2.3. Geodesics of a radially symmetric metric

For the rest of this chapter, it will be convenient to switch from Cartesian coordinates (x_1, x_2) to polar coordinates (r, θ), where $x = (r\cos\theta, r\sin\theta)$. Recall that the Euclidean metric $g = dx_1^2 + dx_2^2$ looks like $g = dr^2 + r^2\,d\theta^2$ in polar coordinates. Hence the metric $g = c(r)^{-2}(dx_1^2 + dx_2^2)$ with radial sound speed $c(r)$ becomes

$$g = c(r)^{-2}\,dr^2 + (r/c(r))^2\,d\theta^2. \tag{2.23}$$

We will work in the region $M = \{(r,\theta)\,;\, r_0 < r \leq r_1\}$ where $r_0 < r_1$ (note that r_0 is not necessarily required to be positive), and consider metrics of the form

$$g = a(r)^2\,dr^2 + b(r)^2\,d\theta^2 \tag{2.24}$$

where $a, b \in C^\infty([r_0, r_1])$ are positive. Clearly this includes metrics (2.23) with radial sound speed, with $a(r) = 1/c(r)$ and $b(r) = r/c(r)$. However, the two forms turn out to be equivalent:

Exercise 2.12. Show that a metric of the form (2.24) can be put in the form (2.23) by a change of variables.

Working with the form (2.24) will be useful in view of the following example.

Example 2.13 (Surfaces of revolution). Let r correspond to the z-coordinate in $\mathbb{R}^3$, and let $h : [r_0, r_1] \to \mathbb{R}$ be a smooth positive function. Let S be the surface of revolution obtained by rotating the graph of $r \mapsto h(r)$ about the z-axis. The surface S is given by $S = \{q(r,\theta)\,;\, r \in (r_0, r_1],\ \theta \in [0, 2\pi]\}$ where

$$q(r,\theta) = (h(r)\cos\theta, h(r)\sin\theta, r).$$

Then S has tangent vectors

$$\partial_r = (h'(r)\cos\theta, h'(r)\sin\theta, 1),$$
$$\partial_\theta = (-h(r)\sin\theta, h(r)\cos\theta, 0).$$

Equip S with the metric g induced by the Euclidean metric in $\mathbb{R}^3$. Since $\partial_r \cdot \partial_r = 1 + h'(r)^2$, $\partial_r \cdot \partial_\theta = 0$ and $\partial_\theta \cdot \partial_\theta = h(r)^2$, one has

$$g = (1 + h'(r)^2)\,dr^2 + h(r)^2\,d\theta^2.$$

Thus surfaces of revolution have metrics of the form (2.24), where $a(r) = \sqrt{1 + h'(r)^2}$ and $b(r) = h(r)$.

The geodesic equations for the metric (2.24) can be determined by computing the Christoffel symbols

$$\Gamma_{jk}^l = \frac{1}{2} g^{lm}(\partial_j g_{km} + \partial_k g_{jm} - \partial_m g_{jk}).$$

A direct computation shows that

$$\Gamma_{11}^1 = \partial_r a/a, \quad \Gamma_{12}^1 = \Gamma_{21}^1 = 0, \quad \Gamma_{22}^1 = -b\partial_r b/a^2,$$
$$\Gamma_{11}^2 = 0, \quad \Gamma_{12}^2 = \Gamma_{21}^2 = \partial_r b/b, \quad \Gamma_{22}^2 = 0.$$

Thus the geodesic equations are

$$\ddot{r} + \frac{\partial_r a}{a}(\dot{r})^2 - \frac{b\partial_r b}{a^2}(\dot{\theta})^2 = 0, \tag{2.25}$$

$$\ddot{\theta} + \frac{2\partial_r b}{b}\dot{r}\dot{\theta} = 0. \tag{2.26}$$

The conserved quantities (speed and angular momentum) corresponding to (2.7) and (2.8) are given as follows:

$$(a(r)\dot{r})^2 + (b(r)\dot{\theta})^2 \text{ is conserved,} \tag{2.27}$$

$$b(r)^2\dot{\theta} \text{ is conserved.} \tag{2.28}$$

In fact, the first quantity is conserved since geodesics have constant speed, and the fact that the second quantity is conserved follows directly by taking its t-derivative and using the second geodesic equation.

As in Theorem 2.6, we would like that when a geodesic reaches its deepest point where $\dot{r} = 0$, it turns back toward the surface (i.e. $\ddot{r} > 0$). Now the equation (2.25) implies that

$$\dot{r} = 0 \implies \ddot{r} = \frac{b\partial_r b}{a^2}(\dot{\theta})^2.$$

Thus, when $\dot{r} = 0$, one has $\ddot{r} > 0$ iff $b' > 0$. This is the analogue of the Herglotz condition. For a radial sound speed as in (2.23), one has $b(r) = r/c(r)$ and the condition $b' > 0$ is equivalent with $\frac{d}{dr}\left(\frac{r}{c(r)}\right) > 0$.

