CHAPTER 3

Geometric preliminaries

In this chapter we discuss certain geometric preliminaries required for studying the geodesic X-ray transform on a general compact Riemannian manifold (M, g) with boundary. We will discuss the concept of a compact non-trapping manifold with strictly convex boundary. We will also introduce the exit time function $\tau(x, v)$, the geodesic vector field X, geodesic flow φ_t, and the scattering relation α and study their basic properties. The chapter will conclude with the important notion of a simple manifold, including several equivalent definitions.

3.1. Non-trapping and strict convexity

Let (M, g) be a compact, connected and oriented Riemannian manifold with smooth boundary ∂M and dimension $n \geq 2$.

Geodesics travel at constant speed, so we fix the speed to be one. We pack positions and velocities together in what we call the *unit sphere bundle* SM. This consists of pairs (x, v), where $x \in M$ and $v \in T_xM$ with norm $|v|_g = 1$, where g is the inner product in the tangent space at x (i.e. the Riemannian metric). Given $(x, v) \in SM$, let $\gamma_{x,v}$ denote the unique geodesic determined by (x, v) so that $\gamma_{x,v}(0) = x$ and $\dot{\gamma}_{x,v}(0) = v$. For any $(x, v) \in SM$ the geodesic $\gamma_{x,v}$ is defined on a maximal interval of existence that we denote by $[-\tau_-(x, v), \tau_+(x, v)]$ where $\tau_\pm(x, v) \in [0, \infty]$, so that

$$\gamma_{x,v} : [-\tau_-(x, v), \tau_+(x, v)] \to M$$

is a smooth curve that cannot be extended to any larger interval as a smooth curve in M.

Definition 3.1. We let

$$\tau(x, v) := \tau_+(x, v).$$

Thus $\tau(x, v)$ is the *exit time* when the geodesic $\gamma_{x,v}$ exits M.

Exercise 3.2. Give examples of compact manifolds (M, g) with boundary and points $(x, v) \in SM$ where the following holds:

(a) The first time when $\gamma_{x,v}$ hits ∂M is different from the exit time $\tau(x, v)$.

(b) $\tau(x, v)$ is not continuous on SM.

(c) $\tau_{\pm}(x, v) = \infty$.

(d) $\tau_{-}(x, v)$ is finite but $\tau_{+}(x, v) = \infty$.

If some geodesic has infinite length, one needs to be careful when studying the geodesic X-ray transform since the integral of a smooth function over such a geodesic may not be finite. For the most part of this book, we will be working on manifolds where this issue does not appear.

Definition 3.3. We say that (M, g) is *non-trapping* if $\tau(x, v) < \infty$ for all $(x, v) \in SM$. Equivalently, there are no geodesics in M with infinite length.

Example 3.4. Compact subdomains in $\mathbb{R}^n$ and in hyperbolic space are non-trapping, and so are the small spherical caps in Example 2.22. Large spherical caps, catenoid type surfaces and flat cylinders have trapped geodesics (see Examples 2.23–2.25).

Unit tangent vectors at the boundary of M constitute the boundary ∂SM of SM and will play a special role. Specifically

$$\partial SM := \{(x, v) \in SM : \ x \in \partial M\}.$$

We will need to distinguish those tangent vectors pointing inside ("influx boundary") and those pointing outside ("outflux boundary"), so we define two subsets of ∂SM

$$\partial_\pm SM := \{(x,v) \in \partial SM : \pm\langle v, \nu(x)\rangle_g \geq 0\}$$

where ν denotes the **inward** unit normal vector to the boundary. The convention of using the inward unit normal instead of the outward unit normal will eliminate some minus signs in the volume form $d\mu$ in Section 4.1 and certain other places. We also denote

$$\partial_0 SM := \partial_+ SM \cap \partial_- SM.$$

Note that one has $\partial_0 SM = S(\partial M)$.

