Try to solve 3 of the problems below

some definitions:

(1) A stochastic process $(X_t)_{t\in[0,\infty)}$ is said to have *independent increments* provided that

$$\forall n \in \mathbb{N} : 0 = t_0 < t_1 < \ldots < t_n$$

$$X_{t_n} - X_{t_{n-1}}, \ldots, X_{t_1} - X_{t_0}$$

are independent. We say that the *increments are stationary* if for all $0 \le s < t$, $u > 0$

$$X_{t+u} - X_{s+u} \sim X_t - X_s$$

(where $\sim$ stands for 'having the same distribution as').

(2) **Poisson process**
The stochastic process $N := (N_t)_{t\in[0,\infty)}$ is called a *Poisson process with intensity* $\lambda > 0$ if

(a) $\mathbb{P}(\{\omega \in \Omega : N_0(\omega) = 0\}) = 1$,

(b) N has independent, stationary increments,

(c) $\mathbb{P}(\{\omega \in \Omega : N_t(\omega) = k\}) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}, \quad k \in \mathbb{N} \cup \{0\}$.

problems:

(1) Let $E := \{0, 1, 2, \ldots\}$ and $\mathcal{E} := 2^E$, the power set of E.

(a) Show that for the *Poisson process* $(N_t)_{t\in[0,\infty)}$ *with intensity* $\lambda > 0$ the map defined by

$$P_h(k, B) := \mathbb{P}(N_{s+h} \in B | N_s = k), \quad h > 0, s > 0, k \in E, B \in \mathcal{E}$$

is a transition function.

(b) **Poisson is Markov** Take for $\mathbb{F}$ the natural filtration of N. Show that N is a Markov process w.r.t. $\mathbb{F}$ with transition function defined above.
Hint: In oder to compute $\mathbb{E}[f(N_s)|\mathcal{F}_t]$ write $\mathbb{E}[f((N_s-N_t)+N_t)|\mathcal{F}_t]$ and use Proposition 7.4.6 from the lecture notes 'An introduction to probability theory'.

(2) **Markov property - equivalences: the proof**
Complete the proof of Theorem 2.3 $(i) \implies (ii)$ (or $(iii) \implies (ii)$) by showing that the set

$$\mathcal{H} := \{Y : Y \text{ integrable and } \mathbb{E}[Y|\mathcal{F}_t] = \mathbb{E}[Y|X_t] \text{ holds}\}$$

satisfies the following properties from the **Monotone Class Theorem for functions** (see Theorem A.1 from the appendix of the lecture notes):

(i) linear combinations of elements of $\mathcal{H}$ are again in $\mathcal{H}$,

(ii) If $(f_n)_{n=1}^\infty \subseteq \mathcal{H}$ such that $0 \le f_n \uparrow f$, and f is bounded, then it follows $f \in \mathcal{H}$.

(3) **conditional expectation**
Let $([0,1],\mathcal{B}([0,1]),\lambda)$ be the probability space where $\mathcal{B}([0,1])$ denotes the Borel σ-algebra on $[0,1]$ and λ the Lebesgue measure. Find the conditional expectation $\mathbb{E}[X|Y]$ if $X,Y : [0,1]\to\mathbb{R}$ are given by $X(x)=\sin(4\pi x)$ and $Y(x)=x\mathbb{1}_{[0,0.5]}(x)$.
(In order to get $\mathbb{E}[X|Y]:=\mathbb{E}[X|\sigma(Y)]$ find out first $\sigma(Y)$.)

(4) **Markov property - a special case**
Assume that the relation from **Definition 2.1**,

$$\mathbb{P}(A\cap B|X_t)=\mathbb{P}(A|X_t)\mathbb{P}(B|X_t)$$

holds for $A\in\mathcal{F}_t$ and $B\in\sigma(X_s;s\geq t)$. Show that if

$$A=B\in\mathcal{F}_t\cap\sigma(X_s;s\geq t)$$

then

$$\mathbb{P}(A|X_t)=\mathbb{1}_A\quad a.s.$$

(5) **Gaussian process:** We say the process $(X_t)_{t\geq 0}$ is a Gaussian process, if for any $n\in\mathbb{N}$ and $0\leq t_1<t_2<...<t_n$ the 'random vector' $(X_{t_1},...,X_{t_n})$ has a Gaussian distribution.

(i) Explain why then also the vector of the increments

$$(X_{t_1},X_{t_2}-X_{t_1}...,X_{t_n}-X_{t_{n-1}})$$

is Gaussian (see, for example, Proposition 9.4.7 of 'An introduction to probability theory' from Koppa).

(ii) Assume that X has independent increments, $X_0=0$, and that for any $s>t\geq 0$ the increment X_s-X_t has expectation 0 and variance $s-t$. Determine the distribution of $(X_{t_1},X_{t_2},X_{t_3})$, where $0\leq t_1<t_2<t_3$, by finding out its expectation and covariance matrix.