Definition 2.14. A metric $g = a(r)^2\,dr^2 + b(r)^2\,d\theta^2$, where $a, b \in C^\infty([r_0, r_1])$ are positive, satisfies the *Herglotz condition* if

$$b'(r) > 0, \qquad r \in [r_0, r_1].$$

The following result is the analogue of Theorem 2.6.

Theorem 2.15 (Geodesics). *Let g satisfy the Herglotz condition as in Definition 2.14. Let $(r(t), \theta(t))$ be a unit speed geodesic with $r(0) = r_1$ and $\dot{r}(0) < 0$. There are two types of geodesics: either $r(t)$ strictly decreases to $\{r = r_0\}$ in finite time, or the geodesic stays in M and goes back to $\{r = r_1\}$ in finite time. Geodesics of the second type have a unique closest point (ρ, α) to the origin, and they consist of two symmetric branches where first $r(t)$ strictly decreases from r_1 to ρ, and then $r(t)$ strictly increases from ρ to r_1. Moreover, for any $(\rho, \alpha) \in M$ there is a unique such geodesic $\gamma_{\rho,\alpha}(t) = (r(t), \theta(t))$ with $\dot{\theta}(0) > 0$, and it satisfies*

$$\dot{r} = \mp\frac{1}{a(r)b(r)}\sqrt{b(r)^2 - b(\rho)^2}, \tag{2.29}$$

$$\theta(t) = \alpha \mp b(\rho)\int_\rho^{r(t)} \frac{a(r)}{b(r)}\frac{1}{\sqrt{b(r)^2 - b(\rho)^2}}\,dr, \tag{2.30}$$

where $-$ corresponds to the first branch where $r(t)$ decreases, and $+$ corresponds to the second branch where $r(t)$ increases.

Proof. Since the geodesic has unit speed, (2.27) implies that

$$(a(r)\dot{r})^2 + (b(r)\dot{\theta})^2 = 1. \tag{2.31}$$

Moreover, (2.28) implies that

$$b(r)^2\dot{\theta} = p \tag{2.32}$$

for some constant p. Combining the above two equations gives that $(a(r)\dot{r})^2 + (p/b(r))^2 = 1$, and thus

$$(a(r)\dot{r})^2 = 1 - \frac{p^2}{b(r)^2}. \tag{2.33}$$

Let I be the maximal interval of existence of the geodesic $(r(t), \theta(t))$ in M, so I is of the form $[0, T)$, $[0, T]$ or $[0, \infty)$ for some $T > 0$. Now, since $\dot{r}(0) < 0$, there are two possible cases: either $\dot{r}(t) < 0$ for all $t \in I$, or $\dot{r}(\bar{t}) = 0$ for some $\bar{t} \in I$. Assume that we are in the first case. Taking the t-derivative in (2.33) gives

$$2a(r)\dot{r}\frac{d}{dt}(a(r)\dot{r}) = 2p^2 b(r)^{-3} b'(r)\dot{r}, \qquad t \in I.$$

Since $\dot{r}(t) < 0$ for all $t \in I$, we may divide by $\dot{r}$ and obtain

$$\frac{d}{dt}(a(r)\dot{r}) = \frac{p^2 b(r)^{-3} b'(r)}{a(r)}, \qquad t \in I.$$

Using the Herglotz condition we have $b'(r) > 0$ for all $r \in [r_0, r_1]$. Thus there are $c_0, \varepsilon_0 > 0$ so that

$$a(r)\dot{r} \geq c_0 + \varepsilon_0 t, \qquad t \in I. \tag{2.34}$$

Now if $T = \infty$ one would get $\dot{r}(\bar{t}) = 0$ for some $\bar{t} \in I$, which is a contradiction. Hence in the first case where $\dot{r}(t) < 0$ for all $t \in I$, the geodesic must reach $\{r = r_0\}$ in finite time and $r(t)$ is strictly decreasing.

