In the last lecture we characterized the range of the Radon transform on $\mathcal{S}(\mathbb{R}^2)$. There is a similar characterization on $C_c^\infty(\mathbb{R}^2)$.

Thm 1.18 R is bijective $C_c^\infty(\mathbb{R}^2) \to \mathcal{D}_H(\mathbb{R}\times S^1)$, where

$$\mathcal{D}_H(\mathbb{R}\times S^1) = \mathcal{S}_H(\mathbb{R}\times S^1) \cap C_c^\infty(\mathbb{R}\times S^1)$$
$$= \{\varphi \in C_c^\infty(\mathbb{R}\times S^1) \,;\, \varphi \text{ satisfies } (1.3) \text{ and } (1.4)\}$$

(Here $C_c^\infty(\mathbb{R}\times S^1) = \{\varphi \in C^\infty(\mathbb{R}\times S^1) \,;\, \exists A>0 : \varphi(s,\omega)=0 \text{ whenever } |s|>A\}$.)

This follows from Thm 1.14 and the following:

Thm 1.19 (Helgason support theorem) Let f be continuous on $\mathbb{R}^2$ such that $|x|^k f \in L^\infty(\mathbb{R}^2)$ for any $k \geq 0$. If $A>0$ and

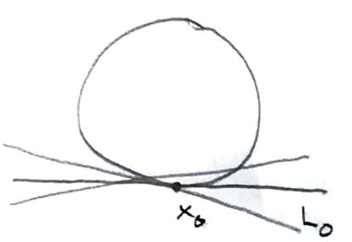
$$Rf(s,\omega) = 0 \quad \text{whenever } |s|>A \text{ and } \omega \in S^1,$$

then $f(x)=0$ whenever $|x|>A$.

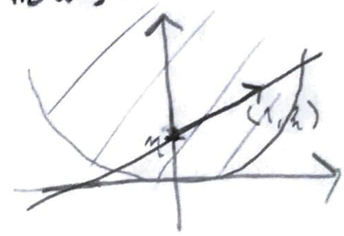
Pf of Thm 1.18 The main point is to show that given any $\varphi \in \mathcal{D}_H(\mathbb{R}\times S^1)$, there is $f \in C_c^\infty(\mathbb{R}^2)$ with $Rf=\varphi$. But by Thm 1.14 there is $f \in \mathcal{S}(\mathbb{R}^2)$ with $Rf=\varphi$. Since $\varphi \in C_c^\infty(\mathbb{R}\times S^1)$, there is $A>0$ so that $Rf(s,\omega)=0$ when $|s|>A$. Thus by Thm 1.19 one has $f \in C_c^\infty(\mathbb{R}^2)$. $\square$

We will not need Thm 1.19 later. However, we will prove a closely related result.

Thm 1.20 (Local uniqueness) Let B be a ball in $\mathbb{R}^2$, and let $f \in C_c(\mathbb{R}^2)$ be supported in $\overline{B}$. Let $x_0 \in \partial B$ and let L_0 be the tangent line to ∂B through x_0. If f integrates to zero along any line L in a small neighborhood of L_0, then $f=0$ near x_0.



Pf We assume that $f \in C_c^\infty(\mathbb{R}^2)$ is supported in $\overline{B}$ (the general case $f \in C_c(\mathbb{R}^2)$ is an exercise). After a translation and rotation, assume $x_0 = 0$, $\overline{B} \subset \{x_n \geq 0\}$, and that L_0 is the x-axis. It is convenient to reparametrize the Radon transform as follows:



$$Pf(\zeta,\eta) = \int_{-\infty}^{\infty} f(t, \zeta t+\eta)\, dt, \quad \zeta,\eta \in \mathbb{R}.$$

