CHAPTER 2

Radial sound speeds

In this chapter we will discuss geometric inverse problems in a disk with radial sound speed. The fact that the sound speed is radial is a strong symmetry condition, which allows one to determine the behaviour of geodesics and solve related inverse problems quite explicitly.

We first discuss geodesics of a radial sound speed satisfying the important *Herglotz condition*, using the Hamiltonian approach to geodesics and Cartesian coordinates. We then prove the classical result of [**Her07, WZ07**] that travel times uniquely determine a radial sound speed of this type. Next we switch to polar coordinates and study geodesics of a radially symmetric metric, and prove that the geodesic X-ray transform is injective. The main point is that the geodesic equations can be integrated explicitly using quadrature, and a function can be determined from its integrals over geodesics using suitable changes of coordinates and inverting Abel type transforms. Finally, we give examples of manifolds (surfaces of revolution) where the geodesic X-ray transform is injective or is not injective.

2.1. Geodesics of a radial sound speed

The fact that the geodesics of a radial sound speed can be explicitly determined is related to the existence of multiple conserved quantities in the Hamiltonian approach to geodesics. We first recall this approach.

2.1.1. Geodesics as a Hamilton flow. Let $M \subset \mathbb{R}^n$, let x be standard Cartesian coordinates, and let $g = (g_{jk}(x))_{j,k=1}^n$ be a Riemannian metric on M. A curve $x(t) = (x^1(t), \ldots, x^n(t))$ is a geodesic iff it satisfies the geodesic equations

$$\ddot{x}^l(t) + \Gamma_{jk}^l(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0, \tag{2.1}$$

where Γ_{jk}^l are the Christoffel symbols given by

$$\Gamma_{jk}^l = \frac{1}{2} g^{lm}(\partial_j g_{km} + \partial_k g_{jm} - \partial_m g_{jk}).$$

We will assume that all geodesics have unit speed, i.e.

$$|\dot{x}(t)|_g = \sqrt{g_{jk}(x(t))\dot{x}^j(t)\dot{x}^k(t)} = 1.$$

In this section we will also use the Euclidean length of vectors $x \in \mathbb{R}^2$, written as

$$|x|_e = \sqrt{x_1^2 + x_2^2}.$$

We recall that the geodesic equations are often derived via the *Lagrangian approach* to classical mechanics (they arise as the Euler-Lagrange equations satisfied by minimizers of the length functional $L(x) = \int_a^b |\dot{x}(t)|_g \, dt$.) We will now switch to the *Hamiltonian approach*, which considers the position $x(t)$ and momentum $\xi(t)$, where $\xi(t)$ is the covector corresponding to $\dot{x}(t)$, simultaneously.

Writing

$$\xi_j(t) := g_{jk}(x(t))\dot{x}^k(t), \qquad f(x,\xi) := \sqrt{g^{jk}(x)\xi_j\xi_k},$$

a short computation shows that the geodesic equations (for unit speed geodesics) are equivalent with the Hamilton equations

$$\begin{cases} \dot{x}(t) &= \nabla_\xi f(x(t),\xi(t)), \\ \dot{\xi}(t) &= -\nabla_x f(x(t),\xi(t)). \end{cases} \tag{2.2}$$

Here $f(x,\xi) = |\xi|_g$ (*speed*, or square root of kinetic energy) is called the *Hamilton function*, and it is defined on the cotangent space

$$T^*M = \{(x,\xi)\,;\, x \in M,\, \xi \in \mathbb{R}^n\} = M \times \mathbb{R}^n \subset \mathbb{R}^{2n}.$$

The operators ∇_x and ∇_ξ are the standard (Euclidean) gradient operators with respect to the x and ξ variables.

EXERCISE 2.1. Show that (2.1) is equivalent with (2.2).

Writing $\gamma(t) = (x(t),\xi(t))$ and using the Hamilton vector field H_f on T^*M, defined by

$$H_f := \nabla_\xi f \cdot \nabla_x - \nabla_x f \cdot \nabla_\xi = (\nabla_\xi f, -\nabla_x f)$$

we may write the Hamilton equations as

$$\dot{\gamma}(t) = H_f(\gamma(t)).$$

DEFINITION 2.2. A function $u = u(x,\xi)$ is a *conserved quantity* if it is constant along the Hamilton flow, i.e. $t \mapsto u(x(t),\xi(t))$ is constant for any curve $(x(t),\xi(t))$ solving (2.2).