Assume now that we are in the second case where $\dot{r}(t) < 0$ for $0 \leq t < \bar{t}$ and $\dot{r}(\bar{t}) = 0$ for some $\bar{t} \in I$. Let $\rho = r(\bar{t})$ and $\alpha = \theta(\bar{t})$. Since both $\eta(s) = (r(\bar{t}+s), \theta(\bar{t}+s))$ and $\zeta(s) = (r(\bar{t}-s), 2\alpha - \theta(\bar{t}-s))$ solve the geodesic equations with the same initial data when $s = 0$, the geodesic has two branches that are symmetric with respect to $t = \bar{t}$. Note that we must have $p = \pm b(\rho)$ upon evaluating (2.33) at $t = \bar{t}$. If additionally $\dot{\theta}(0) > 0$ then by (2.32) one has $p > 0$, so in fact $p = b(\rho)$.

Moreover, given any $(\rho, \alpha) \in M$ we may consider the geodesic with $(r(0), \theta(0)) = (\rho, \alpha)$ and $(\dot{r}(0), \dot{\theta}(0)) = (0, 1/b(\rho))$ where the value for $\dot{\theta}(0)$ is obtained from (2.31) (the geodesic must have unit speed). The arguments above show that this geodesic has two symmetric branches, and reaches $\{r = r_1\}$ in finite time by (2.34). The required geodesic $\gamma_{\rho,\alpha}$ is obtained from $(r(t), \theta(t))$ after a translation in t.

The equation for $\dot{r}(t)$ follows from (2.33), where $p = b(\rho)$. Finally, (2.32) with $p = b(\rho)$ gives

$$\theta(t') = \alpha + b(\rho)\int_{\bar{t}}^{t'} \frac{1}{b(r(t))^2}\, dt.$$

We change variables $t = t(r)$ and use that by (2.29) one has

$$\frac{dt}{dr}(r) = \frac{1}{\dot{r}(t(r))} = \mp\frac{a(r)b(r)}{\sqrt{b(r)^2 - b(\rho)^2}}.$$

This proves (2.30). □

2.4. Geodesic X-ray transform

In this section we prove the result of [**Rom67**] (see also [**Rom87, Sha97**]) showing invertibility of the geodesic X-ray transform for radially symmetric metrics satisfying the Herglotz condition. Let

$$g = a(r)^2\, dr^2 + b(r)^2\, d\theta^2$$

be a metric in $M = \{(r, \theta)\,;\, r_0 < r \leq r_1\}$ satisfying the Herglotz condition $b'(r) > 0$ for $r \in [r_0, r_1]$. For $f \in C^\infty(M)$, we wish to study the problem of recovering f from its integrals over maximal geodesics starting from $\{r = r_1\}$. By Theorem 2.15 there are two types of geodesics: those that go to $\{r = r_0\}$ in finite time, and those that never reach $\{r = r_0\}$ and curve back to $\{r = r_1\}$ in finite time. We only consider integrals of f over geodesics of the second type. This corresponds to having measurements only on $\{r = r_1\}$ and not on $\{r = r_0\}$, which is relevant for instance in seismic imaging where $\{r = r_1\}$ corresponds to the surface of the Earth.

By Theorem 2.15, for any $(\rho, \alpha) \in M$ there is a unique unit speed geodesic $\gamma_{\rho,\alpha}(t)$ joining two points of $\{r = r_1\}$ and having (ρ, α) as its closest point to the origin. Denote by $\tau(\rho, \alpha)$ the length of this geodesic. Given $f \in C^\infty(M)$, we define its *geodesic ray transform* by

$$If(\rho,\alpha) = \int_0^{\tau(\rho,\alpha)} f(\gamma_{\rho,\alpha}(t))\,dt, \qquad (\rho,\alpha) \in M.$$

The main result in this section shows that under the Herglotz condition the geodesic X-ray transform is injective, i.e. f is uniquely determined by If.

THEOREM 2.16 (Injectivity). *Let g satisfy the Herglotz condition in Definition 2.14. If $f \in C^\infty(M)$ satisfies $If(\rho, \alpha) = 0$ for all $(\rho, \alpha) \in M$, then $f = 0$.*

To prove the theorem, we first note that by Theorem 2.15 one has

$$\gamma_{\rho,\alpha}(t) = (r(t), \alpha \mp \psi(\rho, r(t)))$$

where

$$\psi(\rho, r(t)) := b(\rho) \int_\rho^{r(t)} \frac{a(r)}{b(r)} \frac{1}{\sqrt{b(r)^2 - b(\rho)^2}}\,dr. \tag{2.35}$$

Moreover,

$$\frac{dr}{dt} = \mp \frac{1}{a(r)b(r)} \sqrt{b(r)^2 - b(\rho)^2}.$$

Here the sign $-$ corresponds to the first branch of the geodesic where $r(t)$ decreases from r_1 to ρ, and $+$ corresponds to the second branch where $r(t)$ increases.

