Solutions for Demonstration 6

Problem 1. Let $f\colon [0,\infty) \to \mathbb{R}$ be a càdlàg function and let $T > 0$. Then, there exists a partition $0 = t_0 < t_1 < \cdots < t_n = T$ such that $\max_{1\leqslant k\leqslant n} \sup_{s,t\in[t_{k-1},t_k)} |f(s) - f(t)| < 1$. Then we have

$$\begin{aligned}\sup_{0\leqslant t\leqslant T} |f(t)| &= \max\left\{ \max_{1\leqslant k\leqslant n} \sup_{[t_{k-1},t_k)} |f(t)|, |f(T)|\right\}\\ &\leqslant |f(T)| + \sum_{1\leqslant k\leqslant n} \sup_{s,t\in[t_{k-1},t_k)} |f(t) - f(s)| + \sum_{1\leqslant k\leqslant n} |f(t_{k-1})| < \infty.\end{aligned}$$

Problem 2. Let X be a Lévy process in law and $T_t f(x) = \mathbb{E} f(x + X_t)$, $f \in C_0(\mathbb{R}^d)$. We will show that $(T_t)_{t\geqslant 0}$ is a conservative Feller semi-group.

(a) Since any $f \in C_0(\mathbb{R}^d)$ is bounded, the DCT asserts the continuity of $x \mapsto T_t f(x)$ on $\mathbb{R}^d$. Moreover, $f(x + X_t) \to 0$ as $|x| \to \infty$ and the DCT again implies $T_t f(x) \to 0$ as $|x| \to \infty$. Hence $T_t f \in C_0(\mathbb{R}^d)$ whenever $f \in C_0(\mathbb{R}^d)$.

(b) Fix $t \geqslant 0$ and $x \in \mathbb{R}^d$. We show that $\sup_{f\in C_0(\mathbb{R}^d),\, 0\leqslant f\leqslant 1} T_t f(x) = 1$. First, it is clear that

$$\sup_{f\in C_0(\mathbb{R}^d),\, 0\leqslant f\leqslant 1} T_t f(x) \leqslant 1. \tag{1}$$

Let $\varepsilon \in (0, 1)$. Since $\mathbb{P}(|x + X_t| > n) \to 0$ as $n \to \infty$, there is an $n_\varepsilon \in \mathbb{N}$ such that $\mathbb{P}(|x + X_t| \leqslant n_\varepsilon) \geqslant 1 - \varepsilon$. We consider a continuous function $f_\varepsilon \in C_0(\mathbb{R}^d)$ which is equal to 1 for all $|x| \leqslant n_\varepsilon$, is 0 for all $|x| > n_\varepsilon + 1$ and $0 \leqslant f_\varepsilon \leqslant 1$. Then

$$T_t f_\varepsilon(x) = \mathbb{E} f_\varepsilon(x + X_t) \geqslant \mathbb{E} f_\varepsilon(x + X_t)\mathbb{1}_{\{|x+X_t|\leqslant n_\varepsilon\}} = \mathbb{P}(|x + X_t| \leqslant n_\varepsilon) \geqslant 1 - \varepsilon.$$

Combining this with (1) yields the desired conclusion.

(c) We show $T_{s+t} f(x) = T_t T_s f(x)$ for any $s, t \geqslant 0$, $x \in \mathbb{R}^d$ and $f \in C_0(\mathbb{R}^d)$. Indeed,

$$\begin{aligned}T_t T_s f(x) &= \mathbb{E}\big(T_s f(x + X_t)\big) = \mathbb{E}\left(\widetilde{\mathbb{E}} f(x + X_t + \widetilde{X}_s)\right)\\ &= \mathbb{E}\left(\widetilde{\mathbb{E}} f(x + X_t + \widetilde{X}_{s+t} - \widetilde{X}_t)\right) = \mathbb{E} f(x + X_{s+t} - X_t + X_t) = T_{s+t} f(x),\end{aligned}$$

where $\widetilde{X}$ is an independent copy of X and $\widetilde{\mathbb{E}}$ is the corresponding expectation.

(d) For $t \geqslant 0$ and $f \in C_0(\mathbb{R}^d)$, one has

$$\|T_t f\| = \sup_{x\in\mathbb{R}^d} |\mathbb{E} f(x + X_t)| \leqslant \mathbb{E} \sup_{x\in\mathbb{R}^d} |f(x + X_t)| \leqslant \|f\|.$$

(e) For $f \in C_0(\mathbb{R}^d)$, one has $\mathbb{E} f(x + X_0) = f(x)$ for any $x \in \mathbb{R}^d$. Hence $T_0 f = f$.