Now (2.2) implies that

$$\begin{aligned} & u \text{ is conserved} \\ \iff \quad & \frac{d}{dt}u(x(t),\xi(t)) = 0 \\ \iff \quad & H_f u(x(t),\xi(t)) = 0. \end{aligned}$$

Since

$$H_f f = (\nabla_\xi f, -\nabla_x f) \cdot (\nabla_x f, \nabla_\xi f) = 0,$$

the Hamilton function f (speed) is always conserved.

Let now $M \subset \mathbb{R}^2$, and consider a metric of the form

$$g_{jk}(x) = c(x)^{-2}\delta_{jk}$$

where $c \in C^\infty(M)$ is positive. Then $f(x,\xi) = c(x)|\xi|_e$ and, writing $\hat{\xi} = \frac{\xi}{|\xi|_e}$,

$$H_f = c(x)\hat{\xi} \cdot \nabla_x - |\xi|_e \nabla_x c(x) \cdot \nabla_\xi.$$

Define the *angular momentum*

$$L(x,\xi) = \xi \cdot x^\perp, \qquad x^\perp = (-x_2, x_1).$$

When is L conserved? We compute

$$H_f L = c(x)\hat{\xi} \cdot (-\xi^\perp) - |\xi|_e \nabla_x c(x) \cdot x^\perp = -|\xi|_e \nabla_x c(x) \cdot x^\perp.$$

Thus $H_f L = 0$ iff $\nabla c(x) \cdot x^\perp = 0$, which is equivalent with the fact that c is radial:

LEMMA 2.3. *The angular momentum L is conserved iff*

$$c = c(r), \qquad r = |x|_e.$$

If $M \subset \mathbb{R}^2$ and $c(x)$ is radial, then the Hamilton flow on T^*M (a four-dimensional manifold) has two independent conserved quantities (the speed f and angular momentum L). One says that the flow is *completely integrable*, which implies that the geodesic equations can be solved quite explicitly by quadrature using f and L. See e.g. [**Tay11**, Chapter 1] for more details on these facts.

2.1.2. Geodesics of a radial sound speed. We will now begin to analyze geodesics in this setting. Let $M = \overline{\mathbb{D}} \setminus \{0\}$ where $\mathbb{D}$ is the unit disk in $\mathbb{R}^2$. Assume that

$$g_{jk}(x) = c(r)^{-2}\delta_{jk}, \qquad r = |x|_e,$$

where $c \in C^\infty([0,1])$. Note that the origin is a special point and $g_{jk}(x)$ is not necessarily smooth there, hence we will consider geodesics only away from the origin.

We write

$$r(t) = |x(t)|_e, \qquad \hat{x} = \frac{x}{|x|_e}.$$

Then $f(x,\xi) = c(r)|\xi|_e$ and the Hamilton equations (2.2) become

$$\left\{ \begin{array}{ll} \dot{x}(t) & = c(r(t))\hat{\xi}(t), \\ \dot{\xi}(t) & = -|\xi(t)|_e c'(r(t))\hat{x}(t). \end{array} \right. \tag{2.3}$$

Consider geodesics starting on $\partial\mathbb{D}$, i.e. $r(0) = 1$, and write

$$\xi(0) = \frac{1}{c(1)}(-\sqrt{1-p^2}x(0) + px(0)^\perp), \qquad 0 < p < 1. \tag{2.4}$$

Note that $\xi(0)$ points inward, and hence also $\dot{x}(0) = c(1)^2\xi(0)$ points inward. The normalization yields $|\dot{x}(0)|_g = |\xi(0)|_g = 1$, so that the geodesic has unit speed.

