(1) **separating points**
Show that there exists a sequence $(f_n)_{n=1}^{\infty} \subseteq C_0(\mathbb{R})$ which is dense in $C_0(\mathbb{R})$ and separates the points: for any $x, y \in \mathbb{R}^{\partial} := \mathbb{R} \cup \partial$, (where ∂ is a point outside of $\mathbb{R}$, which is used for the one-point compactification of $\mathbb{R}$) $\exists k \in \mathbb{N}$, $k \geq 1$ such that

$$f_k(x) \neq f_k(y).$$

Hints:

(a) By Stone-Weierstrass, for any $N \geq 1$, the polynomials considered on $[-N, N]$ are dense in $C[-N, N]$. But then also

$$P_N = \{p : [-N, N] \to \mathbb{R} : p \text{ polynomial with rational coefficients } \}$$

is countable and dense in $C[-N, N]$.

(b) Extend each $p \in \bigcup_{N \geq 1} P_N$ to a continuous function $\hat{p}$ on $\mathbb{R}$ such that the extension is in $C_0(\mathbb{R})$.

(c) Explain why $\{\hat{p} : p \in \bigcup_{N \geq 1} P_N\}$ is countable and show this set is dense in $C_0(\mathbb{R})$.

(d) Describe which function $\hat{p}$ could be used for example to separate the points $x, y \in [-N, N]$ and which $\hat{p}$ one could use to separate $x \in [-N, N]$ and $y = \partial$.

(2) **exponential stopping and resolvents**
Let $X := \{X_t; t \geq 0\}$ be a Lévy process and τ an exponentially distributed random variable with parameter $p > 0$, independent from X. Consider the random variable given by $X_\tau(\omega) := X_{\tau(\omega)}(\omega)$. Find out the connection between the expression

$$\mathbb{E}f(x + X_\tau), \quad f \in C_0(\mathbb{R})$$

and the resolvent $\mathcal{R}_p$ of the semigroup given by $T(t)f(x) = \mathbb{E}f(x + X_t)$.

(3) **Up- and downcrossings: the secret behind the càdlàg versions**
Explain the proof of Theorem 1 from
https://almostsure.wordpress.com/2009/12/06/upcrossings-downcrossings-and-martingale-convergence/