Geometric inverse problems in 2D

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Contents

Preface

This monograph is devoted to geometric inverse problems in two dimensions. *Inverse problems* arise in various fields of science and engineering, frequently in connection with imaging methods where one attempts to produce images of the interior of an unknown object by making indirect measurements outside. A standard example is X-ray computed tomography (CT) in medical imaging. There one sends X-rays through the patient and measures how much the rays are attenuated along the way. From these measurements one would like to determine the attenuation coefficient of the tissues inside. If the X-rays are sent along a two-dimensional cross-section (identified with $\mathbb{R}^2$) of the patient, the X-ray measurements correspond to the *Radon transform* Rf of the unknown attenuation function f in $\mathbb{R}^2$. Here, Rf just encodes the integrals of f along all straight lines in $\mathbb{R}^2$. The easy *direct problem* in X-ray CT would be to determine the Radon transform Rf when f is known. However, in order to produce images one needs to solve the *inverse problem*: determine f when Rf is known (i.e. invert the Radon transform).

One can divide the mathematical analysis of the Radon transform inverse problem in several parts, including the following:

- (Uniqueness) If $Rf_1 = Rf_2$, does it follow that $f_1 = f_2$?
- (Stability) If Rf_1 and Rf_2 are close, are f_1 and f_2 close in suitable norms? Is there stability with respect to noise or measurement errors?
- (Reconstruction) Is there an efficient algorithm for reconstructing f from the knowledge of Rf?
- (Range characterization) Which functions are of the form Rf for some f?
- (Partial data) Can one determine (some information on) f from partial knowledge of Rf?

In this monograph we will study inverse problems in *geometric* settings. For X-ray type problems this will mean that straight lines are replaced by more general curves. A particularly clean setting, which is still relevant for several applications, is given by geodesic curves of a smooth Riemannian metric. We will focus on this setting and formulate our questions on compact Riemannian manifolds (M, g) with smooth boundary. This corresponds to working with compactly supported functions in the Radon transform problem.

We will now briefly describe the main geometric inverse problems studied here. Our first question is a direct generalization of the Radon transform problem.

> **1. Geodesic X-ray transform.** Is it possible to determine an unknown function f in (M, g) from the knowledge of its integrals over maximal geodesics?

This is a fundamental inverse problem that is related to several other inverse problems, in particular in seismic imaging applications. A classical related problem is to determine the interior structure of Earth by measuring travel times of earthquakes. In a mathematical idealization, we may suppose that Earth is a ball $M \subset \mathbb{R}^3$ and that waves generated by earthquakes follow the geodesics of a Riemannian metric g determined by the sound speed in different substructures. If an earthquake is generated at a point $x \in \partial M$, then the first arrival time of that earthquake to a seismic station at $y \in \partial M$ is the geodesic distance $d_g(x, y)$. The *travel time tomography* problem, originating in geophysics in the early 20th century, is to determine the metric g (i.e. the sound speed in M) from the geodesic distances between boundary points. The same problem arose much later in pure mathematics and differential geometry. It can be formulated as follows:

> **2. Boundary rigidity problem.** Is it possible to determine the metric in (M, g), up to a boundary fixing isometry, from the knowledge of the boundary distance function $d_g|_{\partial M \times \partial M}$?

The geodesic X-ray transform problem is in fact precisely the linearization of the boundary rigidity problem for metrics in a fixed conformal class. If one removes the restriction to a fixed conformal class, the linearization of the boundary rigidity problem is a *tensor tomography problem.* To describe such a problem, let (M, g) be a compact Riemannian n-manifold with smooth boundary, and let $m \geq 0$. The geodesic X-ray transform on symmetric m-tensor fields is an operator I_m defined by

$$I_m f(\gamma) = \int_\gamma f_{j_1 \cdots j_m}(\gamma(t))\dot{\gamma}^{j_1}(t) \cdots \dot{\gamma}^{j_m}(t)\, dt, \qquad \gamma \text{ is a maximal geodesic in } M,$$

where $f = f_{j_1 \cdots j_m}\, dx^{j_1} \otimes \cdots \otimes dx^{j_m}$ is a smooth symmetric m-tensor field on M. Here and throughout this monograph we employ the Einstein summation convention where a repeated lower and upper index is summed. In the above case this means that

$$f_{j_1 \cdots j_m}\, dx^{j_1} \otimes \cdots \otimes dx^{j_m} = \sum_{j_1, \ldots, j_m = 1}^{n} f_{j_1 \cdots j_m}\, dx^{j_1} \otimes \cdots \otimes dx^{j_m}.$$

If $m \geq 1$ the operator I_m always has a nontrivial kernel: one has $I_m(\sigma \nabla h) = 0$ whenever h is a smooth symmetric $(m-1)$-tensor field with $h|_{\partial M} = 0$, ∇ is the total covariant derivative, and σ denotes the symmetrization of a tensor. Tensors of the form $\sigma \nabla h$ are called *potential tensors.* If $m = 1$, this just means that $I_1(dh) = 0$ whenever $h \in C^\infty(M)$ satisfies $h|_{\partial M} = 0$. Any 1-tensor field f has a solenoidal decomposition $f = v + dh$ where v is *solenoidal* (i.e. divergence-free) and $h|_{\partial M} = 0$. Thus it is only possible to determine the solenoidal part of a 1-tensor f from $I_1 f$. This decomposition generalizes to tensors of arbitrary order, leading to the following inverse problem.