We wish to study how deep the geodesic goes into M, which boils down to understanding $r(t)$. Computing the derivative of $r(t)$ gives

$$\dot{r} = \frac{x \cdot \dot{x}}{|x|_e} = \frac{c(r)}{r|\xi|_e}(x \cdot \xi). \tag{2.5}$$

In particular, we see that $\dot{r}(t)$ has the same sign as $x(t) \cdot \xi(t)$. The latter quantity can be analyzed by (2.3). We compute

$$\begin{aligned} \frac{d}{dt}(x \cdot \xi) &= \dot{x} \cdot \xi + x \cdot \dot{\xi} = |\xi|_e(c - rc'(r)) \\ &= c^2|\xi|_e \frac{d}{dr}\left(\frac{r}{c(r)}\right)\Big|_{r=r(t)}. \end{aligned} \tag{2.6}$$

Next we make use of the conserved quantities:

$$f \text{ conserved} \implies c(r(t))|\xi(t)|_e = 1 \implies |\xi(t)|_e = \frac{1}{c(r(t))}, \tag{2.7}$$

$$L \text{ conserved} \implies \xi(t) \cdot x(t)^\perp = \xi(0) \cdot x(0)^\perp. \tag{2.8}$$

Then (2.6) becomes

$$\frac{d}{dt}(x \cdot \xi) = c(r)\frac{d}{dr}\left(\frac{r}{c(r)}\right)\Big|_{r=r(t)}. \tag{2.9}$$

REMARK 2.4. We note that one can derive a useful ODE for $r(t)$. By (2.5) one has $\dot{r} = c(\hat{x} \cdot \hat{\xi})$. Decompose $\hat{\xi} = (\hat{\xi} \cdot \hat{x})\hat{x} + (\hat{\xi} \cdot \hat{x}^\perp)\hat{x}^\perp$. Noting that $|\hat{x} \cdot \hat{\xi}| = \sqrt{1 - (\hat{\xi} \cdot \hat{x}^\perp)^2} = \sqrt{1 - \left(\frac{pc(r)}{rc(1)}\right)^2}$ by (2.7), (2.8) and (2.4), we see that $r(t)$ solves the equation

$$\dot{r} = \pm c(r)\sqrt{1 - \left(\frac{pc(r)}{rc(1)}\right)^2}, \qquad \pm \xi \cdot \hat{x} \geq 0. \tag{2.10}$$

This is an autonomous ODE for $r(t)$ (all other dependence on t has been eliminated).

To simplify the behaviour of geodesics we would like that $\dot{r}(t)$ has a unique zero at some $t = t_p$, is negative for $t < t_p$, and positive for $t > t_p$. This means that geodesics curve back toward the boundary after they reach their deepest point. Since $\dot{r}(t)$ has the same sign as $x(t) \cdot \xi(t)$, the identity (2.9) shows that this is guaranteed by the following important condition.

DEFINITION 2.5. We say that a radial sound speed $c \in C^\infty([0, 1])$ satisfies the *Herglotz condition* if

$$\frac{d}{dr}\left(\frac{r}{c(r)}\right) > 0, \qquad r \in [0, 1]. \tag{2.11}$$

Assuming this condition we can describe the behaviour of geodesics.

THEOREM 2.6. *Assume that $c \in C^\infty([0, 1])$ satisfies the Herglotz condition. Let $0 < p < 1$, and consider the geodesic with $x(0) \in \partial\mathbb{D}$ and $\xi(0)$ given by (2.4). There is a unique time $t_p > 0$ such that*

$$\dot{r}(t) < 0 \text{ for } 0 \leq t < t_p, \qquad \dot{r}(t_p) = 0, \qquad \dot{r}(t) > 0 \text{ for } t_p < t \leq 2t_p.$$

One has $0 < r(t) < 1$ for $0 < t < 2t_p$ and $r(0) = r(2t_p) = 1$. Moreover, the geodesic is symmetric with respect to $t = t_p$ so that $x(t_p + s) = R_p(x(t_p - s))$ where R_p is reflection about $\hat{x}(t_p)$.

PROOF. By (2.4) one has

$$x(0) \cdot \xi(0) = -c(1)^{-1}\sqrt{1 - p^2} < 0, \tag{2.12}$$

and (2.5) implies that $\dot{r}(0) < 0$. Thus $x(t)$ stays in $\overline{\mathbb{D}} \setminus \{0\}$ at least for a short time. Note also that by (2.7) (conservation of f) and the positivity of c, one has $|\xi(t)|_e \geq \varepsilon_0 > 0$ whenever the geodesic is defined.