(f) Notice that if $f \in C_0(\mathbb{R}^d)$, then f is uniformly continuous on $\mathbb{R}^d$. Hence, for $\varepsilon > 0$ one can find $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$. Then

$$\begin{aligned}\|T_t f - f\| &= \sup_{x\in\mathbb{R}^d} |T_t f(x) - f(x)| \leqslant \sup_{x\in\mathbb{R}^d} \mathbb{E}|f(x + X_t) - f(x)|\\ &\leqslant \sup_{x\in\mathbb{R}^d} \mathbb{E}|f(x + X_t) - f(x)|\mathbb{1}_{\{|X_t|<\delta\}} + \sup_{x\in\mathbb{R}^d} \mathbb{E}|f(x + X_t) - f(x)|\mathbb{1}_{\{|X_t|\geqslant\delta\}}\\ &\leqslant \varepsilon + 2\|f\|\mathbb{P}(|X_t| \geqslant \delta) \to \varepsilon \ \text{ as } t \downarrow 0.\end{aligned}$$

By the arbitrariness of ε, we obtain $\|T_t f - f\| \to 0$ as $t \downarrow 0$. □

Problem 3. Let N and $\widetilde{N}$ be two independent Poisson processes with intensities λ and $\widetilde{\lambda}$ respectively. Let $a, b \in \mathbb{R}$, and define $X_t = aN_t + b\widetilde{N}_t$. W.l.o.g., assume that $ab \neq 0$.

First, we show that X is a Lévy process in law.

- $X_0 = 0$ a.s.;
- $\mathbb{P}(|X_{t+s} - X_s| > \varepsilon) \leqslant \mathbb{P}(|N_{t+s} - N_s| > \frac{\varepsilon}{2|a|}) + \mathbb{P}(|\widetilde{N}_{t+s} - \widetilde{N}_s| > \frac{\varepsilon}{2|b|}) \to 0$ as $t \downarrow 0$. Hence X is continuous in probability.
- Check the stationary increments:

$$\begin{aligned}\varphi_{X_{s+t}-X_s}(u) &= \mathbb{E}\left(\mathrm{e}^{\mathrm{i}u(X_{s+t}-X_s)}\right) = \mathbb{E}\left(\mathrm{e}^{\mathrm{i}ua(N_{s+t}-N_s)}\right)\mathbb{E}\left(\mathrm{e}^{\mathrm{i}ub(\widetilde{N}_{s+t}-\widetilde{N}_s)}\right) \\ &= \mathbb{E}\left(\mathrm{e}^{\mathrm{i}uaN_t}\right)\mathbb{E}\left(\mathrm{e}^{\mathrm{i}ub\widetilde{N}_t}\right) = \mathbb{E}\left(\mathrm{e}^{\mathrm{i}uX_t}\right) = \varphi_{X_t}(u).\end{aligned}$$

- Check the independent increments: let $0 \leqslant t_0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n$ and denote

$$\mathbf{X} = (X_{t_n} - X_{t_{n-1}}, \ldots, X_1 - X_0) = a(N_{t_n} - N_{t_{n-1}}, \ldots, N_1 - N_0) + b(\widetilde{N}_{t_n} - \widetilde{N}_{t_{n-1}}, \ldots, \widetilde{N}_1 - \widetilde{N}_0).$$

Then we have for any $\mathbf{u} = (u_k)_{k=1}^n \in \mathbb{R}^n$ that

$$\varphi_{\mathbf{X}}(\mathbf{u}) = \mathbb{E}\,\mathrm{e}^{\mathrm{i}\mathbf{u}\cdot\mathbf{X}} = \prod_{k=1}^n \mathbb{E}\,\mathrm{e}^{\mathrm{i}u_k a(N_{t_k}-N_{t_{k-1}})}\,\mathbb{E}\,\mathrm{e}^{\mathrm{i}u_k b(\widetilde{N}_{t_k}-\widetilde{N}_{t_{k-1}})} = \prod_{k=1}^n \varphi_{X_{t_k}-X_{t_{k-1}}}(u_k).$$

Secondly, we compute the characteristic functions:

$$\varphi_{aN_t}(u) = \mathbb{E}\,\mathrm{e}^{\mathrm{i}uaN_t} = \mathrm{e}^{\lambda t(\mathrm{e}^{\mathrm{i}ua}-1)},\ \varphi_{b\widetilde{N}_t}(u) = \mathbb{E}\,\mathrm{e}^{\mathrm{i}ub\widetilde{N}_t} = \mathrm{e}^{\tilde{\lambda}t(\mathrm{e}^{\mathrm{i}ub}-1)}.$$

Hence it follows from the independence of N and $\widetilde{N}$ that

$$\varphi_{X_t}(u) = \varphi_{aN_t}(u)\varphi_{b\widetilde{N}_t}(u) = \mathrm{e}^{\lambda t(\mathrm{e}^{\mathrm{i}ua}-1)+\tilde{\lambda}t(\mathrm{e}^{\mathrm{i}ub}-1)}. \tag{2}$$