> **3. Tensor tomography problem.** Is it possible to determine the solenoidal part of an m-tensor field f in (M, g) from the knowledge of $I_m f$?

A variant of the geodesic X-ray transform, arising in applications such as SPECT (single photon emission computed tomography), includes an attenuation factor. In this case, $f \in C^\infty(M)$ is a source function and $a \in C^\infty(M)$ is an attenuation coefficient, and one can measure integrals like

$$I^a f(\gamma) = \int_\gamma e^{\int_0^t a(\gamma(s))\,ds} f(\gamma(t))\,dt, \quad \gamma \text{ is a maximal geodesic.}$$

This is the *attenuated geodesic X-ray transform* of f, and a typical inverse problem is to determine f from $I^a f$ when a is assumed to be known. Clearly this reduces to the standard geodesic X-ray transform when $a = 0$. Similar questions appear in mathematical physics, where the attenuation coefficient is replaced by a *connection* or a *Higgs field* on some vector bundle over M. This roughly corresponds to replacing the function $a(x)$ by a matrix valued function or a 1-form.

> **4. Attenuated geodesic X-ray transform.** Is it possible to determine a function f in (M, g) from its attenuated geodesic X-ray transform, when the attenuation is given by a connection and a Higgs field?

This question also arises as the linearization of the *scattering rigidity problem* for a connection/Higgs field. One can ask related questions for tensor fields and also for more general weighted X-ray transforms.

Finally, we consider a geometric inverse problem of a somewhat different nature. Consider the Dirichlet problem for the Laplace equation in (M, g),

$$\begin{cases} \Delta_g u &= 0 \text{ in } M, \\ \quad u &= f \text{ on } \partial M. \end{cases}$$

Here Δ_g is the Laplace-Beltrami operator on (M, g), given in local coordinates by

$$\Delta_g u = |g|^{-1/2} \partial_{x_j} (|g|^{1/2} g^{jk} \partial_{x_k} u)$$

where (g^{jk}) is the inverse matrix of $g = (g_{jk})$, and $|g| = \det(g_{jk})$. This is a uniformly elliptic operator, and there is a unique solution $u \in C^\infty(M)$ for any $f \in C^\infty(\partial M)$. The *Dirichlet-to-Neumann map* Λ_g takes the Dirichlet data of u to Neumann data,

$$\Lambda_g : f \mapsto \partial_\nu u|_{\partial M}$$

where $\partial_\nu u|_{\partial M} = du(\nu)|_{\partial M}$ with ν denoting the inner unit normal to ∂M.

The above problem is related to Electrical Impedance Tomography, where the objective is to determine the electrical properties of a medium by making voltage and current measurements on its boundary. Here the metric g corresponds to the electrical resistivity of the medium, and for a prescribed boundary voltage f one measures the corresponding current flux $\partial_\nu u$ at the boundary. Thus the electrical measurements are encoded by the Dirichlet-to-Neumann map Λ_g. There are natural gauge invariances: the map Λ_g remains unchanged under a boundary fixing isometry of (M, g), and when $\dim(M) = 2$ there is an additional invariance due to conformal changes of the metric. This leads to the following inverse problem.

> **5. Calderón problem.** Is it possible to determine the metric in (M, g), up to gauge, from the knowledge of the Dirichlet-to-Neumann map Λ_g?

In this monograph we will discuss known results for the above problems when (M, g) is *two-dimensional.* The reason for restricting to the two-dimensional setting is that the available results and methods are slightly different in three and higher dimensions. Moreover, the two-dimensional theory is at the moment fairly well developed in the context of *simple manifolds.* A compact Riemannian manifold (M, g) with smooth boundary is called simple if

- the boundary ∂M is *strictly convex* (the second fundamental form of ∂M is positive definite),
- M is *nontrapping* (any geodesic reaches the boundary in finite time), and
- M has *no conjugate points*.

Examples of simple manifolds include strictly convex domains in Euclidean space, strictly convex simply connected domains in nonpositively curved manifolds, strictly convex subdomains of the hemisphere, and small metric perturbations of these.

In this book we will show that questions 1-4 above have a positive answer on two-dimensional simple manifolds, and question 5 has a positive answer on any two-dimensional manifold. In particular, this gives a positive answer in two dimensions to the boundary rigidity problem posed by Michel in [**Mic82**]. The original proof of this result by Pestov and Uhlmann [**PU05**] employs striking connections between the above problems: in fact, it uses the solution of the geodesic X-ray transform problem and the Calderón problem in order to solve the boundary rigidity problem.

We will also see that there are counterexamples to questions 1-4 if one goes outside the class of simple manifolds. However, it is an outstanding open problem whether questions 1-4 have positive answers in the class of strictly convex nontrapping manifolds (i.e. whether the no conjugate points assumption can be removed).

