(1) **rv's independent from a σ-algebra**
Assume that $Y = (Y_1, ..., Y_d) : \Omega \to \mathbb{R}^d$ is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G} \subseteq \mathcal{F}$ a sub-σ-algebra. Show that

$$\mathbb{E}\Big[\exp\{i(x_1Y_1 + ... + x_dY_d)\}\mathbb{1}_G\Big] = \mathbb{P}(G)\mathbb{E}\exp\{i(x_1Y_1 + ... + x_dY_d)\},\ \forall x \in \mathbb{R}^d,\ \forall G \in \mathcal{G}.$$

implies that Y is independet from $\mathbb{1}_G$ for any $G \in \mathcal{G}$. (We then say that Y and $\mathcal{G}$ are independent.)
Hint: You may use the fact that the random variables $(X_1, \ldots, X_d)$ and X_{d+1} are independent $\iff$

$$\mathbb{E}\exp\{i(x_1X_1 + \ldots + x_{d+1}X_{d+1})\} = \mathbb{E}\exp\{i(x_1X_1 + \ldots + x_dX_d)\}\mathbb{E}\exp\{ix_{d+1}X_{d+1}\},$$

for all $(x_1, ..., x_{d+1}) \in \mathbb{R}^{d+1}$.

(2) **infinitesimal generator of the Poisson process**
Let $E := \mathbb{N}$ and $2^{\mathbb{N}} := \mathcal{B}(E)$ and assume that $T(t)f(x) = \mathbb{E}f(N_t + x)$, where $(N_t)_{t\geq 0}$ is a Poisson process with parameter $\lambda > 0$. Show using the distribution of N_t that A is given by

$$Af(x) = \lambda(f(x+1) - f(x)), \quad x \in \mathbb{N}$$

and $D(A) = \{f : \mathbb{N} \to \mathbb{R} : f \text{ bounded}\}$.

What is your *guess* for E, A and a subset $S \subseteq D(A)$ in case of a compound Poisson process X, given by

$$X_t = \sum_{k=1}^{N_t} Y_k \quad (\text{with} \sum_{k=1}^{0} Y_k := 0)$$

where $(Y_k)_{k\geq 1}$ is an i.i.d. sequence with $Y_k : (\Omega, \mathcal{F}) \to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, independent from $(N_t)_{t\geq 0}$?

(3) **Lévy-Khintchine representation and infinitesimal generator**

(a) If $(N_t)_{t\geq 0}$ is a Poisson process with intensity $\lambda > 0$ then we know that the characteristic function is given by

$$\varphi_{N_t}(u) = \mathbb{E}e^{iuN_t} = e^{\lambda t(e^{iu}-1)}, \quad u \in \mathbb{R}$$

Put $T(t)f(x) := \mathbb{E}f(N_t + x)$ and compute for any function f of the class

$$\{f : x \mapsto e^{iux} :\ u \in \mathbb{R}\}$$

the limit

$$\lim_{t\downarrow 0} \frac{T(t)f(x) - f(x)}{t}.$$

We know that for a real-valued bounded and measurable function g we would get

$$Ag(x) = \lambda(g(x+1) - g(x)).$$

What do you notice?

(b) If $(W_t)_{t\geq 0}$ is the standard Brownian motion, then

$$\varphi_{W_t}(u) = e^{-\frac{1}{2}tu^2}, \quad u \in \mathbb{R}.$$

Is it true that for any $f(x) = e^{iux}$ and $T(t)f(x) := \mathbb{E}f(W_t + x)$ it holds

$$\lim_{t\downarrow 0} \frac{T(t)f - f}{t} = \frac{1}{2}f''?$$

(4) **the semigroup of a diffusion**
One can show that the solution X of the stochastic differential equation

$$X_t = x + \int_0^t a(X_s)ds + \int_0^t \sigma(X_s)dB_s,$$

where $\{B_s : s \geq 0\}$ denotes the standard one-dimensional Brownian motion and $a, \sigma : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous and bounded functions, is a Markov process w.r.t the augmented natural filtration $\{F_t^X; t \geq 0\}$. Let

$$C_c^2(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} : f \text{ compact support, } f, f'f'' \text{ continuous } \}.$$

Use Itô's formula on $f(X_t)$, $f \in C_c^2(\mathbb{R})$ and show that $T(t)f(x) := \mathbb{E}f(X_t)$ is given by

$$T(t)f(x) = f(x) + \int_0^t \mathbb{E}Af(X_s)ds,$$

where

$$Af(x) = a(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x), \quad x \in \mathbb{R}.$$

Hint: If $(Y_s)_{s\in[0,T]}$ is progressively measurable and such that $\mathbb{E}\int_0^T Y_s^2 ds < \infty$, then $\mathbb{E}\int_0^T Y_s dB_s = 0$.