Thirdly, we can compute the characteristic functions for a compound Poisson process as follows: Assume that $Z_t = \sum_{k=1}^{\bar{N}_t} Y_k$ is a compound Poisson process, where $\bar{N}$ is a Poisson process with intensity $\bar{\lambda}$. Then

$$\begin{aligned}\varphi_{Z_t}(u) &= \mathbb{E}\,\mathrm{e}^{\mathrm{i}uZ_t} = \mathbb{E}\left(\sum_{n=0}^\infty \mathrm{e}^{\mathrm{i}u\sum_{k=1}^n Y_k}\,\mathbb{1}_{\{\bar{N}_t=n\}}\right) \overset{(Y_k)\perp N}{=} \sum_{n=0}^\infty \mathbb{E}\left(\mathrm{e}^{\mathrm{i}u\sum_{k=1}^n Y_k}\right)\mathbb{P}(\bar{N}_t = n) \\ &= \sum_{n=0}^\infty \left[\mathbb{E}\left(\mathrm{e}^{\mathrm{i}uY_1}\right)\right]^n \mathbb{P}(\bar{N}_t = n) = \sum_{n=0}^\infty \left[\mathbb{E}\left(\mathrm{e}^{\mathrm{i}uY_1}\right)\right]^n \mathrm{e}^{-\bar{\lambda}t}\,\frac{(\bar{\lambda}t)^n}{n!} \\ &= \mathrm{e}^{\bar{\lambda}t\left[\mathbb{E}(\mathrm{e}^{\mathrm{i}uY_1})-1\right]}.\end{aligned} \tag{3}$$

Now, we compare (2) and (3) and realize that if Y_1 is a random variable having the distribution

$$\mathbb{P}(Y_1 = a) = \frac{\lambda}{\lambda+\tilde{\lambda}},\quad \mathbb{P}(Y_1 = b) = \frac{\tilde{\lambda}}{\lambda+\tilde{\lambda}}, \tag{4}$$

and let

$$\bar{\lambda} = \lambda + \tilde{\lambda},$$

then

$$\varphi_{Z_t} = \varphi_{X_t}.$$

Hence X is a compound Poisson process, where the corresponding Poisson process has intensity $\lambda + \tilde{\lambda}$, and the sequence of i.i.d. random variable (Y_k) has the common distribution as in (4).

Problem 4. Let $(X_n)_{n=0}^{\infty}$ be an $(\mathcal{F}_n)_{n=0}^{\infty}$ martingale, and let $\tau\colon \Omega \to \mathbb{N}\cup\{\infty\}$ be an $(\mathcal{F}_n)_{n=0}^{\infty}$ stopping time. Define the stopped process X^{τ} by

$$X_n^{\tau} := X_{\tau\wedge n}.$$

We show that X^{τ} is an $(\mathcal{F}_n)_{n=0}^{\infty}$ martingale. Indeed, observe that

$$X_{\tau\wedge n} = \sum_{k=0}^{n-1} X_k \mathbb{1}_{\{\tau=k\}} + X_n \mathbb{1}_{\{\tau\geqslant n\}}.$$

- One has

$$\mathbb{E}|X_{\tau\wedge n}| \leqslant \sum_{k=0}^{n-1} \mathbb{E}|X_k| + \mathbb{E}|X_n| < \infty.$$

- Since τ is a stopping time, it implies that $\{\tau \geqslant n\} = \Omega\backslash\{\tau \leqslant n-1\} \in \mathcal{F}_{n-1}$. Hence $X_{\tau\wedge n}$ is $\mathcal{F}_n$- measurable.

- For any $n \geqslant 0$, a.s.,

$$\begin{aligned}
\mathbb{E}\left[X_{\tau\wedge(n+1)}|\mathcal{F}_n\right] &= \mathbb{E}\left[\sum_{k=0}^{n} X_k \mathbb{1}_{\{\tau=k\}} + X_{n+1}\mathbb{1}_{\{\tau\geqslant n+1\}} \,\middle|\, \mathcal{F}_n\right] \\
&= \sum_{k=0}^{n} X_k \mathbb{1}_{\{\tau=k\}} + \mathbb{E}\left[X_{n+1}\mathbb{1}_{\{\tau\geqslant n+1\}} \,\middle|\, \mathcal{F}_n\right] \\
&= \sum_{k=0}^{n} X_k \mathbb{1}_{\{\tau=k\}} + \mathbb{1}_{\{\tau\geqslant n+1\}}\mathbb{E}\left[X_{n+1} \,\middle|\, \mathcal{F}_n\right] \\
&= \sum_{k=0}^{n} X_k \mathbb{1}_{\{\tau=k\}} + \mathbb{1}_{\{\tau\geqslant n+1\}} X_n \\
&= \sum_{k=0}^{n-1} X_k \mathbb{1}_{\{\tau=k\}} + \mathbb{1}_{\{\tau\geqslant n\}} X_n \\
&= X_{\tau\wedge n}.
\end{aligned}$$