Changing variables $t = t(r)$, we have

$$\begin{aligned} If(\rho,\alpha) &= \int_0^{\tau(\rho,\alpha)} f(r(t), \theta(t)) \\ &= \int_0^{\frac{1}{2}\tau(\rho,\alpha)} f(r(t), \alpha - \psi(\rho, r(t)))\,dt + \int_{\frac{1}{2}\tau(\rho,\alpha)}^{\tau(\rho,\alpha)} f(r(t), \alpha + \psi(\rho, r(t)))\,dt \\ &= \int_\rho^{r_1} \frac{a(r)b(r)}{\sqrt{b(r)^2 - b(\rho)^2}} f(r, \alpha - \psi(\rho, r))\,dr \\ &\quad + \int_\rho^{r_1} \frac{a(r)b(r)}{\sqrt{b(r)^2 - b(\rho)^2}} f(r, \alpha + \psi(\rho, r))\,dr. \end{aligned} \tag{2.36}$$

Assume for the moment that f is radial, $f = f(r)$. This is analogous to the result in Theorem 2.7 of determining a radial sound speed $c(r)$ from travel times, and the proof will use a similar method. If $f = f(r)$, we obtain

$$If(\rho,\alpha) = 2 \int_\rho^{r_1} \frac{a(r)b(r)}{\sqrt{b(r)^2 - b(\rho)^2}} f(r)\,dr. \tag{2.37}$$

We change variables

$$s = b(r)^2. \tag{2.38}$$

This is a valid change of variables since $b(r)$ is strictly increasing by the Herglotz condition. One has

$$If(\rho, \alpha) = 2 \int_{b(\rho)^2}^{b(r_1)^2} \frac{a(r(s))b(r(s))r'(s)}{(s - b(\rho)^2)^{1/2}} f(r(s))\, ds.$$

This is an Abel transform as in Theorem 2.8, where x corresponds to $b(\rho)^2$. If $If(\rho, \alpha) = 0$ for $r_0 < \rho < r_1$, it follows from Theorem 2.8 that

$$a(r(s))b(r(s))r'(s)f(r(s)) = 0, \qquad b(r_0)^2 < s < b(r_1)^2.$$

Since a, b and r' are positive, we get $f(r(s)) = 0$ for all s and thus $f(r) = 0$ for $r_0 < r < r_1$ as required.

We next consider the general case where $f = f(r, \theta) \in C^\infty(M)$. For any fixed r, the function $f(r, \cdot\,)$ is a smooth 2π-periodic function in $\mathbb{R}$, and it has the Fourier series

$$f(r, \theta) = \sum_{k=-\infty}^{\infty} f_k(r) e^{ik\theta}. \tag{2.39}$$

Here the Fourier coefficients $f_k(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(r, \theta) e^{-ik\theta}\, d\theta$ are smooth functions in $(r_0, r_1]$, and the Fourier series converges absolutely and uniformly in $\{\bar{r} \leq r \leq r_1\}$ whenever $r_0 < \bar{r} < r_1$.

Inserting (2.39) in (2.36), we have

$$If(\rho, \alpha) = \sum_{k=-\infty}^{\infty} \left[\int_{\rho}^{r_1} \frac{a(r)b(r)}{\sqrt{b(r)^2 - b(\rho)^2}} f_k(r) 2 \cos(k\psi(\rho, r))\, dr \right] e^{ik\alpha}.$$

Denote the expression in brackets by $A_k f_k(\rho)$. Thus, if $If(\rho, \alpha) = 0$ for $(\rho, \alpha) \in M$, then the Fourier coefficients $A_k f_k(\rho)$ vanish for each k and for $r_0 < \rho < r_1$. It remains to show that each generalized Abel transform A_k is injective. Note that if $k = 0$, then A_0 is exactly the Abel transform in (2.37) and this was already shown to be injective.

For $k \neq 0$, we make the same change of variables as in (2.38) and write

$$g_k(s) = 2a(r(s))b(r(s))r'(s)f_k(r(s)).$$

Then $A_k f_k(\rho) = T_k g_k(b(\rho)^2)$, where

$$T_k g_k(x) = \int_x^{b(r_1)^2} \frac{K_k(x, s)}{(s - x)^{1/2}} g_k(s)\, ds$$

where $x = x(\rho) = b(\rho)^2$ takes values in the range $b(r_0)^2 < x \leq b(r_1)^2$, and

$$K_k(x, s) = \cos(k\psi(\rho(x), r(s))).$$

Since a, b, and r' are positive, the injectivity of A_k is equivalent with the injectivity of T_k.

We now record some properties of the functions K_k.

