(1) **a filtration, which is not right-continuous**
Let $C[0,\infty)$ be the space of continuous functions on $[0,\infty)$. Let

$$\mathcal{F}_t := \sigma\{\{g \in C[0,\infty) : g(s) \in B\} : s \in [0,t], B \in \mathcal{B}(\mathbb{R})\},$$

i.e. $\mathcal{F}_t$ is the smallest σ-algebra, such that all coordinate mappings

$$\pi_s : C[0,\infty) \to \mathbb{R} : g \mapsto g(s), \quad s \in [0,t]$$

are measurable. Show that the strict inclusion $\mathcal{F}_t \subset \mathcal{F}_{t+}$ holds by constructing a set which is in $\mathcal{F}_{t+}$ but not in $\mathcal{F}_t$.

Hint: You can, for example, consider the set

$$A_{t,m} := \{g \in C[0,\infty) : g(t + \tfrac{1}{n}) \geq g(t),\ \forall n = m, m+1, ...\}$$

and show that $A := \bigcup_{m=1}^{\infty} A_{t,m} \in \mathcal{F}_{t+}$ but not in $\mathcal{F}_t$.

(2) **Brownian motion semigroup**
If $W = (W_t)_{t\in[0,\infty)}$ is the (standard) Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ (defined on sheet -2-), then for $t \geq 0, x \in \mathbb{R}$ and $B \in \mathcal{B}(\mathbb{R})$ we put

$$P_t(x,B) := \mathbb{P}(W_t + x \in B).$$

(One can show that $\{P_t(x,B)\}$ is a family of transition functions; and from Kolmogorov's continuity criterium it follows that there exists a modification $\bar{W}$ of W such that for almost all $\omega \in \Omega$ the paths

$$t \mapsto \bar{W}_t(\omega)$$

are continuous. Assume both as known.)
Check whether it holds $\lim_{t\downarrow 0} \int_{\mathbb{R}} f(y)P_t(x,dy) = f(x)$ for all $x \in \mathbb{R}$ if

(a) $f = \sin$,
(b) $f = 1\!\!1_{(-\infty,0]}$.

Hint: The answer is very easy if one uses that (and explains why)

$$\int_{\mathbb{R}} f(y)P_t(x,dy) = \mathbb{E}f(W_t + x) = \mathbb{E}f(\bar{W}_t + x).$$

(3) **augmented natural filtration**
Show that $\mathcal{F}_t^{\mathbb{P}} = \{G \subseteq \Omega : \exists H \in \mathcal{F}_t^X : H\Delta G \in \mathcal{N}^{\mathbb{P}}\}$.

Hint: One can prove the equality using the following steps:

(a) Show that $\mathcal{G} := \{G \subseteq \Omega : \exists H \in \mathcal{F}_t^X : H\Delta G \in \mathcal{N}^{\mathbb{P}}\}$ is a σ-algebra.
(b) The relation $(F = H\Delta G \iff G = H\Delta F)$ implies $\mathcal{G} \subseteq \mathcal{F}_t^{\mathbb{P}}$.
(c) Notice that $\mathcal{N}^{\mathbb{P}} \subseteq \mathcal{G}$ and $\mathcal{F}_t^X \subseteq \mathcal{G}$ and conclude $\mathcal{F}_t^{\mathbb{P}} \subseteq \mathcal{G}$.

(4) **properties of the characteristic function of a Lévy process**

Assume that X with $X_t : (\Omega, \mathcal{F}) \to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is a *Lévy process in law,* i.e.

(i) $X_0 = 0$ a.s.

(ii) X has independent and homogeneous increments

(iii) X is continuous in probability, i.e. for all $\delta > 0$:

$$\lim_{s \downarrow t} \mathbb{P}(|X_s - X_t| > \delta) = 0.$$

For $u \in \mathbb{R}^d$ and $t \geq 0$ define

$$f_t(u) := \mathbb{E}e^{i\langle u, X_t \rangle},$$

with $\langle u, X_t \rangle := \sum_{k=1}^d u_k X_t^{(k)}$. Show that the following holds:

(a) $\forall u \in \mathbb{R}^d: \;\; f_0(u) = 1,$

(b) $\forall u \in \mathbb{R}^d, \;\; t, s \geq 0: \;\; f_{t+s}(u) = f_t(u) f_s(u),$

(c) $\forall u \in \mathbb{R}^d, \;\; \forall t \geq 0: \;\; f_t(u) \neq 0.$

(5) **martingale**

Let $\{X_t : t \geq 0\}$ be a Lévy process in law like in the previous exercise. Fix $\theta \in \mathbb{R}^d$. Show with the help of (4)(c) that $\{Z_t : t \geq 0\}$ given by

$$Z_t := \frac{e^{i\langle \theta, X_t \rangle}}{\mathbb{E}e^{i\langle \theta, X_t \rangle}}$$

is a martingale w.r.t. $\{\mathcal{F}_t^X ; t \geq 0\}$.