By assumption, $Pf = 0$ near $(0,0)$.
Since $f \in C_c^\infty(\mathbb{R}^2)$, we may compute

$$\partial_\zeta Pf(\zeta,\eta) = \int_{-\infty}^{\infty} t\,\partial_{x_2} f(t,\zeta t+\eta)\, dt = \partial_\eta \left(\int_{-\infty}^{\infty} t f(t,\zeta t+\eta)\, dt \right)$$
$$= \partial_\eta P(x_1 f)(\zeta,\eta).$$

Since $Pf = 0$ near $(0,0)$, we have $\partial_\eta (P(x_1 f)) = 0$ near $(0,0)$, so $P(x_1 f)(\zeta,\eta) = c(\zeta)$. But taking η negative and using that $\mathrm{supp}(f) \subset \overline{B}$ gives $P(x_1 f) = 0$ near $(0,0)$. Repeating this argument gives

$$P(x_1^k f) = 0 \quad \text{near } (0,0) \text{ for any } k \geq 0.$$

Choosing $\zeta = 0$ gives

$$\int_{-\infty}^{\infty} t^k f(t,\eta)\, dt = 0 \quad \text{for } \eta \text{ near } 0 \text{ whenever } k \geq 0.$$

This means that all moments of $f(\cdot,\eta)$ vanish, which implies that $f(\cdot,\eta) = 0$ for η near 0 (exercise). $\square$

1.3 The normal operator

We will now study the normal operator $R^* R$, where the adjoint R^* is computed with respect to the natural L^2 inner products on $\mathbb{R}^2$ and $\mathbb{R} \times S^1$. In fact, R^* is a backprojection operator: if $f \in C_c^\infty(\mathbb{R}^2)$ and $h \in C^\infty(\mathbb{R} \times S^1)$,

$$(Rf, h)_{L^2(\mathbb{R}\times S^1)} = \int_{-\infty}^{\infty} \int_{S^1} Rf(s,\omega)\,\overline{h(s,\omega)}\,d\omega\,ds$$

$$= \int_{-\infty}^{\infty} \int_{S^1} \int_{-\infty}^{\infty} f(s\omega + t\omega^{\perp})\,\overline{h(s,\omega)}\,dt\,d\omega\,ds$$

$$\overset{y = s\omega + t\omega^{\perp}}{=} \int_{\mathbb{R}^2} f(y)\,\overline{\left(\int_{S^1} h(y\cdot\omega, \omega)\,d\omega\right)}\,dy$$

Thus R^* is given by

$$R^* : C^\infty(\mathbb{R}\times S^1) \to C^\infty(\mathbb{R}^2), \quad R^* h(y) = \int_{S^1} h(y\cdot\omega, \omega)\,d\omega.$$

The following result shows that R^*R corresponds to multiplication by $\frac{4\pi}{|\xi|}$ on the Fourier side, and gives an inversion formula.

<u>Thm 1.23</u> (Normal operator)

$$R^*R = 4\pi |D|^{-1} = \mathcal{F}^{-1}\left\{ \frac{4\pi}{|\xi|} \mathcal{F}(\cdot) \right\},$$

$$f = \frac{1}{4\pi} |D| R^*Rf.$$

<u>Remark 1.24</u> We write $|D|^\alpha f = (-\Delta)^{\alpha/2} f = \mathcal{F}^{-1}\{ |\xi|^\alpha \hat{f} \}$.

<u>Pf</u> $$(R^*Rf, g)_{L^2(\mathbb{R}^2)} = (Rf, Rg)_{L^2(\mathbb{R}\times S^1)}$$

$$= \int_{S^1} \left[\int_{-\infty}^{\infty} Rf(s,\omega)\,\overline{Rg(s,\omega)}\,ds \right] d\omega$$

$$\overset{\text{Parseval}}{=} \frac{1}{2\pi} \int_{S^1} \left[\int_{-\infty}^{\infty} (Rf)^\sim(\sigma,\omega)\,\overline{(Rg)^\sim(\sigma,\omega)}\,d\sigma \right] d\omega$$

