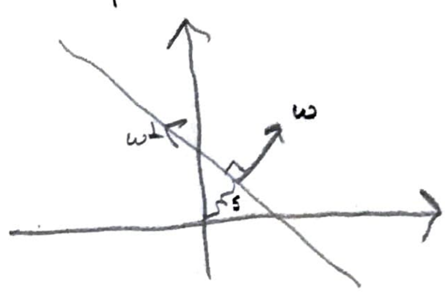
1. Radon transform in the plane

1.1 Uniqueness and stability

The $\begin{cases}\text{X-ray} \\ \text{Radon}\end{cases}$ transform of a function f in $\mathbb{R}^n$ encodes the integrals of f over $\begin{cases}\text{lines} \\ (n-1)\text{-planes}\end{cases}$ in $\mathbb{R}^n$.

We assume $n=2$, so the two coincide. We parametrize lines in $\mathbb{R}^2$ by normal vector ω and distance s from the origin.



<u>Def. 1.1</u> If $f \in C_c^\infty(\mathbb{R}^2)$, the Radon transform Rf of f is

$$Rf(s,\omega) = \int_{-\infty}^{\infty} f(s\omega + t\omega^\perp)\,dt, \qquad s \in \mathbb{R},\ \omega \in S^1.$$

Here $(\omega_1, \omega_2)^\perp = (-\omega_2, \omega_1)$.

Arises in X-ray computed tomography (imaging along 2D slices).

<u>Inverse problem</u>: determine f from knowledge of Rf.

If $f \in C_c^\infty(\mathbb{R}^2)$, then $Rf \in C^\infty(\mathbb{R} \times S^1)$ and each $Rf(\cdot,\omega)$ is compactly supported. Translation invariance (exercise):

$$R(f(\cdot - s_0\omega))(s,\omega) = Rf(s-s_0,\omega).$$

There is a well-known relation between Rf and the Fourier transform $\hat{f} = \mathscr{F}\{f\}$. For $h \in C_c^\infty(\mathbb{R}^n)$, write

$$\hat{h}(\xi) = \mathscr{F}\{h\}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} h(x)\,dx, \quad \xi \in \mathbb{R}^n.$$

Properties:

1. $\mathcal{F}$ is bounded $L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$.
2. $\mathcal{F}$ is bijective $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, where $\mathcal{S}(\mathbb{R}^n)$ is the <u>Schwartz space</u> of all $f \in C^\infty(\mathbb{R}^n)$ with $x^\alpha \partial^\beta f \in L^\infty(\mathbb{R}^n)$ for all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$.
3. Any $f \in \mathcal{S}(\mathbb{R}^n)$ can be recovered from $\hat{f}$ by the <u>Fourier inversion formula</u>

$$f(x) = \mathcal{F}^{-1}\{\hat{f}\}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi, \quad x \in \mathbb{R}^n.$$

4. For $f, g \in \mathcal{S}(\mathbb{R}^n)$ one has the <u>Parseval identity</u>

$$\int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{g}(\xi)}\, d\xi = (2\pi)^n \int_{\mathbb{R}^n} f(x)\overline{g(x)}\, dx$$

and <u>Plancherel formula</u>

$$\|\hat{f}\|_{L^2(\mathbb{R}^n)} = (2\pi)^{n/2} \|f\|_{L^2(\mathbb{R}^n)}.$$

5. Fourier transform converts derivatives to polynomials:

$$(D_j f)^\wedge(\xi) = \xi_j \hat{f}(\xi), \qquad D_j = \frac{1}{i}\frac{\partial}{\partial x_j}.$$

Let $(Rf)^\sim(\cdot, \omega)$ be the Fourier transform of $Rf(\cdot,\omega)$.

<u>Thm 1.4</u> (Fourier slice theorem) If $f \in C_c^\infty(\mathbb{R}^2)$, then

$$(Rf)^\sim(\sigma, \omega) = \hat{f}(\sigma\omega), \qquad \sigma \in \mathbb{R},\ \omega \in S^1.$$

<u>Pf</u>

$$(Rf)^\sim(\sigma,\omega) = \int_{-\infty}^{\infty} e^{-i\sigma s}\left[\int_{-\infty}^{\infty} f(s\omega + t\omega^\perp)\, dt\right] ds$$

$$\overset{y = s\omega + t\omega^\perp}{=} \int_{\mathbb{R}^2} e^{-i\sigma\omega\cdot y} f(y)\, dy = \hat{f}(\sigma\omega). \qquad \square$$

<u>Thm 1.5</u> (Uniqueness) If $Rf_1 = Rf_2$, then $f_1 = f_2$.

