Solutions for Demonstration 7

Problem 1. We show that there exists a countable dense subset $(f_k)_{k\geqslant 1}$ of $C_0(\mathbb{R})$ separating points of $\mathbb{R}^\partial := \mathbb{R}\cup\partial$, where ∂ is a point outside $\mathbb{R}$ and plays a role of the one-point compactification of $\mathbb{R}$. In the sequel, we consider the supremum norm.

Step 1: For each $N\in\mathbb{N}$, $N\geqslant 1$, denote

$$\begin{aligned} C[-N,N] &: \text{ the family of all continuous functions on } [-N,N] \\ P_N(\mathbb{R}) &: \text{ the family of all polynomials defined on } [-N,N] \text{ with real coefficients} \\ P_N(\mathbb{Q}) &: \text{ the family of all polynomials defined on } [-N,N] \text{ with rational coefficients.} \end{aligned}$$

First, it is straightforward to see that $P_N(\mathbb{Q})$ is dense in $P_N(\mathbb{R})$. Secondly, the Stone-Weierstrass theorem [1] verifies that $P_N(\mathbb{R})$ is dense in $C[-N,N]$. Hence $P_N(\mathbb{Q})$ is dense in $C[-N,N]$.

Step 2: Observe that $P_N(\mathbb{Q})$ is countable. Then

$$P(\mathbb{Q}) := \bigcup_{N=1}^{\infty} P_N(\mathbb{Q})$$

is countable.

For each polynomial $p_N\in P_N(\mathbb{Q})\subset P(\mathbb{Q})$, we extend it to $\hat{p}_N$ by (draw a graph)

$$\hat{p}_N(x) := \begin{cases} p_N(x) & \text{if } |x|\leqslant N \\ p_N(N)(N+1-x) & \text{if } x\in[N,N+1] \\ p_N(-N)(N+1+x) & \text{if } x\in[-N-1,-N] \\ 0 & \text{if } |x|>N+1 \end{cases}$$

Then $\hat{p}_N\in C_0(\mathbb{R})$. Moreover, $|\hat{p}_N(x)|\leqslant \max\{|p_N(-N)|,|p_N(N)|\}$ for $|x|\geqslant N$.

Step 3: Denote by $\widehat{P}(\mathbb{Q})$ the family of such $\hat{p}_N$, $p\in P_N(\mathbb{Q})$, $N\geqslant 1$. Then $\widehat{P}(\mathbb{Q})$ is also countable. We now show that $\widehat{P}(\mathbb{Q})$ is dense in $C_0(\mathbb{R})$. Indeed, for each $\varepsilon>0$, there exists $N_\varepsilon\in\mathbb{N}$ such that $|f(x)|\leqslant\varepsilon$ for all $|x|\geqslant N_\varepsilon$. By *Step 1*, there is a $p_{N_\varepsilon}\in P_{N_\varepsilon}(\mathbb{Q})$ such that

$$\sup_{x\in[-N_\varepsilon,N_\varepsilon]} |p_{N_\varepsilon}(x)-f(x)|\leqslant\varepsilon.$$

Consequently, $|p_{N_\varepsilon}(\pm N_\varepsilon)|\leqslant 2\varepsilon$. For the corresponding extension $\hat{p}_{N_\varepsilon}\in\widehat{P}(\mathbb{Q})$ in *Step 2*, we have

$$\begin{aligned} \sup_{x\in\mathbb{R}}|\hat{p}_{N_\varepsilon}(x)-f(x)| &\leqslant \underbrace{\sup_{|x|\geqslant N_\varepsilon}|\hat{p}_{N_\varepsilon}(x)|}_{\leqslant\max\{\hat{p}_{N_\varepsilon}(-N_\varepsilon),\hat{p}_{N_\varepsilon}(N_\varepsilon)\}\leqslant 2\varepsilon} + \sup_{|x|\geqslant N_\varepsilon}|f(x)| + \sup_{x\in[-N_\varepsilon,N_\varepsilon]}|p_{N_\varepsilon}(x)-f(x)| \\ &\leqslant 2\varepsilon+\varepsilon+\varepsilon = 4\varepsilon. \end{aligned}$$

Hence $\widehat{P}(\mathbb{Q})$ is dense in $C_0(\mathbb{R})$.

