càdlàg paths
$f : [0,\infty) \to \mathbb{R}$ is càdlàg if

$$\forall t > 0 \qquad \lim_{s \uparrow t} f(s) := f(t_-) \quad \text{exists}$$
$$\forall t \geq 0 \qquad \lim_{s \downarrow t} f(s) \quad \text{exists}$$

Define $\Delta f(t) := f(t) - f(t_-)$.
Lemma If $f : [0,\infty) \to \mathbb{R}$ is càdlàg, then on each $[a,b) \subseteq [0,\infty)$ and for each $\delta > 0$ there exists a finite partition

$$p = \{a = t_0 < t_1 < ... < t_n = b\}$$

for which

$$\sup\{|f(u) - f(v)| : u, v \in [t_k, t_{k+1}),\, k = 0, ..., n-1\} < \delta.$$

problems:

(1) **càdlàg $\implies$ locally bounded**
Let $f : [0,\infty) \to \mathbb{R}$ be a càdlàg function. Show that for any $T > 0$,

$$\sup_{0 \leq t \leq T} |f(t)| < \infty.$$

Hint: One can use the above Lemma for this.

(2) **Lévy is Feller**
Assume that X is a Lévy process in law. Start showing that $\{T_t : t \geq 0\}$ given by

$$T_t f(x) := \mathbb{E} f(x + X_t), \quad f \in C_0(\mathbb{R}^d)$$

is a conservative Feller semigroup, i.e. that it holds:

(a) $T(t) : C_0(\mathbb{R}^d) \to C_0(\mathbb{R}^d)$ is positive $\forall t \geq 0$ (the linearity is clear).
(b) $\sup_{f \in C_0(\mathbb{R}^d), 0 \leq f \leq 1} T_t f(x) = 1$.
(c) For $s, t \geq 0, x \in \mathbb{R}^d$ and $f \in C_0(\mathbb{R}^d) : T_{t+s} f(x) = T_t T_s f(x)$.
(d) For $t \geq 0$ and $f \in C_0(\mathbb{R}^d) : \|T_t f\| \leq \|f\|$.
(e) For $f \in C_0(\mathbb{R}^d) : T_0 f = f$.
(f) For $f \in C_0(\mathbb{R}^d) : \|T_t f - f\| \to 0$ for $t \downarrow 0$.

(3) **compound Poisson?**
If we have two independent Poisson processes N and $\tilde{N}$ and $a, b \in \mathbb{R}$, is then X given by

$$X_t = aN_t + b\tilde{N}_t$$

a compound Poisson process? If yes, how would the representation with the help of and i.i.d. sequence (Y_n) and *one* Poisson process look like?

(4) **stopping discrete-time martingales**

Assume that $X = (X_n)_{n=0}^{\infty}$ is a $(\mathcal{F}_n)_{n=0}^{\infty}$ martingale. Assume that $\tau : \Omega \to \mathbb{N} \cup \{\infty\}$ is an $(\mathcal{F}_n)_{n=0}^{\infty}$ stopping time. Show that then the stopped process X^τ given by

$$X_n^\tau := X_{\tau \wedge n}$$

is also a martingale, i.e. it holds

(a) $\mathbb{E}|X_{\tau\wedge n}| < \infty$

(b) $X_{\tau\wedge n}$ is $\mathcal{F}_n$ measurable

(c) $\mathbb{E}[X_{\tau\wedge(n+1)}|\mathcal{F}_n] = X_{\tau\wedge n}$ a.s. for $n = 0, 1, 2, ...$

Hint: Explain why $\{\tau \geq n\} \in \mathcal{F}_{n-1}$ and use

$$\begin{aligned} X_{\tau\wedge n} &= X_0 1\!\!1_{\{\tau=0\}} + X_1 1\!\!1_{\{\tau=1\}} + ... + X_{n-1} 1\!\!1_{\{\tau=n-1\}} + X_n 1\!\!1_{\{\tau\geq n\}} \\ &= X_\tau 1\!\!1_{\{\tau\leq n-1\}} + X_n 1\!\!1_{\{\tau\geq n\}} \end{aligned}$$