Lemma 2.17. *For any $k \in \mathbb{Z}$, $K_k(x, s)$ is smooth in $\{b(r_0)^2 \leq x \leq s \leq b(r_1)^2\}$ and satisfies $K_k(x, x) = 1$ for all x.*

Proof. Changing variables $s = b(r)^2$, we have

$$\psi(\rho, r) = b(\rho) \int_{b(\rho)^2}^{b(r)^2} \frac{q(s)}{(s - b(\rho)^2)^{1/2}}\, ds$$

where $q(s) = \frac{a(r(s))r'(s)}{b(r(s))}$ is smooth. We further make another change of variables $s = b(\rho)^2 + (b(r)^2 - b(\rho)^2)t$ to obtain that

$$\psi(\rho, r) = (b(r)^2 - b(\rho)^2)^{1/2} G(\rho, r)$$

where

$$G(\rho, r) = b(\rho) \int_0^1 \frac{q(b(\rho)^2 + (b(r)^2 - b(\rho)^2)t)}{t^{1/2}}\, dt.$$

Here G is smooth since q and b are smooth. Using that $\cos x = \eta(x^2)$ where $\eta(t)$ is smooth on $\mathbb{R}$ (this can be seen by looking at the Taylor series of $\cos x$), it follows that $K_k(x,s) = \eta(k^2 \psi(\rho(x), r(s))^2)$ is smooth. Finally, note that $x = s$ corresponds to $\rho = r$, which shows that $K_k(x,x) = \cos(k\psi(\rho(x), \rho(x))) = 1$. □

The equation $T_k g_k = F$ is a *singular Volterra integral equation of the first kind* (see [**GV91**] for a detailed treatment of such equations). The injectivity of T_k now follows from the next result that extends Theorem 2.8 (which considers the special case $K \equiv 1$). This concludes the proof of Theorem 2.16.

THEOREM 2.18. *Let $K \in C^1(T)$ where $T := \{(x,t)\,;\, \alpha \le x \le t \le \beta\}$, and assume that $K(x,x) = 1$ for $x \in [\alpha, \beta]$. Given any $f \in \mathcal{A}((\alpha, \beta])$, there is a unique solution $u \in L^1_{\mathrm{loc}}((\alpha, \beta])$ of*

$$\int_x^\beta \frac{K(x,t)}{(t-x)^{1/2}} u(t)\, dt = f(x). \tag{2.40}$$

Moreover, if $K \in C^\infty(T)$ and if $f(x) = (\beta - x)^{1/2} h(x)$ for some $h \in C^\infty((\alpha, \beta])$, then $u \in C^\infty((\alpha, \beta])$.

PROOF. We define

$$H(x,t) := K(x,t) - 1.$$

Note that $H(x,x) = 0$ by the assumption on K. The equation (2.40) may be written as

$$Au + Bu = f \tag{2.41}$$

where $Au(x) = \int_x^\beta \frac{u(t)}{(t-x)^{1/2}}\, dt$ is the Abel transform, and

$$Bu(x) := \int_x^\beta \frac{H(x,t)}{(t-x)^{1/2}} u(t)\, dt.$$

If $B \equiv 0$ then (2.41) is a standard Abel integral equation and it can be solved using Theorem 2.8. More generally, we will show that the perturbation B can be handled by a Volterra iteration.

We first show that B maps any function $u \in L^1_{\mathrm{loc}}((\alpha, \beta])$ into $\mathcal{A}((\alpha, \beta])$, i.e. that $ABu \in W^{1,1}_{\mathrm{loc}}((\alpha, \beta])$. We use Fubini's theorem and the change of variables $s = x + (t-x)r$ to compute

$$\begin{aligned} ABu(x) &= \int_x^\beta \int_s^\beta \frac{H(s,t)}{(s-x)^{1/2}(t-s)^{1/2}} u(t)\, dt\, ds \\ &= \int_x^\beta \int_x^t \frac{H(s,t)}{(s-x)^{1/2}(t-s)^{1/2}} u(t)\, ds\, dt \\ &= \int_x^\beta \left[\int_0^1 \frac{H(x + (t-x)r, t)}{r^{1/2}(1-r)^{1/2}}\, dr \right] u(t)\, dt. \end{aligned}$$

