)f_u - f_u}{t} - Af_u \right| = \lim_{t\downarrow 0} \sup_{x\in\mathbb{R}} \left| \frac{T(t)f_u - f_u}{t} - \frac{1}{2}\frac{d^2 f_u}{dx^2} \right|$$
$$= \lim_{t\downarrow 0} \sup_{x\in\mathbb{R}} \left| \frac{\mathrm{e}^{\mathrm{i}ux}(\mathrm{e}^{-\frac{tu^2}{2}}-1)}{t} + \frac{u^2}{2}\,\mathrm{e}^{\mathrm{i}ux} \right| \leqslant \lim_{t\downarrow 0} \left| \frac{\mathrm{e}^{-\frac{tu^2}{2}}-1}{t} + \frac{u^2}{2} \right| = 0.$$

Problem 4. Let $x \in \mathbb{R}$ and assume that X^x solves the following SDE

$$X_t^x = x + \int_0^t a(X_s^x)ds + \int_0^t \sigma(X_s^x)dB_s,$$

where a, σ are Lipschitz and bounded functions. For $f \in C_c^2(\mathbb{R})$ we define

$$Af(x) := a(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x)$$

and let $T(t)f(x) := \mathbb{E}f(X_t^x)$. Applying Itô's formula gives

$$f(X_t^x) = f(X_0^x) + \int_0^t f'(X_s^x)a(X_s^x)ds + \int_0^t f'(X_s^x)\sigma(X_s^x)dB_s + \frac{1}{2}\int_0^t f''(X_s^x)\sigma^2(X_s^x)ds$$
$$= f(x) + \int_0^t Af(X_s^x)ds + \int_0^t f'(X_s^x)\sigma(X_s^x)dB_s.$$

Since f has compact support and σ is bounded, it implies that $f'\sigma$ is bounded. Hence $\int_0^\cdot f'(X_s^x)\sigma(X_s^x)dB_s$ is a martingale with zero mean. Then

$$T(t)f(x) = \mathbb{E}f(X_t^x) = f(x) + \mathbb{E}\int_0^t Af(X_s^x)ds = f(x) + \int_0^t \mathbb{E}Af(X_s^x)ds,$$

where in the last equality we may interchange the expectation and the integral because of Fubini's theorem and boundedness of Af. □