DEFINITION 3.5. The *geodesic X-ray transform* of a function $f \in C^\infty(M)$ on a compact non-trapping manifold (M,g) with smooth boundary is the function If defined by

(3.1)
$$If(x,v) = \int_0^{\tau(x,v)} f(\gamma_{x,v}(t))\,dt, \qquad (x,v) \in \partial_+ SM.$$

The idea is that if M is non-trapping, then any geodesic γ going through some point $(y,w) \in SM$ has an initial point $(x,v) = \gamma_{y,w}(-\tau_-(y,w))$. One must have $(x,v) \in$

∂SM, since if one had $(x, v) \in SM^{\mathrm{int}}$ then the geodesic could be extended further in both directions. Moreover, one must have $(x, v) \in \partial_+ SM$ since any geodesic starting at a point in $\partial SM \setminus \partial_+ SM$ could be extended further for small negative times.

The argument in the preceding paragraph shows that on non-trapping manifolds, there is a one-to-one correspondence between the set of unit speed geodesics and the set $\partial_+ SM$ of their initial points. Parametrizing geodesics by their initial points in $\partial_+ SM$ means that we are using the *fan beam* parametrization of geodesics.

REMARK 3.6. Note that the fan beam parametrization is different from the *parallel beam* parametrization that we used in Chapter 1, and also from the parametrization used in Section 2.4 for geodesics of a radial sound speed under the Herglotz condition based on their closest point to the origin.

Since f is smooth and the point $\gamma_{x,v}(t)$ depends smoothly on (x, v), the formula (3.1) shows that the regularity properties of If are decided by the regularity properties of the exit time function $\tau(x, v)$. If the boundary of M is not strictly convex, it can happen that τ is discontinuous. On the other hand, if ∂M is strictly convex then τ will be continuous and in fact smooth in most places, and the theory will be particularly clean.

For a precise definition of when the boundary ∂M is strictly convex, we will use the *second fundamental form* of ∂M that describes how ∂M sits inside M. Recall that the (scalar) second fundamental form is the bilinear form on $T(\partial M)$ given by

$$\Pi_x(v, w) := -\langle \nabla_v \nu, w \rangle_g,$$

where $x \in \partial M$ and $v, w \in T_x \partial M$. Here ∇ is the Levi-Civita connection of g, and on the right hand side ν is extended arbitrarily as a smooth vector field in M (recall that $\nabla_X Y|_x$ only depends on $X|_x$ and the value of Y along any curve $\eta(t)$ with $\dot{\eta}(0) = X|_x$, so that $\Pi_x(v, w)$ does not depend on the choice of the extension of ν).

Definition 3.7. We shall say that ∂M is *strictly convex* if Π_x is positive definite for all $x \in \partial M$.

The combination of non-trapping with strict convexity of the boundary will produce several desirable properties. In fact, many results in this book will be stated either for compact non-trapping manifolds with strictly convex boundary, or for simple manifolds which satisfy the additional condition that geodesics do not have conjugate points.

We already encountered the notion of strict convexity in Section 2.5, where this notion was related to the behaviour of tangential geodesics. We wish to show that a similar characterization exists in the general case. To do this, it is convenient to introduce the following notions.

Lemma 3.8 (Closed extension). *Let (M, g) be a compact manifold with smooth boundary. There is a closed (=compact without boundary) connected manifold (N, g) having the same dimension as M so that (M, g) is isometrically embedded in (N, g).*

Proof. (Special case) The lemma has an easy proof in the special case where M is a subset of $\mathbb{R}^n$. In that case it is enough to consider some cube $N = [-R, R]^n$ with $M \subset N^{\mathrm{int}}$, and to extend g smoothly as a $2R$-periodic positive definite symmetric matrix function in N. Identifying the opposite sides of N, we see that (N, g) becomes a torus with (M, g) embedded in its interior. Then (N, g) is the required extension. $\square$

Exercise 3.9. Prove Lemma 3.8 in general, by considering the double of the manifold M.

If (N, g) is a closed extension of (M, g), we continue to write $\gamma_{x,v}(t)$ for the geodesic in (N, g). One benefit of working with a closed extension is that now $\gamma_{x,v}(t)$ is well defined and smooth for all $t \in \mathbb{R}$.

Lemma 3.10 (Boundary defining function). *Let (M, g) be a compact manifold with smooth boundary, and let (N, g) be a closed extension. There is a function $\rho \in C^\infty(N)$, called a* boundary defining function, *so that $\rho(x) = d(x, \partial M)$ near ∂M in M, and*

$$M = \{x \in N : \rho \geq 0\},$$

$$\partial M = \{x \in N : \rho = 0\},$$

$$N \setminus M = \{x \in N : \rho < 0\}.$$

One has $\nabla\rho(x) = \nu(x)$ for all $x \in \partial M$.

