We will prove

Thm 1.32 (Recovery of singularities) If $f \in \mathcal{E}'(\mathbb{R}^2)$, then

$$\text{sing supp}(R^* Rf) = \text{sing supp}(f).$$

Recall that $\mathcal{E}'(\mathbb{R}^2) = \bigcup_{s \in \mathbb{R}} H^s_{comp}(\mathbb{R}^2)$ is the set of compactly supported distributions, so $\mathcal{F}\{\mathcal{E}'(\mathbb{R}^2)\} \subset C^\infty(\mathbb{R}^2)$, and

$$\text{sing supp}(f) = \mathbb{R}^n \setminus \{x_0 \in \mathbb{R}^n \,;\, f \text{ is } C^\infty \text{ near } x_0\}.$$

Recall also that $R^* R f = \mathcal{F}^{-1}\left\{ \frac{4\pi}{|\xi|} \hat{f}(\xi) \right\}$.

The proof is based on the fact that $R^* R$ is an elliptic pseudodifferential operator (ΨDO), and such operators preserve singularities. We first give an example to motivate ΨDOs.

Example 1.34 Let $A = a(x, D)$ be a differential operator of order m, i.e.

$$Af(x) = a(x, D)f(x) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha f(x)$$

where $a_\alpha \in C^\infty(\mathbb{R}^n)$. Here $D^\alpha = \left(\frac{1}{i}\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{1}{i}\frac{\partial}{\partial x_n}\right)^{\alpha_n}$.

If each a_α is constant, we may use the formula $(D_j f)^\wedge(\xi) = \xi_j \hat{f}(\xi)$ to compute

$$(Af)^\wedge(\xi) = \underbrace{\left(\sum_{|\alpha| \le m} a_\alpha \xi^\alpha \right)}_{a(\xi) \text{ (symbol of } A)} \hat{f}(\xi)$$

The Fourier inversion formula gives

$$Af(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(\xi) \hat{f}(\xi) \, d\xi.$$

More generally, if $a_\alpha \in C^\infty(\mathbb{R}^n)$ we may compute

$$Af(x) = A[\mathcal{F}^{-1}\{\hat{f}(\xi)\}]$$

$$= \sum_{|\alpha|\le m} a_\alpha(x)\, D^\alpha \left[(2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{f}(\xi)\,d\xi\right]$$

$$= (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} \underbrace{\Big(\sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha\Big)}_{a(x,\xi)\ (\text{symbol of } A)} \hat{f}(\xi)\,d\xi \qquad (1.7)$$

Thus any diff. operator of order m has the Fourier representation (1.7), where the symbol $a(x,\xi)$ is a polynomial of order m in ξ. Generalization:

Def. 1.35 For $m \in \mathbb{R}$, let S^m (the set of symbols of order m) be the set of $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ so that

$$\forall \alpha, \beta\ \exists C_{\alpha\beta} : \quad |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha\beta}(1+|\xi|)^{m-|\beta|}$$

For $a \in S^m$, define an operator $A = Op(a)$ as in (1.7),

$$Af(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)\hat{f}(\xi)\,d\xi.$$

Let $\Psi^m = \{Op(a) ; a \in S^m\}$ be the set of ΨDOs of order m. We say that $A = Op(a)$ is elliptic if $\exists c, R > 0$:

$$a(x,\xi) \ge c(1+|\xi|)^m, \qquad x \in \mathbb{R}^n,\ |\xi| \ge R.$$

We will use the following basic facts:

- Each $A \in \Psi^m$ maps $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$.
- Each $A \in \Psi^m$ maps $H^s(\mathbb{R}^n)$ to $H^{s-m}(\mathbb{R}^n)$, where $s \in \mathbb{R}$.

In particular, each $A \in \Psi^m$ is well defined on $\mathcal{E}'(\mathbb{R}^n)$.

ΨDOs have the very important property that they, like differential operators, do not create singularities.

Thm 1.36 (Pseudolocality) Any $A \in \Psi^m$ satisfies ③

$$\operatorname{sing\,supp}(Au) \subset \operatorname{sing\,supp}(u).$$

Pf Suppose $x_0 \notin \operatorname{sing\,supp}(u)$, so we need to show that $x_0 \notin \operatorname{sing\,supp}(Au)$. Since $x_0 \notin \operatorname{sing\,supp}(u)$, there is $\psi \in C_c^\infty(\mathbb{R}^n)$ with $\psi = 1$ near x_0 so that $\psi u \in C_c^\infty(\mathbb{R}^n)$. Write

$$Au = \underbrace{A(\psi u)}_{\in \mathcal{S}(\mathbb{R}^n) \text{ since } \psi u \in \mathcal{S}(\mathbb{R}^n)} + A((1-\psi)u).$$

It is enough to show that $A((1-\psi)u)$ is C^∞ near x_0. Choose $\varphi \in C_c^\infty(\mathbb{R}^n)$ so that $\varphi = 1$ near x_0 and $\operatorname{supp}(\varphi) \subset \{\psi = 1\}$. Define

$$Bu = \varphi A((1-\psi)u).$$

It is enough to show that B maps $\mathcal{E}'(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$ (i.e. that B is smoothing).