Thus $ABu(x) = \int_x^\beta G(x,t)u(t)\,dt$ where $G \in C^1(T)$ since $K \in C^1(T)$. It follows that $ABu \in W^{1,1}_{\mathrm{loc}}((\alpha,\beta])$. By Theorem 2.8 we may write

$$Bu = ARu, \qquad u \in L^1_{\mathrm{loc}}((\alpha,\beta]),$$

where $Ru = -\frac{1}{\pi}\frac{d}{dx}ABu$. Since $H(x,x) = 0$ we have $G(x,x) = 0$, and thus using the above formula for ABu we have

$$Ru(x) = -\frac{1}{\pi}\int_x^\beta \partial_x G(x,t)u(t)\,dt.$$

In particular, the integral kernel of R is in $C^0(T)$, and it follows that

$$|Ru(x)| \leq C\int_x^\beta |u(t)|\,dt. \tag{2.42}$$

Since $Bu = ARu$, the equation (2.41) is equivalent with

$$A(u + Ru) = f.$$

Since $f \in \mathcal{A}((\alpha,\beta])$, one has $f = Au_0$ for some $u_0 \in L^1_{\mathrm{loc}}((\alpha,\beta])$ by Theorem 2.8. Because A is injective, (2.41) is further equivalent with the equation

$$u + Ru = u_0. \tag{2.43}$$

It is enough to show that (2.43) has a unique solution $u \in L^1_{\mathrm{loc}}((\alpha,\beta])$ for any $u_0 \in L^1_{\mathrm{loc}}((\alpha,\beta])$. For uniqueness, if $u + Ru = 0$, then (2.42) implies that

$$|u(x)| \leq C\int_x^\beta |u(t)|\,dt.$$

Gronwall's inequality implies that $u \equiv 0$. To prove existence, we iterate the bound (2.42) which yields

$$\begin{aligned}|R^j u(x)| &\leq C\int_x^\beta |R^{j-1}u(t_1)|\,dt_1 \leq \cdots \\ &\leq C^j\int_x^\beta\int_{t_1}^\beta\cdots\int_{t_{j-1}}^\beta |u(t_j)|\,dt_j\cdots dt_1 \\ &\leq C^j\frac{(\beta-x)^{j-1}}{(j-1)!}\|u\|_{L^1([x,\beta])}.\end{aligned}$$

Thus whenever $\alpha < \gamma < \beta$ one has

$$\|R^j u\|_{L^1([\gamma,\beta])} \leq \frac{(C(\beta-\gamma))^j}{j!}\|u\|_{L^1([\gamma,\beta])}. \tag{2.44}$$

The series

$$u := \sum_{j=0}^\infty (-R)^j u_0$$

converges in $L^1_{\mathrm{loc}}((\alpha,\beta])$ by (2.44), and the resulting function u solves (2.43).

We have proved that given any $f \in \mathcal{A}((\alpha,\beta])$ the equation (2.40) has a unique solution $u \in L^1_{\mathrm{loc}}((\alpha,\beta])$. Let now $K \in C^\infty(T)$ and $f(x) = (\beta-x)^{1/2}h(x)$ for some $h \in C^\infty((\alpha,\beta])$. By Theorem 2.8 one has $f = Au_0$ for some $u_0 \in C^\infty((\alpha,\beta])$, and it is enough to show that the solution u of (2.43) is smooth. But if $K \in C^\infty(T)$ the operator R above has C^∞ integral kernel, hence Ru is smooth, and thus also $u = -Ru + u_0$ is smooth. This concludes the proof of the theorem. $\square$

2.5. Examples and counterexamples

In this section we give some examples of manifolds where the geodesic X-ray transform is injective, and some examples where it is not injective. We first begin with some remarks on the Herglotz condition.

Let $g = a(r)^2\,dr^2 + b(r)^2\,d\theta^2$ be a metric in $M = \{r_0 < r \leq r_1\}$, where $a, b \in C^\infty([r_0, r_1])$ are positive. We first give a definition.

Definition 2.19. The circle $\{r = \bar{r}\}$ is *strictly convex* (resp. *strictly concave*) as a submanifold of (M, g) if for any geodesic $(r(t), \theta(t))$ with $r(0) = \bar{r}$, $\dot{r}(0) = 0$ and $\dot{\theta}(0) \neq 0$, one has $\ddot{r}(0) > 0$ (resp. $\ddot{r}(0) < 0$).

Strict convexity means that any tangential geodesic to the circle $\{r = \bar{r}\}$ curves away from this circle toward $\{r = r_1\}$, with exactly first order contact with the circle when $t = 0$. More precisely, we should say that the circle is strictly convex when viewed from $\{r = r_1\}$ (there is a choice of orientation involved). Strict convexity is equivalent to the fact that $\{r = \bar{r}\}$ has positive definite second fundamental form in (M, g). Conversely, strict concavity means that tangential geodesics to the circle $\{r = \bar{r}\}$ have first order contact and curve toward $\{r = r_0\}$.