Exercise 3.11. Prove Lemma 3.10.

The following result shows that the second fundamental form of ∂M is given by the Riemannian Hessian of ρ, defined in terms of the total covariant derivative ∇ by

$$\mathrm{Hess}(\rho) = \nabla^2\rho = (\partial_{x_j x_k}\rho - \Gamma^l_{jk}\partial_{x_l}\rho)\, dx^j \otimes dx^k.$$

Moreover, strict convexity of the boundary can indeed be characterized by the behaviour of tangential geodesics.

LEMMA 3.12 (Strictly convex boundary). *If* (M, g) *is a compact manifold with smooth boundary and* ρ *is as in Lemma 3.10, then for any* $(x, v) \in \partial_0 SM$ *one has*

$$-\Pi_x(v,v) = \mathrm{Hess}_x(\rho)(v,v) = \frac{d^2}{dt^2}\rho(\gamma_{x,v}(t))\Big|_{t=0}.$$

Thus ∂M *is strictly convex iff any geodesic in* N *starting from some point* $(x, v) \in \partial_0 SM$ *satisfies* $\frac{d^2}{dt^2}\rho(\gamma_{x,v}(t))\big|_{t=0} < 0$. *In particular, any geodesic tangent to* ∂M *stays outside* M *for small positive and negative times, and any maximal* M*-geodesic going from* ∂M *into* M *stays in* M^{int} *except for its endpoints.*

The proof will follow from the next lemma, which will also be useful later.

LEMMA 3.13. *Let* ρ *be as in Lemma 3.10, and consider the smooth function*

$$h : SN \times \mathbb{R} \to \mathbb{R}, \quad h(x, v, t) = \rho(\gamma_{x,v}(t)).$$

If $(x, v) \in SN$ *and if* t_0 *is such that* $x_0 := \gamma_{x,v}(t_0) \in \partial M$*, then one has*

$$\begin{aligned} h(x, v, t_0) &= 0, \\ \frac{\partial h}{\partial t}(x, v, t_0) &= \langle \nu(x_0), \dot{\gamma}_{x,v}(t_0) \rangle, \\ \frac{\partial^2 h}{\partial t^2}(x, v, t_0) &= \langle \nabla_{\dot{\gamma}_{x,v}(t_0)} \nabla \rho, \dot{\gamma}_{x,v}(t_0) \rangle = \mathrm{Hess}_{x_0}(\dot{\gamma}_{x,v}(t_0), \dot{\gamma}_{x,v}(t_0)). \end{aligned}$$

Proof. Write $\gamma(t) = \gamma_{x,v}(t)$. Since $\rho|_{\partial M} = 0$ one has $h(x, v, t_0) = 0$. Moreover, using that $\nabla \rho|_{\partial M} = \nu$ we compute

$$\frac{\partial h}{\partial t}(x, v, t_0) = d\rho|_{x_0}(\dot{\gamma}(t_0)) = \langle \nu(x_0), \dot{\gamma}(t_0) \rangle.$$

Finally, one has

$$\begin{aligned} \frac{\partial^2 h}{\partial t^2}(x, v, t_0) &= \frac{d}{dt}(d\rho|_{\gamma(t)}(\dot{\gamma}(t)))\Big|_{t=t_0} = \frac{d}{dt}\langle \nabla \rho|_{\gamma(t)}, \dot{\gamma}(t) \rangle\Big|_{t=t_0} \\ &= \langle \nabla_{\dot{\gamma}(t)} \nabla \rho, \dot{\gamma}(t) \rangle + \langle \nabla \rho, \nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) \rangle|_{t=t_0}. \end{aligned}$$

The last term is zero since γ is a geodesic (i.e. $\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = 0$). The definition of the total covariant derivative ∇ gives that $\langle \nabla_{\dot{\gamma}(t)} \nabla \rho, \dot{\gamma}(t) \rangle|_{t=t_0} = \nabla^2 \rho(\dot{\gamma}(t_0), \dot{\gamma}(t_0))$, which finishes the proof. $\square$