<u>Pf</u> $Rf_1 = Rf_2 \implies (Rf_1)^\sim(\sigma,\omega) = (R\tilde{f}_2)(\sigma,\omega)$

$$\overset{\text{Thm 1.4}}{\implies} \hat{f}_1(\sigma\omega) = \hat{f}_2(\sigma\omega)\ \forall \sigma, \omega \implies \hat{f}_1 = \hat{f}_2 \implies f_1 = f_2. \qquad \square$$

We wish to prove stability (quantitative uniqueness). For $s \in \mathbb{R}$, define the Sobolev norms

$$\|f\|_{H^s(\mathbb{R}^2)} = \|(1+|\xi|^2)^{s/2}\hat{f}(\xi)\|_{L^2(\mathbb{R}^2)},$$

$$\|Rf\|_{H_T^s(\mathbb{R}\times S^1)} = \|(1+\sigma^2)^{s/2}(Rf)^\sim(\sigma,\omega)\|_{L^2(\mathbb{R}\times S^1)}.$$

Exercise 1.6 If $m \geq 0$ is an integer, show that

$$\|f\|_{H^{2m}(\mathbb{R}^2)} \sim \sum_{|\alpha|\leq 2m} \|\partial^\alpha f\|_{L^2(\mathbb{R}^2)},$$

$$\|Rf\|_{H_T^{2m}(\mathbb{R}\times S^1)} \sim \sum_{j=0}^{2m} \|\partial_s^j Rf\|_{L^2(\mathbb{R}\times S^1)}$$

where $A \sim B$ means that $cA \leq B \leq CA$ for some constants $C, c > 0$ independent of f.

Thus the $H^{2m}(\mathbb{R}^2)$ (resp. $H_T^{2m}(\mathbb{R}\times S^1)$) norm measures the size of the first $2m$ derivatives (resp. first $2m$ derivatives in the s variable, but not in ω) in L^2.

Thm 1.7 (Stability) If $s \in \mathbb{R}$, then

$$\|f_1 - f_2\|_{H^s(\mathbb{R}^2)} \leq \frac{1}{\sqrt{2}} \|Rf_1 - Rf_2\|_{H_T^{s+1/2}(\mathbb{R}\times S^1)}.$$

Pf Let $f = f_1 - f_2$. Then

$$\|f\|^2_{H^s(\mathbb{R}^2)} = \|(1+|\xi|^2)^{s/2}\hat{f}\|^2_{L^2(\mathbb{R}^2)} \overset{\xi=\sigma\omega}{=} \int_0^\infty \int_{S^1} (1+\sigma^2)^s |\hat{f}(\sigma\omega)|^2 \sigma \, d\omega \, d\sigma$$

$$\overset{\text{symmetry}}{=} \frac{1}{2}\int_{-\infty}^\infty \int_{S^1} (1+\sigma^2)^s |\hat{f}(\sigma\omega)|^2 |\sigma| \, d\omega \, d\sigma$$

$$\overset{\text{Fourier slice}}{=} \frac{1}{2}\int_{-\infty}^\infty \int_{S^1} (1+\sigma^2)^s |(Rf)^\sim(\sigma,\omega)|^2 \underbrace{|\sigma|}_{\leq (1+\sigma^2)^{1/2}} \, d\omega \, d\sigma$$

$$\leq \frac{1}{2}\|(1+\sigma^2)^{\frac{s+1/2}{2}}(Rf)^\sim(\sigma,\omega)\|^2_{L^2(\mathbb{R}\times S^1)}$$

$$= \frac{1}{2}\|Rf\|^2_{H_T^{s+1/2}(\mathbb{R}\times S^1)}.$$

$\square$

The last result has a converse:

Thm 1.8 (Continuity) Let $s \in \mathbb{R}$ and let $K \subset \mathbb{R}^2$ be compact. There is $C_K > 0$ so that $\forall f \in C_c^\infty(\mathbb{R}^2)$ with $\mathrm{supp}(f) \subset K$, one has

$$\| Rf \|_{H_T^{s+1/2}(\mathbb{R}\times S^1)} \leq C_K \|f\|_{H^s(\mathbb{R}^2)}.$$

