Try to solve 4 of the 5 problems below ((3) is really short!)

(1) **an optional time or even a stopping time ?**
Assume we have $\mathbb{F}$-adapted stochastic processes $X^{(k)} = (X_t^{(k)})_{t\geq 0}$, $k = 1,..,4$ with $X_0^k = 0$. We consider the random times

$$\tau_k := \inf\{t > 0 : X_t^{(k)} > 1\}, \quad k = 1, 2, 3, 4.$$

We call the map $t \mapsto X_t^{(k)}(\omega)$ a path of $X^{(k)}$.

Find out whether τ_k is an optional time or a stopping time if

(a) $X^{(1)}$ has non-decreasing right-continuous paths which are either constant or jump up by 1 (like the paths of a Poisson process).
(b) $X^{(2)}(t) = \lim_{s\uparrow t} X^{(1)}(s)$ for $t > 0$.
(c) $X^{(3)}$ has continuous paths.
(d) $X^{(4)}$ has non-decreasing right-continuous paths which are either constant or jump up.

Hint: Use $\{\tau_k < t\} = \bigcup_{s\in\mathbb{Q},s<t}\{X_s^{(k)} > 1\}$.

(2) **characterization of $\mathcal{F}_{\tau+}$**
We define for an optional time τ the σ-algebra

$$\mathcal{F}_{\tau+} := \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_{t+}\, \forall t \in [0, \infty)\}.$$

Show

$$\mathcal{F}_{\tau+} = \{A \in \mathcal{F} : A \cap \{\tau < t\} \in \mathcal{F}_t\, \forall t \in [0, \infty)\}.$$

(3) **a filtration property**
Let $\mathbb{F} = \{\mathcal{F}_t; t \geq 0\}$ be a filtration. Is it true that

$$\bigcap_{s>t} \mathcal{F}_s = \bigcap_{n=1}^{\infty} \mathcal{F}_{t+\frac{1}{n}}?$$

(4) $\mathcal{F}_{\tau\wedge\eta}$
Let τ and η be $\mathbb{F}$-stopping times. Assume as known that then also $\tau\wedge\eta$ is a stopping time. Show that

$$\mathcal{F}_{\tau\wedge\eta} = \mathcal{F}_\tau \cap \mathcal{F}_\eta.$$

Hint: See the part of Karatzas & Shreve: *Brownian motion and Stochastic Calculus* in Koppa.

(5) **a filtration, which is not right-continuous**

Let $C[0,\infty)$ be the space of continuous functions on $[0,\infty)$. Let

$$\mathcal{F}_t := \sigma\{\{g \in C[0,\infty) : g(s) \in B\} : s \in [0,t], B \in \mathcal{B}(\mathbb{R})\},$$

i.e. $\mathcal{F}_t$ is the smallest σ-algebra, such that all coordinate mappings

$$\pi_s : C[0,\infty) \to \mathbb{R} : g \mapsto g(s), \quad s \in [0,t]$$

are measurable. Show that the strict inclusion $\mathcal{F}_t \subset \mathcal{F}_{t+}$ holds by constructing a set which is in $\mathcal{F}_{t+}$ but not in $\mathcal{F}_t$.

Hint: You can, for example, consider the set

$$A_{t+\varepsilon} := \bigcap_{\text{rational } s \in [t, t+\varepsilon)} \{g \in C[0,\infty) : g(s) - g(t) > 0\}$$

and show that $A := \bigcap_{n=1}^{\infty} A_{t+\frac{1}{n}} \in \mathcal{F}_{t+}$ but not in $\mathcal{F}_t$.