We compute the <u>integral kernel</u> of B: if $u \in C_c^\infty(\mathbb{R}^n)$, then

$$\begin{aligned} Bu(x) &= (2\pi)^{-n} \varphi(x) \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi) ((1-\psi)u)^\wedge(\xi)\, d\xi \\ &= (2\pi)^{-n} \varphi(x) \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi) \left[\int_{\mathbb{R}^n} e^{-iy\cdot\xi} (1-\psi(y)) u(y)\, dy \right] d\xi \\ &= \int_{\mathbb{R}^n} \underbrace{\left[(2\pi)^{-n} \int_{\mathbb{R}^n} \varphi(x) e^{i(x-y)\cdot\xi} a(x,\xi)(1-\psi(y))\, d\xi \right]}_{K(x,y)} u(y)\, dy \end{aligned}$$

Main point: $|x-y| \geq c > 0$ on $\operatorname{supp}(K)$, due to support properties of φ and ψ. We can use this and the fact that

$$e^{i(x-y)\cdot\xi} = |x-y|^{-2N} (-\Delta_\xi)^N (e^{i(x-y)\cdot\xi}),$$

together with integration by parts, to show that

$$K(x,y) = (2\pi)^{-n} |x-y|^{-2N} \int_{\mathbb{R}^n} \varphi(x) e^{i(x-y)\cdot\xi} ((-\Delta_\xi)^N a(x,\xi)(1-\psi(y)))\, d\xi.$$

Here $|(-\Delta_\xi)^N a(x,\xi)| \leq C(1+|\xi|)^{m-2N}$, which is in $L^1(\mathbb{R}^n)$ if $m-2N < -n$, i.e. $N > \frac{m+n}{2}$. With this choice, the integral above is absolutely convergent and $|K(x,y)| \le C$.

A similar argument applies in order to prove that $|\partial_x^\alpha \partial_y^\beta K(x,y)| \le C_{\alpha\beta}$.

Thus the integral kernel $K(x,y)$ is in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, and it follows that B maps $\mathcal{E}'(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$. $\square$

We now go back to the normal operator. Thm 1.24 states that $R^*R = \mathcal{F}^{-1}\{\frac{4\pi}{|\xi|}\mathcal{F}(\cdot)\}$, where the symbol $\frac{4\pi}{|\xi|}$ would be in S^{-1} except that it is not smooth when $\xi = 0$. This is handled as follows.

Thm 1.38 $R^*R = Q + S$ where $Q \in \Psi^{-1}$ is elliptic and S is smoothing, i.e. S maps $\mathcal{E}'(\mathbb{R}^2)$ to $C^\infty(\mathbb{R}^2)$.

Pf Let $\psi \in C_c^\infty(\mathbb{R}^2)$ satisfy $\psi(\xi) = 1$ for $|\xi| \le 1/2$ and $\psi(\xi) = 0$ for $|\xi| \ge 1$. Write

$$Qf = 4\pi\mathcal{F}^{-1}\left\{\frac{1-\psi(\xi)}{|\xi|}\hat{f}(\xi)\right\}, \qquad Sf = 4\pi\mathcal{F}^{-1}\left\{\frac{\psi(\xi)}{|\xi|}\hat{f}(\xi)\right\}.$$

Then $Q \in \Psi^{-1}$ with symbol $q(x,\xi) = 4\pi\frac{1-\psi(\xi)}{|\xi|}$, and Q is elliptic. Moreover, S is smoothing by Lemma 1.39 below since $\frac{\psi(\xi)}{|\xi|}$ is in $L^1_{comp}(\mathbb{R}^2)$. $\square$

Lemma 1.39 If $m \in L^1_{comp}(\mathbb{R}^n)$, the operator

$$S : f \mapsto \mathcal{F}^{-1}\{m(\xi)\hat{f}(\xi)\}$$

maps $\mathcal{E}'(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$.

Pf $f \in \mathcal{E}'(\mathbb{R}^n) \Rightarrow \hat{f} \in C^\infty(\mathbb{R}^n) \Rightarrow m\hat{f} \in L^1_{comp}(\mathbb{R}^n)$
$\Rightarrow \mathcal{F}^{-1}\{m\hat{f}\} \in C^\infty(\mathbb{R}^n)$. $\square$

We finally prove the recovery of singularities result.

Pf of Thm 1.32 By Thm 1.38,

$$R^* R f = Q f + C^\infty.$$

Thus $\operatorname{sing\,supp}(R^* R f) = \operatorname{sing\,supp}(Qf)$. It follows from Thm 1.36 that $\operatorname{sing\,supp}(Qf) \subset \operatorname{sing\,supp}(f)$. To prove $\operatorname{sing\,supp}(f) \subset \operatorname{sing\,supp}(Qf)$, we use the ellipticity of Q and construct a parametrix (approximate inverse).

Recall that $Qf = \mathcal{F}^{-1}\left\{ \frac{4\pi(1-\psi(z))}{|z|} \hat{f}(z) \right\}$. Define

$$Ef = \mathcal{F}^{-1}\left\{ \frac{(1-\chi(z))|z|}{4\pi} \hat{f}(z) \right\}$$

where $\chi \in C_c^\infty(\mathbb{R}^2)$ satisfies $\chi = 1$ for $|z| \le 2$. Then $E \in \Psi^1$ and

$$\begin{aligned} EQf &= \mathcal{F}^{-1}\left\{ \frac{(1-\chi(z))|z|}{4\pi} \, \frac{4\pi(1-\psi(z))}{|z|} \hat{f}(z) \right\} \\ &= \mathcal{F}^{-1}\left\{ (1-\chi(z)) \hat{f}(z) \right\} \\ &= f - \mathcal{F}^{-1}\left\{ \chi(z) \hat{f}(z) \right\} \end{aligned}$$

Thus $EQf = f + S_1 f$, where S_1 maps $\mathcal{E}'(\mathbb{R}^2)$ to $C^\infty(\mathbb{R}^2)$ by Lemma 1.39. Thm 1.36 applied to E gives

$$\operatorname{sing\,supp}(f) = \operatorname{sing\,supp}(EQf) \subset \operatorname{sing\,supp}(Qf). \qquad \square$$