some definitions:

(1) **Brownian motion**
The stochastic process $W := (W_t)_{t\in[0,\infty)}$ is called a *(standard) Brownian motion* if

(a) $\mathbb{P}(\{\omega \in \Omega : W_0(\omega) = 0\}) = 1$,

(b) its increments are independent and stationary,

(c) $W_t \sim \mathcal{N}(0,t)$ $\Big($ i.e. $\mathbb{P}(\{\omega \in \Omega : W_t(\omega) \in (a,b)\}) = \frac{1}{\sqrt{2\pi t}}\int_a^b e^{-\frac{z^2}{2t}}dz$ $\Big)$.

(2) **martingale**
Assume we have a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,\infty)}$. A stochastic process $M := (M_t)_{t\in[0,\infty)}$ with $M_t : \Omega \to \mathbb{C}$ (or $\mathbb{R}$) is a martingale, if it is $\mathbb{F}$-adapted, $\mathbb{E}|M_t| < \infty$ for all $t \geq 0$ and

$$\mathbb{E}[M_t|\mathcal{F}_s] = M_s \quad a.s. \quad \text{for } t \geq s.$$

Try to solve 4 of the problems below

(1) **transition function**
Let $E := \{0, 1, 2, \ldots\}$ and $\mathcal{E} := 2^E$, the power set of E.

(a) Show that for the *Poisson process* $(N_t)_{t\in[0,\infty)}$ *with intensity* $\lambda > 0$ (see the definition on sheet -1-) the map defined by

$$P_h(k,B) := \mathbb{P}(N_{s+h} \in B | N_s = k), \quad h > 0, s > 0, \, k \in E, B \in \mathcal{E}$$

is a transition function.

(b) **Poisson is Markov** Take for $\mathbb{F}$ the natural filtration of N. Show that N is a Markov process w.r.t. $\mathbb{F}$ with transition function defined above.
Hint: In oder to compute $\mathbb{E}[f(N_s)|\mathcal{F}_t]$ write $\mathbb{E}[f((N_s - N_t) + N_t)|\mathcal{F}_t]$ and use Proposition 7.4.6 from the lecture notes 'An introduction to probability theory'.

(2) **a martingale**
Show that for any $a \in \mathbb{R}$ the process given by

$$M_t := \frac{e^{iaW_t}}{\mathbb{E}e^{iaW_t}}$$

is a martingale w.r.t. $\{\mathcal{F}_t^W; t \geq 0\}$, where W is a standard Brownian motion.
Hint: Use exercise (5) below. You may work with complex valued conditional expectation like with the real valued one.

(3) **a right-continuous filtration**
Consider on Ω the filtration $\{\mathcal{F}_t, t \geq 0\}$, and for each $t \geq 0$ define $\mathcal{G}_t = \mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s$. Show that $\{\mathcal{G}_t, t \geq 0\}$ is right-continuous, i.e. it holds

$$\mathcal{G}_t = \bigcap_{s>t} \mathcal{G}_s, \quad t \geq 0.$$

Hint: It is a one-line proof.

(4) **independence**
Let $X = \{X_t; t \geq 0\}$ is a stochastic process such that for all $0 \leq s \leq t$ it holds $X_t - X_s$ and $\mathcal{F}_s^X$ are independent. Show that then for $s \leq t_1 < t_2 < ... < t_n$ the vector and the σ -algebra,

$$(X_{t_n} - X_{t_{n-1}}, ..., X_{t_2} - X_{t_1}) \text{ and } \mathcal{F}_s^X,$$

are independent.

Hint: For any $D \in \mathcal{F}_s^X$ show the following: If $\varphi(x_1, ..., x_{n+1})$ is the characteristic function of the vector $(X_{t_n} - X_{t_{n-1}}, ..., X_{t_2} - X_{t_1}, 1\!\!1_D)$, and $\varphi_1(x_1, ..., x_n)$ that of $(X_{t_n} - X_{t_{n-1}}, ..., X_{t_2} - X_{t_1})$ and $\varphi_2(y)$ that of $1\!\!1_D$, then it holds

$$\varphi(x_1, ..., x_{n+1}) = \varphi_1(x_1, ..., x_n)\varphi_2(x_{n+1}).$$

This can be done by applying the tower property in a suitable way.

(5) **independent increments and σ-algebras**
Assume $X = \{X_t; t \in [0, \infty)\}$ is a stochastic process with independent increments and $X_0(\omega) = 0$ for all $\omega \in \Omega$. Show that

$$\text{if } \; 0 \leq s \leq t < u \; \text{ then } \; \mathcal{F}_s^X \; \text{ is independent from } \; X_u - X_t.$$

For the proof one can use the following steps:

(a) Recall that

$$\mathcal{A} = \{\{X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n\} : n \in \mathbb{N}, B_k \in \mathcal{B}(\mathbb{R}), t_k \in [0, s], k = 1, \ldots, n\},$$

is a π-system and that $\sigma(\mathcal{A}) = \sigma(X_r : r \in [0, s]) =: \mathcal{F}_s^X$.

(b) Fix $B_0 \in \mathcal{B}(\mathbb{R})$. Define the probability measure $\mathbb{Q}$ by

$$\mathbb{Q}(C) := \mathbb{P}(C | \{X_u - X_t \in B_0\}), \quad C \in \mathcal{F}_s^X.$$

Explain why for $0 \leq t_1 < \ldots < t_n \leq s \leq t < u$ it holds

$$\mathbb{Q}(\{X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n\}) = \mathbb{P}(\{X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n\}).$$

(c) Why can one conclude that for all $C \in \mathcal{F}_s^X$ and $B_0 \in \mathcal{B}(\mathbb{R})$

$$\mathbb{P}(C | \{X_u - X_t \in B_0\}) = \mathbb{P}(C),$$

(which means in fact that $\mathcal{F}_s^X$ is independent from $X_u - X_t$)?