Proof of Lemma 3.12. Let $(x, v) \in \partial_0 SM$ and write $\gamma(t) = \gamma_{x,v}(t)$ and $h(x, v, t) = \rho(\gamma(t))$. By Lemma 3.13 one has

$$\begin{aligned} h(x,v,0) &= 0, \\ \frac{\partial h}{\partial t}(x,v,0) &= 0, \\ \frac{\partial^2 h}{\partial t^2}(x,v,0) &= \langle \nabla_v \nabla \rho, v\rangle = \mathrm{Hess}_x(v,v). \end{aligned}$$

But $\nabla \rho|_{\partial M} = \nu$, which shows that $\langle \nabla_v \nabla \rho, v\rangle = -\Pi_x(v,v)$. This proves the required formula.

Now ∂M is strictly convex $\iff \Pi_x(v,v) > 0$ for all $(x,v) \in \partial_0 SM \iff \partial_t^2 h(x,v,0) < 0$ for all $(x,v) \in \partial_0 SM$. By the Taylor formula

$$\rho(\gamma(t)) = h(x,v,t) = -\frac{1}{2}\Pi_x(v,v)t^2 + O(t^3)$$

when $|t|$ is small. This shows that for small positive and negative times $\rho(\gamma(t)) < 0$, i.e. $\gamma_{x,v}(t)$ is in $N \setminus M$. $\square$

3.2. Regularity of the exit time

We will now discuss in detail the regularity of the fundamental exit time function τ on a compact non-trapping manifold (M,g) with strictly convex boundary. Note that by definition $\tau|_{\partial_- SM} = 0$.

EXAMPLE 3.14. Let $M = \overline{\mathbb{D}}$ be the closed unit disk in the plane, and let $g = e$ be the Euclidean metric. Take

$x = (0, -1)$ and let $v_\theta = (\cos\theta, \sin\theta)$. An easy geometric argument shows that

$$\tau(x, v_\theta) = \begin{cases} 2\sin\theta, & \theta \in [0, \pi], \\ 0, & \theta \in [-\pi, 0] \end{cases}$$

Thus τ is continuous on ∂SM but fails to be continuously differentiable in tangential directions. However, the odd extension with respect to $(x, v) \mapsto (x, -v)$,

$$\tilde{\tau}(x, v_\theta) := \begin{cases} 2\sin\theta, & \theta \in [0, \pi], \\ 2\sin\theta, & \theta \in [-\pi, 0] \end{cases}$$

is smooth.

Exercise 3.15. Verify the claims in Example 3.14.

We will now show that the properties of the exit time function in Example 3.14 are valid in general.

Lemma 3.16. *Let (M, g) be a compact non-trapping manifold with strictly convex boundary. Then τ is continuous on SM and smooth on $SM \setminus \partial_0 SM$.*

Proof. The proof that τ is continuous is left as an exercise. Let (N, g) be a closed extension of (M, g) and let ρ be a boundary defining function as in Lemma 3.10. We define a function $h : SN \times \mathbb{R} \to \mathbb{R}$ by setting $h(x, v, t) := \rho(\gamma_{x,v}(t))$. Then

$$\frac{\partial h}{\partial t}(x, v, t) = d\rho(\dot{\gamma}_{x,v}(t)) = \langle \nabla\rho(\gamma_{x,v}(t)), \dot{\gamma}_{x,v}(t)\rangle.$$

Assume that $(x, v) \in SM \backslash \partial_0 SM$, and set $y := \gamma_{x,v}(\tau(x, v)) \in \partial M$. Since y is the final point of the geodesic, one must have $\dot{\gamma}_{x,v}(\tau(x, v)) \in \partial_- SM$ (otherwise the geodesic could be extended further). By strict convexity, one must also have $\dot{\gamma}_{x,v}(\tau(x, v)) \notin \partial_0 SM$ (since otherwise $\tau(x, v) = 0$ and (x, v) would be in $\partial_0 SM$).