$$\overset{\text{Fourier slice}}{=} \frac{1}{2\pi} \int_{S^1} \left[\int_{-\infty}^{\infty} \hat{f}(\sigma\omega)\,\overline{\hat{g}(\sigma\omega)}\,d\sigma \right] d\omega$$

$$\overset{\text{symmetry}}{=} \frac{2}{2\pi} \int_{S^1} \left[\int_{0}^{\infty} \hat{f}(\sigma\omega)\,\overline{\hat{g}(\sigma\omega)}\,d\sigma \right] d\omega$$

$$\overset{\text{polar coordinates}}{=} \frac{2}{2\pi} \int_{\mathbb{R}^2} \frac{1}{|\xi|} \hat{f}(\xi)\,\overline{\hat{g}(\xi)}\,d\xi$$

$$\overset{\text{Parseval}}{=} (\underbrace{4\pi\,\mathcal{F}^{-1}\left\{ \frac{1}{|\xi|} \hat{f}(\xi) \right\}}_{R^*Rf}, g)_{L^2(\mathbb{R}^2)}$$

$\square$

The same argument, based on computing $(|D_s|^{1/2} Rf, |D_s|^{1/2} Rg)_{L^2(\mathbb{R}\times S^1)}$ instead of $(Rf, Rg)_{L^2(\mathbb{R}\times S^1)}$, proves the famous filtered backprojection (FBP) formula which has been used in X-ray CT scanners.

<u>Thm 1.26</u> If $f \in C_c^\infty(\mathbb{R}^2)$, then

$$f = \frac{1}{4\pi} R^* |D_s| Rf$$

where $|D_s| Rf = \mathcal{F}^{-1}\{ |\sigma| (Rf)^\wedge(\sigma, \omega) \}$.

<u>Recovery of singularities</u> Later we study X-ray transforms in more general geometries. In that setting explicit inversion formulas like FBP are often not available, but some structural properties of the normal operator may still be valid. In particular, Thm 1.24 states that the normal operator R^*R is an <u>elliptic pseudodifferential operator</u> (ΨDO) of order -1, and hence the <u>singularities</u> of f can be recovered from Rf. We try to explain these concepts.

<u>Def. 1.27</u> The set of <u>compactly supported distributions</u> is

$$\mathcal{E}'(\mathbb{R}^n) = \bigcup_{s \in \mathbb{R}} H^s_{comp}(\mathbb{R}^n).$$

By Remark 1.12, R is well defined on $\mathcal{E}'(\mathbb{R}^2)$. We recall that the Fourier transform maps $\mathcal{E}'(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$.

<u>Def. 1.28</u> (Singular support) A function u in $\mathbb{R}^n$ is C^∞ near x_0 if $u|_V \in C^\infty(V)$ for some neighborhood V of x_0. We write

$$\text{sing supp}(u) = \mathbb{R}^n \setminus \{ x_0 \in \mathbb{R}^n \,;\, u \text{ is } C^\infty \text{ near } x_0 \}.$$

Example If $D \subset \mathbb{R}^n$ is a bounded C^∞ domain, then

$$\text{sing supp}(\chi_D) = \partial D.$$

In general, the singular support may capture the jumps, interfaces or sharp features of an image. This can already be very useful in applications.

If in X-ray imaging one is only interested in the singularities of f, then instead of using FBP to reconstruct the whole function f from Rf, it is possible to use the even simpler backprojection method: apply R^* to the data Rf. This completely recovers the singularities of f.

Thm 1.32 (Recovery of singularities) If $f \in \mathcal{E}'(\mathbb{R}^2)$ then

$$\text{sing supp}(R^* Rf) = \text{sing supp}(f).$$

Remark 1.33 Since $R^* R$ is smoothing of order 1, one expects that $R^* Rf$ is a smoothed (i.e. blurred) version of f where the main singularities are still visible.