Let T be the maximal time of existence of the geodesic $x(t)$, i.e.

$$T = \sup\{\bar{t} > 0\,;\, x|_{[0,\bar{t})} \text{ stays in } \overline{\mathbb{D}} \setminus \{0\}\}.$$

There are two ways that $x(t)$ can exit $\overline{\mathbb{D}} \setminus \{0\}$: either $x(t)$ can go to 0, or $x(t)$ can go to $\partial\mathbb{D}$. Let us show that the first case cannot happen. If $x|_{[0,\bar{t})}$ stays in $\overline{\mathbb{D}} \setminus \{0\}$ and $x(t_j) \to 0$ as $t_j \to \bar{t}$, then (2.8) implies that $\xi(0) \cdot x(0)^\perp = 0$. But (2.4) gives that $\xi(0) \cdot x(0)^\perp = p/c(1)$, which is impossible since we assumed that $0 < p < 1$. This shows that either $T = \infty$, or T is finite and $x(T) \in \partial\mathbb{D}$.

Now we go back to (2.9) and note that the positivity of c and the Herglotz condition (2.11) imply that

$$\frac{d}{dt}(x(t) \cdot \xi(t)) \geq \varepsilon_0 > 0, \qquad t \in [0, T).$$

Thus $x(t) \cdot \xi(t)$ is strictly increasing. By (2.12) one has $x(0) \cdot \xi(0) < 0$ and

$$x(t) \cdot \xi(t) \geq x(0) \cdot \xi(0) + \varepsilon_0 t, \qquad t \in [0, T). \tag{2.13}$$

Now if $x(t) \cdot \xi(t)$ were negative for $t \in [0, T)$, then by (2.5) $r(t)$ would be strictly decreasing for $t \in [0, T)$, and the maximal time would be $T = \infty$ since $x(t)$ could not go to $\partial\mathbb{D}$. This is a contradiction with (2.13), hence there must be a unique $t_p > 0$ with $x(t_p) \cdot \xi(t_p) = 0$. By (2.5) one has $\dot{r}(t) < 0$ for $t < t_p$, $\dot{r}(t_p) = 0$, and also $\dot{r}(t) > 0$ for $t > t_p$ since $x(t) \cdot \xi(t)$ is strictly increasing.

The other claims follow if we can show the symmetry $x(t_p + s) = R_p(x(t_p - s))$. Since everything is rotationally symmetric, we may assume that $\hat{x}(t_p) = (1, 0)$ and $R_p(x_1, x_2) = (x_1, -x_2)$. Define $\eta(s) = (x(t_p + s), \xi(t_p + s))$ and $\zeta(s) = (R_p(x(t_p - s)), -R_p(\xi(t_p - s)))$. Then both $\eta(s)$ and $\zeta(s)$ satisfy the Hamilton equations (2.3) with the same initial data when $s = 0$ (since $x(t_p) \cdot \xi(t_p) = 0$), and the symmetry condition follows by uniqueness for ODEs. $\square$

2.2. Travel time tomography

We will now consider a variant of the travel time tomography problem discussed in the introduction, and prove the classical result of [**Her07, WZ07**] showing that travel times uniquely determine a radial sound speed satisfying the Herglotz condition.

If $c \in C^\infty([0, 1])$ satisfies the Herglotz condition, then by Theorem 2.6 the unit speed geodesic starting at $x(0) \in \partial\mathbb{D}$ having codirection $\xi(0) = -\frac{1}{c(1)}(\sqrt{1 - p^2} x(0) + p x(0)^\perp)$ where $0 < p < 1$ returns to $\partial\mathbb{D}$ after time $2t_p$. Note that the travel time $2t_p$ does not depend on the choice of $x(0) \in \partial\mathbb{D}$ because of radial symmetry. Thus we may define the travel time function

$$T_c(p) = 2t_p, \qquad 0 < p < 1.$$

THEOREM 2.7 (Travel time tomography). *Assume that $c \in C^\infty([0, 1])$ is positive and satisfies the Herglotz condition. From the knowledge of the value $c(1)$ and the travel times*

$$T_c(p), \qquad 0 < p < 1,$$

one can determine $c(r)$ for $r \in (0, 1]$.