Lemma 2.20. *Let $r_0 < \bar{r} \leq r_1$.*

(a) *$\{r = \bar{r}\}$ is strictly convex as a submanifold of (M, g) iff $b'(\bar{r}) > 0$.*
(b) *The circle $t \mapsto (\bar{r}, t)$ is a geodesic of (M, g) iff $b'(\bar{r}) = 0$.*
(c) *$\{r = \bar{r}\}$ is strictly concave as a submanifold of (M, g) iff $b'(\bar{r}) < 0$.*

Proof. If $(r(t), \theta(t))$ is a geodesic with $r(0) = \bar{r}$ and $\dot{r}(0) = 0$, then by (2.25)

$$\ddot{r}(0) = \frac{b(\bar{r})b'(\bar{r})}{a(\bar{r})^2}(\theta'(0))^2. \tag{2.45}$$

If $\dot{\theta}(0) \neq 0$, then $\ddot{r}(0)$ has the same sign as $b'(\bar{r})$ since b is positive. This proves parts (a) and (c). For part (b), if $b'(\bar{r}) = 0$, then $t \mapsto (\bar{r}, t)$ satisfies the geodesic equations (2.25)–(2.26). Conversely, if $t \mapsto (\bar{r}, t)$ satisfies the geodesic equations, then $\ddot{r}(0) = 0$ and (2.45) implies that $b\partial_r b/a^2|_{r=\bar{r}} = 0$. One must have $b'(\bar{r}) = 0$. □

Thus, if the Herglotz condition is violated, either $b' = 0$ somewhere and there is a *trapped geodesic* (one that never reaches the boundary), or $b' < 0$ somewhere and tangential geodesics curve toward $\{r = r_0\}$. We also obtain the following characterization of the Herglotz condition.

Corollary 2.21. The following conditions are equivalent.

(a) The circles $\{r = \bar{r}\}$ are strictly convex for $r_0 < \bar{r} \leq r_1$.
(b) $b' \geq 0$ and no circle $\{r = \bar{r}\}$ is a trapped geodesic for $r_0 < \bar{r} \leq r_1$.
(c) $b'(r) > 0$ for $r \in (r_0, r_1]$.

We now go back to Example 2.13 and surfaces of revolution. Recall the setup: r correspond to the z-coordinate in $\mathbb{R}^3$, $h : [r_0, r_1] \to \mathbb{R}$ is a smooth positive function, and S is the surface of revolution obtained by rotating the graph of $r \mapsto h(r)$ about the z-axis. The surface S is given by

$$S = \{(h(r)\cos\theta, h(r)\sin\theta, r)\,;\, r \in (r_0, r_1],\ \theta \in [0, 2\pi]\}.$$

The metric on S induced by the Euclidean metric on $\mathbb{R}^3$ has the form

$$g = (1 + h'(r)^2)\,dr^2 + h(r)^2\,d\theta^2.$$

Thus $a(r) = \sqrt{1+h'(r)^2}$ and $b(r) = h(r)$.

Finally we give four illustrative examples: two examples where the geodesic X-ray transform is injective, and two examples where it fails to be injective.

EXAMPLE 2.22 (Small spherical cap). Let $h : [r_0, r_1] \to \mathbb{R}$, $h(r) = \sqrt{1-r^2}$ where $r_0 = -1$ and $r_1 = -\alpha$ where $0 < \alpha < 1$. Then $S = S_\alpha$ corresponds to a punctured spherical cap strictly contained in a hemisphere:

$$S_\alpha = \{x \in S^2\,;\, x_3 \leq -\alpha\} \setminus \{-e_3\}.$$

Clearly $h' > 0$ in $[r_0, r_1]$. Thus the Herglotz condition is satisfied, and by Theorem 2.16 the geodesic X-ray transform on S_α is injective whenever $0 < \alpha < 1$. More precisely, a function f can be recovered from its integrals over geodesics that start and end on the boundary $\{x_3 = -\alpha\}$, with the geodesics going through the south pole excluded. Of course, geodesics in S_α are segments of great circles.

EXAMPLE 2.23 (Large spherical cap). Let $h : [r_0, r_1] \to \mathbb{R}$, $h(r) = \sqrt{1-r^2}$ where $r_0 = -1$ and $r_1 = \beta$ where $0 < \beta < 1$. Then $S = S_\beta$ corresponds to a punctured spherical cap that is larger than a hemisphere:

$$S_\beta = \{x \in S^2\,;\, x_3 \leq \beta\} \setminus \{-e_3\}.$$

Now the Herglotz condition is violated: one has $h'(r) > 0$ for $r < 0$, but $h'(0) = 0$ and $h'(r) < 0$ for $r > 0$. In particular, the geodesic $\{r = 0\}$, which is just the equator, is a trapped geodesic in S_β. The great circles close to the equator are also trapped geodesics, and S_β is an example of a manifold with strong trapping properties.