Thus $\dot{\gamma}_{x,v}(\tau(x, v)) \in \partial SM \backslash \partial_+ SM$, i.e. $\langle \dot{\gamma}_{x,v}(\tau(x, v)), \nu\rangle < 0$. Since $\nabla\rho$ agrees with ν on ∂M, we see that

$$\left.\frac{\partial h}{\partial t}\right|_{t=\tau(x,v)} < 0.$$

Since $h(x, v, \tau(x, v)) = 0$ and h is smooth, the implicit function theorem ensures that τ is smooth in $SM \setminus \partial_0 SM$.

□

The set $\partial_0 SM$ where τ is not smooth is often called the *glancing region*.

Exercise 3.17. Show that τ is continuous in SM.

Exercise 3.18. Show that τ is indeed not smooth at the glancing region $\partial_0 SM$.

The next result shows that the odd extension of $\tau|_{\partial SM}$ is smooth.

LEMMA 3.19. *Let (M,g) be a non-trapping manifold with strictly convex boundary and let*

$$\tilde{\tau}(x,v) := \begin{cases} \tau(x,v), & (x,v) \in \partial_+ SM, \\ -\tau(x,-v), & (x,v) \in \partial_- SM. \end{cases}$$

Then $\tilde{\tau} \in C^\infty(\partial SM)$; in particular $\tau : \partial_+ SM \to \mathbb{R}$ is smooth.

PROOF. As before we let $h(x,v,t) := \rho(\gamma_{x,v}(t))$ for $(x,v) \in \partial SM$ and $t \in \mathbb{R}$. Note that by Lemma 3.13

- $h(x,v,0) = 0$;
- $\frac{\partial}{\partial t}\big|_{t=0} h(x,v,t) = \langle \nu(x), v \rangle$;
- $\frac{\partial^2}{\partial t^2}\big|_{t=0} h(x,v,t) = \mathrm{Hess}_x \rho(v,v)$.

Hence for some smooth function $R(x,v,t)$ we can write

$$h(x,v,t) = \langle \nu(x), v \rangle t + \frac{1}{2}\mathrm{Hess}_x \rho(v,v) t^2 + R(x,v,t) t^3.$$

Since $h(x,v,\tilde{\tau}(x,v)) = 0$, it follows that

$$\langle \nu(x), v \rangle + \frac{1}{2}\mathrm{Hess}_x \rho(v,v) \tilde{\tau} + R(x,v,\tilde{\tau}) \tilde{\tau}^2 = 0. \tag{3.2}$$

Here we used that $\tilde{\tau}(x,v) = 0$ iff $(x,v) \in \partial_0 SM$. Hence if we let

$$F(x,v,t) := \langle \nu(x), v \rangle + \frac{1}{2}\mathrm{Hess}_x \rho(v,v) t + R(x,v,t) t^2$$

we see that F is smooth, $F(x,v,\tilde{\tau}(x,v)) = 0$ and

$$\frac{\partial}{\partial t}\Big|_{t=0} F(x,v,t) = \frac{1}{2}\mathrm{Hess}_x\rho(v,v).$$

But for $(x,v) \in \partial_0 SM$, $\mathrm{Hess}_x\rho(v,v) = -\Pi_x(v,v) < 0$ and thus by the implicit function theorem, $\tilde{\tau}$ is smooth in a neighbourhood of $\partial_0 SM$. Since $\tilde{\tau}$ is smooth in $\partial SM \setminus \partial_0 SM$ the lemma follows. □

REMARK 3.20. Note that we can define $\tilde{\tau}$ on all SM by setting $\tilde{\tau}(x,v) := \tau(x,v) - \tau(x,-v)$. The restriction of this function to ∂SM coincides with the definition of $\tilde{\tau}$ given by Lemma 3.19. It turns out that in fact $\tilde{\tau} \in C^\infty(SM)$; see Lemma 3.24 below.

Define

$$\mu(x,v) := \langle \nu(x), v \rangle, \qquad (x,v) \in \partial SM.$$

This expression appears in Santaló's formula, which is an important change of variables formula on SM. We record the following result for later purposes.