To prove this theorem, we start with the ODE (2.10) which gives that

$$\frac{dr}{dt} = c(r)\sqrt{1 - \left(\frac{pc(r)}{rc(1)}\right)^2}, \qquad t_p \leq t \leq 2t_p.$$

We use this fact and a change of variables to obtain

$$T_c(p) = 2t_p = 2\int_{t_p}^{2t_p} dt = 2\int_{r_p}^{1} \frac{1}{c(r)\sqrt{1 - \left(\frac{pc(r)}{rc(1)}\right)^2}}\, dr \tag{2.14}$$

where $r_p = r(t_p)$. Thus, from the measurements $T_c(p)$ with $0 < p < 1$ we know the integrals (2.14) involving $c(r)$. We wish to recover $c(r)$ from these integrals.

To simplify (2.14), we make the change of variables

$$u = \left(\frac{c(1)r}{c(r)}\right)^2. \tag{2.15}$$

This is a valid change of variables by the Herglotz condition (2.11). Note that since $\dot r(t_p) = 0$, the ODE (2.10) shows that $r_p = r(t_p)$ satisfies

$$\frac{r_p}{c(r_p)} = \frac{p}{c(1)}.$$

Hence $r = r_p$ corresponds to $u = p^2$. Then $T_c(p)$ becomes

$$T_c(p) = \frac{2}{c(1)} \int_{p^2}^1 \frac{dr}{du} \frac{u}{r} \frac{1}{\sqrt{u - p^2}} \, du. \tag{2.16}$$

This is an *Abel integral*, of the kind encountered by Abel [**Abe26**] when determining the profile of a hill by measuring the time it takes for a particle with different initial positions to roll down the hill. This work of Abel is considered to be the first appearance of an integral equation in mathematics.

These Abel integrals can be inverted by the following result, where we also pay attention to various mapping properties of the Abel transform. See [**GV91**] for a detailed treatment of Abel integral equations.

Theorem 2.8 (Abel transform). *Let $\alpha < \beta$, and define the* Abel transform

$$Au(x) := \int_x^\beta \frac{1}{(y-x)^{1/2}} u(y) \, dy, \qquad \alpha < x \leq \beta.$$

The Abel transform takes $L^1_{\mathrm{loc}}((\alpha, \beta])$ to itself. Define the space

$$\mathcal{A}((\alpha, \beta]) := \{ f \in L^1_{\mathrm{loc}}((\alpha, \beta]) \,;\, Af \in W^{1,1}_{\mathrm{loc}}((\alpha, \beta]) \}.$$

The Abel transform is a bijective map between the following spaces:

$$A : L^1_{\mathrm{loc}}((\alpha, \beta]) \to \mathcal{A}((\alpha, \beta]), \tag{2.17}$$

$$A : \mathcal{A}((\alpha, \beta]) \to \{ f \in W^{1,1}_{\mathrm{loc}}((\alpha, \beta]) \,;\, f(\beta) = 0 \}, \tag{2.18}$$

$$A : C^\infty((\alpha, \beta]) \to \{ (\beta - x)^{1/2} h(x) \,;\, h \in C^\infty((\alpha, \beta]) \}. \tag{2.19}$$

Given any $f \in \mathcal{A}((\alpha, \beta])$, the equation $Au = f$ has a unique solution $u \in L^1_{\mathrm{loc}}((\alpha, \beta])$ given by the formula

$$u(y) = -\frac{1}{\pi} \frac{d}{dy} \int_y^\beta \frac{f(x)}{(x-y)^{1/2}} \, dx. \tag{2.20}$$

If additionally $f \in W^{1,1}_{\mathrm{loc}}((\alpha, \beta])$ with $f(\beta) = 0$, one has the alternative formula

$$u(y) = -\frac{1}{\pi} \int_y^\beta \frac{f'(x)}{(x-y)^{1/2}} \, dx. \tag{2.21}$$

Remark 2.9. Here $L^1_{\mathrm{loc}}((\alpha, \beta]) = \{ u \,;\, u|_{[\gamma, \beta]} \in L^1([\gamma, \beta]) \text{ whenever } \alpha < \gamma < \beta \}$, and similarly for $W^{1,1}_{\mathrm{loc}}((\alpha, \beta])$. Recall that in one dimension $W^{1,1}$ coincides with the space of absolutely continuous functions, and hence functions in $W^{1,1}_{\mathrm{loc}}((\alpha, \beta])$ can be evaluated pointwise at β.