Pf Exercise. $\square$

It follows that R extends as a bounded map

$$R : H_K^s(\mathbb{R}^2) \to H_T^{s+1/2}(\mathbb{R}\times S^1)$$

where $H_K^s(\mathbb{R}^2) = \{ f \in H^s(\mathbb{R}^2) \,;\, \mathrm{supp}(f) \subset K \}$. Thus R is smoothing of order $1/2$ with respect to s-derivatives.

1.2 Range and support theorems

Range characterization: which functions in $\mathbb{R}\times S^1$ are of the form Rf for some $f \in C_c^\infty(\mathbb{R}^2)$? An obvious restriction is that Rf is even,

$$Rf(-s,-w) = Rf(s,w). \tag{1.3}$$

Another restriction comes from the moments

$$\mu_k(Rf)(w) = \int_{-\infty}^{\infty} s^k (Rf)(s,w)\, ds, \qquad k \geq 0,\ w \in S^1.$$

One has:

$$\begin{cases} \text{for any } k \geq 0, \ \mu_k(Rf) \text{ is a homogeneous} \\ \text{polynomial of degree } k \text{ in } w. \end{cases} \tag{1.4}$$

This means that $\mu_k(Rf) = \sum_{j_1,\dots,j_k=1}^{2} a_{j_1\cdots j_k} w_{j_1}\cdots w_{j_k}$ for some constants $a_{j_1\cdots j_k}$.

Ex. Prove (1.3) and (1.4).

These two conditions (called Helgason-Ludwig range conditions) are essentially the only restrictions.

Thm 1.13 (Range characterization on $\mathcal{S}(\mathbb{R}^2)$)
R is bijective $\mathcal{S}(\mathbb{R}^2) \to \mathcal{S}_H(\mathbb{R}\times S^1)$, where

$$\mathcal{S}_H(\mathbb{R}\times S^1) = \{ \varphi \in C^\infty(\mathbb{R}\times S^1) \,;\, (1+s^2)^k \partial_s^l \varphi \in L^\infty(\mathbb{R}\times S^1) \text{ for all } k,l \geq 0, \text{ and } \varphi \text{ satisfies (1.3)-(1.4)}\}.$$

Pf (Sketch) If $\varphi \in \mathcal{S}_H(\mathbb{R}\times S^1)$, we want to find $f \in \mathcal{S}(\mathbb{R}^2)$ with $Rf = \varphi$. One should have the Fourier slice theorem:

$$\tilde{\varphi}(\sigma,\omega) = \hat{f}(\sigma\omega)$$

Motivated by this, we define F in $\mathbb{R}^2 \setminus \{0\}$ by

$$F(\xi) = \tilde{\varphi}(|\xi|, \xi/|\xi|), \qquad \xi \in \mathbb{R}^2\setminus\{0\}.$$

We need to show that there is $f \in \mathcal{S}(\mathbb{R}^2)$ with $\hat{f}(\xi) = F(\xi)$ for $\xi \neq 0$. The main point is to show that F extends smoothly near 0. Now

$$F(\sigma\omega) = \tilde{\varphi}(\sigma,\omega) = \int_{-\infty}^{\infty} e^{-i\sigma s}\varphi(s,\omega)\,d\omega$$

$$\Rightarrow \lim_{\sigma\to 0^+} F(\sigma\omega) = \int_{-\infty}^{\infty} \varphi(s,\omega)\,d\omega = \mu_0\varphi(\omega). \qquad (*)$$

The key assumption (1.4) says that $\mu_0\varphi(\omega)$ is homog. degree 0 in ω, i.e. $\mu_0\varphi(\omega)$ is constant. Define $F(0) := \mu_0\varphi(\omega)$. Now by $(*)$

$$|F(\sigma\omega) - F(0)| = |\tilde{\varphi}(\sigma,\omega) - \tilde{\varphi}(0,\omega)| \leq C\sigma$$

uniformly over $\omega \in S^1$ since $\tilde{\varphi}$ is smooth. Thus F extends continuously near 0. The rest of the proof is an exercise. $\square$