Markov -1-

04.11.19

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(1) a) $(\mathbb{N}, 2^{\mathbb{N}})$

$$P_h(k,B) = \mathbb{P}(N_{s+h} \in B \mid N_s = k), \quad h>0, s>0, \quad k\in\mathbb{N},\ B\in 2^{\mathbb{N}}.$$

$$= \mathbb{P}(\underbrace{N_{s+h} - N_s}_{\text{independent}} + \underbrace{N_s}_{\text{measurable}} \in B \mid N_s = k)$$

$$\underset{\text{Theorem A.3}}{=} \mathbb{P}(\underbrace{N_{s+h} - N_s}_{\overset{d}{=} N_h} + k \in B)$$

$$= \sum_{\ell=0}^{\infty} e^{-\lambda h} \frac{(\lambda h)^{\ell}}{\ell!} \delta_{\ell+k}(B)$$

We check the definition of a transition function:

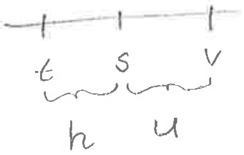
(i) $2^{\mathbb{N}} \ni B \longmapsto P_h(k,B)$ is a probability measure: clear.

(ii) $k \longmapsto \mathbb{P}_h(k,\cdot B)$: measurability w.r.t. the power set always holds: $2^{\mathbb{N}}$-measurability is clear.

(iii) $$\mathbb{P}_0(k,B) = e^{-\lambda\cdot 0} \underbrace{\frac{0^0}{0!}}_{=1} \delta_k(B) + 0 = \delta_k(B)$$

(iv) Chapman-Kolmogorov:

$$P_{h+u}(k,B) \overset{?}{=} \int_{\mathbb{N}} P_h(m,B)\, P_u(k, dm)$$



$$= \sum_{m=0}^{\infty} P_h(m,B)\, \underbrace{P_u(k,\{m\})}_{e^{-\lambda u} \frac{(\lambda u)^{\ell}}{\ell!} \delta_{k+\ell}(\{m\})}$$

$$= \sum_{m=k}^{\infty} \left[\sum_{n=0}^{\infty} e^{-\lambda h} \frac{(\lambda h)^n}{n!} \delta_{m+n}(B) \right] \times e^{-\lambda u} \frac{(\lambda u)^{m-k}}{(m-k)!}$$

$k+\ell = m \Leftrightarrow \ell = m-k$

$$= e^{-\lambda(h+u)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\lambda h)^n}{n!} \frac{(\lambda u)^m}{m!} \delta_{m+n+k}(B)$$

$$= e^{-\lambda(h+u)} \sum_{\ell=0}^{\infty} \left(\sum_{m+n=\ell} \frac{(\lambda h)^n (\lambda u)^m}{n!\, m!} \right) \delta_{\ell+k}(B)$$

$$= e^{-\lambda(h+u)} \sum_{\ell=0}^{\infty} \frac{(\lambda(h+u))^{\ell}}{\ell!} \delta_{\ell+k}(B)$$

$$= P_{h+u}(k, B).$$

b) We use $(\mathcal{F}_t^N)_{t\geq 0}$, the natural filtration of $(N_t)_{t\geq 0}$ and show that for $t \leq s$

$$\underbrace{\mathbb{E}[f(N_s) \mid \mathcal{F}_t^N]} = \int_E f(y) P_{s-t}(N_t, dy)$$

$$= \mathbb{E}\, f(\underbrace{N_s - N_t}_{\text{indep.}} + \underbrace{N_t}_{\text{meas.}} \mid \mathcal{F}_t^N) \underset{\text{Thm A.1}}{=} \mathbb{E}\, f(N_s - N_t + m)\Big|_{m=N_t}$$

$$= \sum_{\ell=0}^{\infty} f(m+\ell)\, e^{-\lambda(s-t)} \frac{(\lambda(s-t))^{\ell}}{\ell!}\Big|_{m=N_t}$$

$$= \sum_{y=0}^{\infty} f(y) \underbrace{\sum_{\ell=0}^{\infty} e^{-\lambda(s-t)} \frac{(\lambda(s-t))^{\ell}}{\ell!} \delta_{N_t+\ell}(\{y\})}_{P_{t-s}(N_t, \{y\})}$$

(2) $\mathcal{H} := \{ Y : Y \text{ integrable and } \mathbb{E}[Y \mid \mathcal{F}_t] = \mathbb{E}[Y \mid X_t] \text{ holds} \}$