Step 4: Verify that $\widehat{P}(\mathbb{Q})$ separates points of $\mathbb{R}^\partial$. Let $x,y\in\mathbb{R}^\partial$, $x\neq y$.

- If $x,y\in[-N,N]$ for some $N\in\mathbb{N}$, then we choose $p_N(u):=u$. Then $p_N\in P_N(\mathbb{Q})$ and its extension $\hat{p}_N\in\widehat{P}(\mathbb{Q})$ satisfies $|\hat{p}_N(x)-\hat{p}_N(y)|=|p_N(x)-p_N(y)|=|x-y|\neq 0$.
- If $x\in[-N,N]$ and $y=\partial$, then we choose $p_N(u):=u^2+1$. Then $p_N\in P_N(\mathbb{Q})$ and its extension $\hat{p}_N\in\widehat{P}(\mathbb{Q})$ satisfies $|\hat{p}_N(x)-\hat{p}_N(y)|=|p_N(x)-0|=x^2+1\neq 0$.

Conclusion: $\widehat{P}(\mathbb{Q})$ is a countable dense subset of $C_0(\mathbb{R})$ and separates points of $\mathbb{R}^\partial$. □

[1] **Stone-Weierstrass approximation theorem**: Let X be a *compact Hausdorff space* and $C(X,\mathbb{R})$ consists of all real-valued continuous function on X. Suppose that $\mathcal{A}$ is a sub-algebra of $C(X,\mathbb{R})$ containing constant functions. Then $\mathcal{A}$ is dense in $C(X,\mathbb{R})$ w.r.t. the supremum-norm if and only if $\mathcal{A}$ separates points of X.

Problem 2. Let X be a Lévy process and τ an exponentially distributed random variable with parameter $p > 0$, independent from X. Let For $f \in C_0(\mathbb{R})$, $x \in \mathbb{R}$ we have

$$\begin{aligned}\mathbb{E}f(x + X_\tau) &= \int_\Omega f(x + X_{\tau(\omega)}(\omega))d\mathbb{P}(\omega) \overset{\tau \perp (X_t)}{=} \int_\Omega \int_{\widetilde{\Omega}} f(x + X_{\tilde{\tau}(\tilde{\omega})}(\omega))d\widetilde{\mathbb{P}}(\tilde{\omega})d\mathbb{P}(\omega)\\ &= p\int_\Omega \int_0^\infty f(x + X_t(\omega))\,\mathrm{e}^{-pt}\,dtd\mathbb{P}(\omega) \overset{Fubini}{=} p\int_0^\infty \mathrm{e}^{-pt}\,T_t f(x)dt = p\mathcal{R}_p f(x),\end{aligned}$$

where $\tilde{\tau}$ is a random variable defined on another probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ and $\tilde{\tau} \overset{d}{=} \tau$.

Remark that in order to show

$$\int_\Omega f(x + X_{\tau(\omega)}(\omega))d\mathbb{P}(\omega) \overset{\tau \perp (X_t)}{=} \int_\Omega \int_{\widetilde{\Omega}} f(x + X_{\tilde{\tau}(\tilde{\omega})}(\omega))d\widetilde{\mathbb{P}}(\tilde{\omega})d\mathbb{P}(\omega),$$

one can use the fact that $\tau_n \downarrow \tau$, where

$$\tau_n := \sum_{k=1}^\infty \frac{k}{2^n} \mathbb{1}_{\{\frac{k-1}{2^n} \leqslant \tau < \frac{k}{2^n}\}},$$

together with the right-continuity of X, and a DCT argument. $\square$