LEMMA 3.21. *Let (M,g) be a compact non-trapping manifold with strictly convex boundary. The function $\mu/\tilde{\tau}$ extends to a smooth positive function on ∂SM whose value* *at* $(x, v) \in \partial_0 SM$ *is*

$$\frac{\Pi_x(v, v)}{2}.$$

PROOF. Using (3.2) we can write

$$\mu(x, v) = -\frac{1}{2}\mathrm{Hess}_x \rho(v, v)\tilde{\tau} - R(x, v, \tilde{\tau})\tilde{\tau}^2$$

and hence for $(x, v) \in \partial SM \setminus \partial_0 SM$ near $\partial_0 SM$ we can write

$$\mu/\tilde{\tau} = -\frac{1}{2}\mathrm{Hess}_x \rho(v, v) - R(x, v, \tilde{\tau})\tilde{\tau}.$$

But the right hand side of the last equation is a smooth function near $\partial_0 SM$ since R and $\tilde{\tau}$ are; its value at $(x, v) \in \partial_0 SM$ is $\Pi_x(v, v)/2$. Finally, observe that μ and $\tilde{\tau}$ are both positive for $(x, v) \in \partial_+ SM \setminus \partial_0 SM$ and both negative for $(x, v) \in \partial_- SM \setminus \partial_0 SM$. $\square$

Even more precise regularity properties of the exit time function τ near $\partial_0 SM$ can be obtained from the next lemma. This will be the main tool when studying regularity properties of solutions to transport equations. The proof is motivated by the theory of Whitney folds [**Hör85**, Appendix C.4].

LEMMA 3.22. *Let* (M, g) *be compact with smooth boundary, let* $(x_0, v_0) \in \partial_0 SM$*, and let* ∂M *be strictly convex* *near x_0. Assume that M is embedded in a compact manifold N without boundary. Then, near (x_0, v_0) in SM, one has*

$$\tau(x,v) = Q(\sqrt{a(x,v)}, x, v),$$
$$-\tau(x,-v) = Q(-\sqrt{a(x,v)}, x, v),$$

where Q is a smooth function near $(0, x_0, v_0)$ in $\mathbb{R} \times SN$, a is a smooth function near (x_0, v_0) in SN, and $a \geq 0$ in SM.

PROOF. This follows directly by applying Lemma 3.23 below to $h(t,x,v) = \rho(\gamma_{x,v}(t))$ near $(0, x_0, v_0)$, where ρ is a boundary defining function for M. □

LEMMA 3.23. *Let $h(t,y)$ be smooth near $(0,y_0)$ in $\mathbb{R} \times \mathbb{R}^N$. If*

$$h(0,y_0) = 0, \qquad \partial_t h(0,y_0) = 0, \qquad \partial_t^2 h(0,y_0) < 0,$$

then one has

$$h(t,y) = 0 \text{ near } (0,y_0) \text{ when } h(0,y) \geq 0 \quad \Longleftrightarrow \quad t = Q(\pm\sqrt{a(y)}, y)$$

where Q is a smooth function near $(0, y_0)$ in $\mathbb{R} \times \mathbb{R}^N$, a is a smooth function near y_0 in $\mathbb{R}^N$, and $a(y) \geq 0$ when $h(0,y) \geq 0$. Moreover, $Q(\sqrt{a(y)}, y) \geq Q(-\sqrt{a(y)}, y)$ when $h(0,y) \geq 0$.

PROOF. We use the same argument as in [**Hör85**, Theorem C.4.2]. Using that $\partial_t^2 h(0, y_0) < 0$, the implicit function theorem gives that

$$\partial_t h(t, y) = 0 \text{ near } (0, y_0) \quad \Longleftrightarrow \quad t = g(y)$$

where g is smooth near y_0 and $g(y_0) = 0$. Write

$$h_1(s, y) := h(s + g(y), y).$$

Then $\partial_s h_1(0, y) = 0$ and $\partial_s^2 h_1(0, y_0) < 0$. Thus by the Taylor formula we have

$$h_1(s, y) = h_1(0, y) - s^2 F(s, y)$$

where F is smooth near $(0, y_0)$ and $F(0, y_0) > 0$. We define

$$r(s, y) := sF(s, y)^{1/2}$$

and note that $r(0, y_0) = 0$, $\partial_s r(0, y_0) > 0$. Thus the map $(s, y) \mapsto (r(s, y), y)$ is a local diffeomorphism near $(0, y_0)$, and there is a smooth function S near $(0, y_0)$ so that

$$r(s, y) = \bar{r} \quad \Longleftrightarrow \quad s = S(\bar{r}, y).$$

Moreover, $\partial_r S(0, y_0) > 0$. Define the function

$$h_2(r, y) := h_1(0, y) - r^2.$$

Now

$$\begin{aligned} h(t,y) &= h_1(t-g(y),y) = h_1(0,y) - (t-g(y))^2 F(t-g(y),y) \\ &= h_2(r(t-g(y),y),y). \end{aligned}$$