(i) $\alpha, \beta \in \mathbb{R}$, $Y, Z \in \mathcal{H}$. Then

$$\mathbb{E}[\alpha Y + \beta Z \mid \mathcal{F}_t] \underset{\substack{\text{linearity} \\ \text{of cond. expectation}}}{=} \alpha\, \mathbb{E}[Y \mid \mathcal{F}_t] + \beta\, \mathbb{E}[Z \mid \mathcal{F}_t]$$

$$\underset{\mathcal{H}}{=} \alpha\, \mathbb{E}[Y \mid X_t] + \beta\, \mathbb{E}[Z \mid X_t]$$

$$= \mathbb{E}[\alpha Y + \beta Z \mid X_t] \quad \text{a.s.}$$

(ii) $(f_n)_{n\geq 1} \subseteq \mathcal{H}$, $0 \leq f_n \uparrow f$, f bounded.

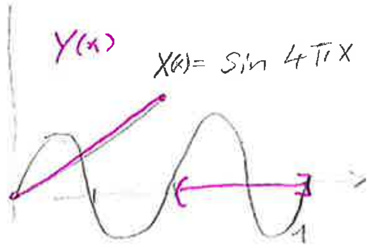
We have $\mathbb{E}[f \mid \mathcal{F}_t] = \lim_n \mathbb{E}[f_n \mid \mathcal{F}_t]$

monotone convergence for cond. expectation

$= \lim_n \mathbb{E}[f_n \mid X_t]$ ($\mathcal{H}$)

$= \mathbb{E}[f \mid X_t]$.

(3)



$\sigma(Y) = \{ A : A = B \cup (\tfrac{1}{2}, 1] \cup \{0\}$, $A = B$ where $B \in \mathcal{B}((0, \tfrac{1}{2}]) \}$

Hence a $\sigma(Y)$ measurable function is a Borel function which is constant on $(\tfrac{1}{2}, 1]$

We conjecture

$$\mathbb{E}[X \mid \sigma(Y)] = \sin(4\pi \cdot)\, \mathbb{1}_{[0,\frac{1}{2}]} =: g$$

It is easy to see that $\mathbb{E}(X \mathbb{1}_A) = \mathbb{E}(g \mathbb{1}_A) \quad \forall A \in \sigma(Y)$.

(4) $A = B \in \mathcal{F}_t \cap \sigma(X_s : s \geq t)$. Then

$$\underbrace{\mathbb{P}(A \cap B \mid X_t)} = \underbrace{\mathbb{P}(A \mid X_t)\, \mathbb{P}(B \mid X_t)}$$

$$= \underbrace{\mathbb{P}(A \mid X_t)} \qquad = \mathbb{P}(A \mid X_t)^2$$

$$\mathbb{E}[\mathbb{1}_A \mid X_t] = \mathbb{E}[\underbrace{\mathbb{1}_A}_{\text{measurable}} \mid \mathcal{F}_t] = \mathbb{1}_A \quad \text{a.s.}$$

(5) $Y = (X_{t_1}, \dots, X_{t_n})^T$ is Gaussian $\iff$

$$Y = \underset{\substack{\uparrow \\ n \times n \\ \text{matrix}}}{A}\, \underbrace{(f_1, \dots, f_n)^T}_{\text{standard normal}} + b \in \mathbb{R}^n$$

Since $Z = (X_{t_1}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}})^T$

we get from $Z = BY$ with $B = \begin{pmatrix} 1 & & 0 \\ -1 & \ddots & \\ 0 & \ddots & \\ & -1 & 1 \end{pmatrix}$

so $Z = BA(f_1, \dots, f_n) + Bb$ is Gaussian as well.

(ii) $\mathbb{E} X_{t_1} X_{t_2} = \underbrace{\mathbb{E} X_{t_1}^2}_{= t_1} + \underbrace{\mathbb{E} X_{t_1}(X_{t_2} - X_{t_1})}_{= 0}$

similarly:

$\mathbb{E} X_{t_1} X_{t_3} = t_1$

$\mathbb{E} X_{t_2} X_{t_3} = t_2$

$Y = (X_{t_1}, X_{t_2}, X_{t_3})$

$$R_Y = \begin{pmatrix} t_1 & t_1 & t_1 \\ t_1 & t_2 & t_2 \\ t_1 & t_2 & t_3 \end{pmatrix}$$