Proof. If $\alpha < \gamma < \beta$, we may use Fubini's theorem to show that

$$\begin{aligned} \int_\gamma^\beta |Au(x)| \, dx &\leq \int_\gamma^\beta \int_x^\beta \frac{|u(y)|}{(y-x)^{1/2}} \, dy \, dx = \int_\gamma^\beta \int_\gamma^y \frac{|u(y)|}{(y-x)^{1/2}} \, dx \, dy \\ &= 2 \int_\gamma^\beta (y-\gamma)^{1/2} |u(y)| \, dy \leq 2(\beta - \gamma)^{1/2} \int_\gamma^\beta |u(y)| \, dy. \end{aligned}$$

This shows that A maps $L^1_{\mathrm{loc}}((\alpha,\beta])$ to itself. We use the definition of A and Fubini's theorem to compute

$$\begin{aligned} A^2u(z) &= \int_z^\beta \frac{Au(x)}{(x-z)^{1/2}}\,dx = \int_z^\beta \int_x^\beta \frac{u(y)}{(x-z)^{1/2}(y-x)^{1/2}}\,dy\,dx \\ &= \int_z^\beta \int_z^y \frac{u(y)}{(x-z)^{1/2}(y-x)^{1/2}}\,dx\,dy. \end{aligned}$$

The last quantity may be written as $\int_z^\beta k(z,y)u(y)\,dy$ where, using the change of variables $x = z + (y-z)w$,

$$k(z,y) = \int_z^y \frac{1}{(x-z)^{1/2}(y-x)^{1/2}}\,dx = \int_0^1 \frac{1}{w^{1/2}(1-w)^{1/2}}\,dw.$$

Thus $k(z,y)$ is a constant, given by the beta function $B(\frac{1}{2},\frac{1}{2}) = \pi$. The constant can be computed directly as follows: changing variables $w = \frac{1}{2} + \frac{1}{2}v$ and $v = \sin\theta$ gives

$$\int_0^1 \frac{1}{w^{1/2}(1-w)^{1/2}}\,dw = \int_{-1}^1 \frac{1}{\sqrt{1-v^2}}\,dv = \int_{-\pi/2}^{\pi/2} d\theta = \pi.$$

This shows that for any $u \in L^1_{\mathrm{loc}}((\alpha,\beta])$ one has

$$A^2u(z) = \pi \int_z^\beta u(y)\,dy. \tag{2.22}$$

Thus $(A(Au))'(z) = -\pi u(z)$, so A maps $L^1_{\mathrm{loc}}((\alpha,\beta])$ into $\mathcal{A}((\alpha,\beta])$.

We next show that the map (2.17) is bijective. By (2.22), if $Au = 0$ it follows that $u \equiv 0$, so A is injective. Now let $f \in \mathcal{A}((\alpha,\beta])$. Setting $u := -\frac{1}{\pi}\frac{d}{dx}Af$ we have $u \in L^1_{\mathrm{loc}}((\alpha,\beta])$ and

$$\pi \int_z^\beta u(y)\,dy = Af(z)$$

since one always has $Af(\beta) = 0$. Combining this with (2.22) we get $Af = A(Au)$, and since A is injective we have $Au = f$. We have proved that (2.17) is bijective and that one has the inversion formula (2.20).

Next let $f \in W^{1,1}_{\mathrm{loc}}((\alpha,\beta])$ with $f(\beta) = 0$, and integrate by parts to obtain

$$\begin{aligned} Af(x) &= \int_x^\beta f(y)\frac{d}{dy}(2(y-x)^{1/2})\,dy \\ &= -2\int_x^\beta (y-x)^{1/2} f'(y)\,dy. \end{aligned}$$

It follows that $Af \in L^1_{\mathrm{loc}}((\alpha,\beta])$ and

$$(Af)'(x) = \int_x^\beta \frac{f'(y)}{(y-x)^{1/2}}\,dy = A(f')(x).$$

By (2.20) the function $u := -\frac{1}{\pi}(Af)'$ satisfies $Au = f$. But now one also has $u = -\frac{1}{\pi}A(f')$, which proves the second inversion formula (2.21). The fact that (2.18) is a bijective map follows immediately.