Thus $h(t,y) = 0$ is equivalent with

$$r(t-g(y),y)^2 = h_1(0,y) = h(g(y),y). \tag{3.3}$$

We claim that

$$h(g(y),y) \geq 0 \text{ near } y_0 \text{ when } h(0,y) \geq 0. \tag{3.4}$$

If (3.4) holds, then we may solve (3.3) to obtain

$$h(t,y) = 0 \text{ near } (0,y_0) \text{ when } h(0,y) \geq 0 \iff r(t-g(y),y) = \pm\sqrt{h(g(y),y)}.$$

The last condition is equivalent with

$$t - g(y) = S(\pm\sqrt{h(g(y),y)}, y).$$

This proves the lemma upon taking $Q(r,y) = g(y) + S(r,y)$ and $a(y) = h(g(y),y)$ (note that $r \mapsto Q(r,y)$ is increasing since $\partial_r S(0,y_0) > 0$). To prove (3.4), we use the Taylor formula

$$h(g(y)+s,y) = h(g(y),y) + \partial_t h(g(y),y)s + G(s,y)s^2$$

where $G(0, y_0) < 0$. Choosing $s = -g(y)$ and using that $\partial_t h(g(y), y) = 0$ shows that $h(g(y), y) \geq h(0, y)$ near $y = y_0$, and thus (3.4) indeed holds. □

LEMMA 3.24. *Let (M, g) be a non-trapping manifold with strictly convex boundary. Then the functions*

$$\tilde{\tau}(x, v) := \tau(x, v) - \tau(x, -v), \quad \textit{and} \quad T(x, v) := \tau(x, v)\tau(x - v)$$

are smooth in SM.

PROOF. Given the properties of τ we just have to prove smoothness near a glancing point $(x_0, v_0) \in \partial_0 SM$. By Lemma 3.22 given $(x, v) \in SM$ near $(x_0, v_0) \in \partial_0 SM$ we have:

$$\tilde{\tau}(x, v) = Q(\sqrt{a(x, v)}, x, v) + Q(-\sqrt{a(x, v)}, x, v).$$

Since we can write $H(r^2, x, v) = Q(r, x, v) + Q(-r, x, v)$, where H is smooth near $(0, x_0, v_0)$, we deduce that

$$\tilde{\tau}(x, v) = H(a(x, v), x, v)$$

thus showing smoothness of $\tilde{\tau}$. The statement for T follows by taking products, rather than sums. □

REMARK 3.25. Using this lemma, it is possible to write the functions Q and a from Lemma 3.22 in terms of $\tilde{\tau}$ and

T. Indeed, since τ satisfies the quadratic equation

$$\tau(\tau - \tilde{\tau}) = T$$

we have

$$\tau = \frac{\tilde{\tau} + \sqrt{\tilde{\tau}^2 + 4T}}{2}$$

with $\tilde{\tau}, T \in C^\infty(SM)$. Thus $Q(t,x,v) = (\tilde{\tau}(x,v) + t)/2$ and $a = \tilde{\tau}^2 + 4T$.

3.3. The geodesic flow and the scattering relation

Let (M,g) be a compact, connected and oriented Riemannian manifold with boundary ∂M and dimension $n \geq 2$. By Lemma 3.8 we may assume that (M,g) is isometrically embedded into a closed manifold (N,g) of the same dimension.

The geodesics of (N,g) are defined for all times in $\mathbb{R}$. We pack them into what is called the *geodesic flow*. For each $t \in \mathbb{R}$ this is a diffeomorphism

$$\varphi_t : SN \to SN$$

defined by

$$\varphi_t(x,v) := (\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t)).$$

This is a *flow*, i.e. $\varphi_{t+s} = \varphi_t \circ \varphi_s$ for all $s,t \in \mathbb{R}$. The flow has an infinitesimal generator called the *geodesic vector*