In fact the geodesic X-ray transform is *not injective* on S_β (even if the south pole is included). To see this, let $f : S^2 \to \mathbb{R}$ be an odd function with respect to the antipodal map, i.e. $f(-x) = -f(x)$, and assume f is supported in $\{-\beta < x_3 < \beta\}$. For example, one can take $f(x) = \varphi(x) - \varphi(-x)$ where φ is a C^∞ function supported in a small neighborhood of e_1 with $\varphi > 0$ near e_1.

Using the support condition for f, the integral of f over a maximal geodesic in (M, g) (a segment of a great circle C in S^2) is equal to the integral of f over the whole great circle C. But since f is odd, its integral over any great circle is zero. This shows that the geodesic X-ray transform If of f in S_β vanishes, but f is not identically zero.

EXAMPLE 2.24 (Catenoid). Let $h : [-1, 1] \to \mathbb{R}$, $h(r) = \cosh(r) = \frac{e^r + e^{-r}}{2}$. The corresponding surface of revolution is the *catenoid*

$$S = \{(\cosh(r)\cos(\theta), \cosh(r)\sin(\theta), r)\,;\, r \in [-1, 1], \theta \in [0, 2\pi]\}.$$

One has $h'(r) = \sinh(r) = \frac{e^r - e^{-r}}{2}$. Thus in particular $h'(0) = 0$ and $h'(r) > 0$ for $r > 0$. Define

$$S_\pm = \{x \in S\,;\, \pm x_3 > 0\}.$$

Then S_+ corresponds to $h : (r_0, r_1] \to \mathbb{R}$ with $r_0 = 0$ and $r_1 = 1$. By Theorem 2.16 the geodesic X-ray transform in S_+ is injective, when considering geodesics that start and end on $S_+ \cap \{x_3 = 1\}$. By symmetry, also the geodesic X-ray transform on S_- is injective for geodesics that start and end on $S_- \cap \{x_3 = -1\}$. Since $S = S_+ \cup S_- \cup S_0$ where $S_0 = S \cap \{x_3 = 0\}$ has zero measure, it follows that also the geodesic X-ray transform on S is injective (any smooth function on S can be recovered from its integrals starting and ending on ∂S).

Note that since $h'(0) = 0$, the geodesic S_0 is a trapped geodesic in S. The manifold S has also other trapped geodesics that start on ∂S and orbit S_0 for infinitely long time. The catenoid is an example of a negatively curved manifold with weak trapping properties (the trapped set is hyperbolic). Because the trapping is weak, the geodesic X-ray transform is still invertible in this case.

Example 2.25 (Catenoid type surface with flat cylinder glued in the middle). Let $h : [-1, 1] \to \mathbb{R}$ with $h(r) = 1$ for $r \in [-\frac{1}{2}, \frac{1}{2}]$, $h'(r) > 0$ for $r > \frac{1}{2}$, and $h'(r) < 0$ for $r < -\frac{1}{2}$, and let S be the surface of revolution obtained by rotating $h|_{[-1,1]}$. Then $S \cap \{-\frac{1}{2} \leq x_3 \leq \frac{1}{2}\}$ is a flat cylinder.

Consider a smooth function f in S given by

$$f(h(r)\cos\theta, h(r)\sin\theta, r) = \eta(r)$$

where $\eta \in C_c^\infty(-\frac{1}{2}, \frac{1}{2})$ is nontrivial and satisfies $\int_{-1/2}^{1/2} \eta(r)\, dr = 0$. Then f integrates to zero over any geodesic starting and ending on ∂S. To see this, note that f vanishes outside the flat cylinder, and any geodesic that enters the flat cylinder must be a geodesic of the cylinder. Since $h \equiv 1$ in the cylinder, the metric is $dr^2 + d\theta^2$, one has $a = b = 1$, the geodesic equations are $\ddot{r} = \ddot{\theta} = 0$, and unit speed geodesics are of the form $\zeta(t) = (r(t), \theta(t)) = (\alpha t + \beta, \gamma t + \delta)$ where $(\dot{r})^2 + (\dot{\theta})^2 = \alpha^2 + \gamma^2 = 1$. Thus it follows that

$$\int_\zeta f\, dt = \int \eta(\alpha t + \beta)\, dt = 0.$$

Thus S is an example of a manifold that has a large flat part (the cylinder) with many trapped geodesics, and the geodesic X-ray transform is not injective. The reason for non-injectivity is that S contains part of $\mathbb{R} \times S^1$, and the X-ray transform on $\mathbb{R}$ is not injective (there are nontrivial functions that integrate to zero on $\mathbb{R}$).