Finally, if $u \in C^\infty((\alpha,\beta])$ we change variables $y = x + (\beta - x)s$ and obtain

$$Au(x) = \int_x^\beta \frac{u(y)}{(y-x)^{1/2}}\,dy = (\beta-x)^{1/2}\int_0^1 \frac{u(x+(\beta-x)s)}{s^{1/2}}\,ds.$$

Since u is smooth, one has $Au(x) = (\beta - x)^{1/2} h(x)$ where $h \in C^\infty((\alpha, \beta])$. Conversely, if $f(x) = (\beta - x)^{1/2} h(x)$ where $h \in C^\infty((\alpha, \beta])$, the change of variables $x = y + (\beta - y)s$ gives

$$\int_y^\beta \frac{f(x)}{(x-y)^{1/2}}\, dx = (\beta - y) \int_0^1 \frac{(1-s)^{1/2} h(y + (\beta - y)s)}{s^{1/2}}\, ds.$$

If u is defined by (2.20), we see that $u \in C^\infty((\alpha, \beta])$ and u solves $Au = f$. Thus (2.19) is a bijective map. $\square$

We now return to (2.16). Since the value $c(1)$ is known, using (2.16) and Theorem 2.8 we can determine the function $f(u) := \frac{dr}{du} \frac{u}{r(u)}$ from the knowledge of $T_c(p)$ for $0 < p < 1$. We rewrite this as $\frac{d}{du} \log r(u) = \frac{f(u)}{u}$, which shows that we can recover the function

$$r(u) = \exp\left(- \int_u^1 \frac{f(v)}{v}\, dv \right).$$

By taking the inverse function, we can determine $u(r)$. By (2.15), we have determined the function $c(r) = c(1) r / \sqrt{u(r)}$. This completes the proof of Theorem 2.7.

Remark 2.10. If we assume that the sound speed extends smoothly to $M := \overline{\mathbb{D}}$, then Theorem 2.7 can be reformulated using the notation of Chapter 3 as follows: if g_1 and g_2 are two Riemannian metrics on M corresponding to radial sound speeds satisfying the Herglotz condition, if $g_1|_{\partial M} = g_2|_{\partial M}$ and if $\tau_{g_1}|_{\partial_+ SM} = \tau_{g_2}|_{\partial_+ SM}$ (the travel times of maximal geodesics for g_1 and g_2 agree), then $g_1 = g_2$.

In the boundary rigidity problem, one considers measurements given by the boundary distance function $d_g|_{\partial M \times \partial M}$ instead of the travel time function τ_g. It follows from equation (11.1) that if $d_{g_1}|_{\partial M \times \partial M} = d_{g_2}|_{\partial M \times \partial M}$ and the boundary is strictly convex, then $g_1|_{\partial M} = g_2|_{\partial M}$. Moreover, if the manifolds are simple then by Proposition 11.7 one has $\tau_{g_1}|_{\partial_+ SM} = \tau_{g_2}|_{\partial_+ SM}$. Thus in the setting of simple metrics, Theorem 2.7 also solves the boundary rigidity problem for radial sound speeds.

Remark 2.11. Theorem 2.7 assumes that $c(1)$, i.e. $g|_{\partial M}$, is known. Often one can determine $g|_{\partial M}$ by looking at short geodesics. However, in the present setting one gets something slightly different. In (2.16), write $f(u) = \frac{dr}{du} \frac{u}{r(u)}$ and note that f is smooth in $[p^2, 1]$. The change of variables $u = p^2 + (1-p^2)s$ yields

$$\int_{p^2}^1 \frac{f(u)}{\sqrt{u - p^2}}\, du = (1 - p^2)^{1/2} \int_0^1 \frac{f(p^2 + (1-p^2)s)}{s^{1/2}}\, ds.$$

Thus we obtain

$$\lim_{p \to 1} \frac{T_c(p)}{\sqrt{1-p^2}} = \frac{4 f(1)}{c(1)}.$$

From (2.15) we see that $\frac{du}{dr} = c(1)^2 \left(\frac{2r}{c(r)^2} - \frac{2r^2 c'(r)}{c(r)^3} \right)$. This implies that $f(1) = \frac{dr}{du}(1) = (2 - \frac{2c'(1)}{c(1)})^{-1} = \frac{c(1)}{2(c(1) - c'(1))}$. Hence, by looking at travel times of short geodesics one recovers the quantity $c(1) - c'(1)$ from $T_c(p)$.