CHAPTER 1

The Radon transform in the plane

In this chapter we will study basic properties of the Radon transform in the plane. In this setting it is possible to give precise results on uniqueness, stability, reconstruction, and range characterization for the related inverse problem. We will also discuss the normal operator and show that it is an elliptic pseudodifferential operator. These results will act as model cases for the corresponding geodesic X-ray transform results in later chapters. The results are rather classical, and we refer to [**Hel99**] and [**Nat01**] for more detailed treatments.

1.1. Uniqueness and stability

The *X-ray transform* If of a function f in $\mathbb{R}^n$ encodes the integrals of f over all straight lines, whereas the *Radon transform* Rf encodes the integrals of f over $(n-1)$-dimensional affine planes. We will focus on the case $n = 2$, where the two transforms coincide. There are many ways to parametrize the set of lines in $\mathbb{R}^2$. We will parametrize lines by their normal vector ω and signed distance s from the origin.

Definition 1.1. If $f \in C_c^\infty(\mathbb{R}^2)$, the *Radon transform* of f is the function

$$Rf(s,\omega) := \int_{-\infty}^{\infty} f(s\omega + t\omega^\perp)\,dt, \quad s \in \mathbb{R},\ \omega \in S^1.$$

Here $\omega^\perp$ is the vector in S^1 obtained by rotating ω counterclockwise by $90°$.

Remark 1.2. The parametrization of lines by (s,ω) as above is called the *parallel beam geometry*, which is commonly used for the Radon transform in the plane. When studying the geodesic X-ray transform in later chapters we will however use a different parametrization, the *fan beam geometry*, which is customary in that context.

The Radon transform arises in medical imaging in the context of *X-ray computed tomography.* In this imaging method, X-rays are sent through the patient from various locations and angles, and one measures how much the rays are attenuated. The measurements correspond to integrals of the unknown attenuation coefficient in the body along straight lines. Moreover, the imaging is often carried out in two-dimensional cross sections of the body, and the idealized measurements (corresponding to X-rays sent from all locations and angles) correspond exactly to the two-dimensional Radon transform. This leads to the basic inverse problem in X-ray computed tomography.

Inverse problem: determine the attenuation function f in $\mathbb{R}^2$ from X-ray measurements encoded by the Radon transform Rf.

It is easy to see that given any $f \in C_c^\infty(\mathbb{R}^2)$, one has $Rf \in C^\infty(\mathbb{R} \times S^1)$ and for each $\omega \in S^1$ the function $Rf(\,\cdot\,, \omega)$ is compactly supported in $\mathbb{R}$. Moreover, the Radon transform enjoys the following invariance under translations:

$$R(f(\,\cdot\, - s_0\omega))(s, \omega) = Rf(s - s_0, \omega).$$

EXERCISE 1.3. Prove the properties for R stated in the previous paragraph.

The translation invariance suggests that the Radon transform should behave well under Fourier transforms. Indeed, there is a well-known relation between Rf and the Fourier transform $\hat{f} = \mathscr{F} f$ given by the *Fourier slice theorem*. Here, for $h \in C_c^\infty(\mathbb{R}^n)$ we use the convention

$$\hat{h}(\xi) = \mathscr{F} h(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} h(x)\,dx, \qquad \xi \in \mathbb{R}^n.$$

Recall the following facts regarding the Fourier transform in $\mathbb{R}^n$ (see e.g. [**Hör85**, Chapter 7] for more details):

1. The Fourier transform is bounded $L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$.
2. The Fourier transform is bijective $\mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$, where $\mathscr{S}(\mathbb{R}^n)$ is the *Schwartz space* consisting of all $f \in C^\infty(\mathbb{R}^n)$ so that $x^\alpha \partial^\beta f \in L^\infty(\mathbb{R}^n)$ for all $\alpha, \beta \in \mathbb{N}_0^n$.
3. Any $f \in \mathscr{S}(\mathbb{R}^n)$ can be recovered from its Fourier transform $\hat{f}$ by the Fourier inversion formula
$$f(x) = \mathscr{F}^{-1}\hat{f}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi)\,d\xi, \qquad x \in \mathbb{R}^n.$$
4. For $f, g \in \mathscr{S}(\mathbb{R}^n)$ one has the Parseval identity
$$\int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi = (2\pi)^n \int_{\mathbb{R}^n} f(x)\overline{g(x)}\,dx$$
and the Plancherel formula
$$\|\hat{f}\|_{L^2(\mathbb{R}^n)} = (2\pi)^{n/2}\|f\|_{L^2(\mathbb{R}^n)}.$$
5. Fourier transform converts derivatives to polynomials:

ier_transform_derivative
$$(D_j f)\hat{} = \xi_j \hat{f}(\xi) \tag{1.1}$$
where $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$.

EXERCISE 1.4. Show that R maps $\mathscr{S}(\mathbb{R}^2)$ to $C^\infty(\mathbb{R} \times S^1)$. A more precise result will be given in Theorem 1.15.

We will denote by $(Rf)\tilde{}(\,\cdot\,, \omega)$ the Fourier transform of Rf with respect to s.

thm_fourier_slice

THEOREM 1.5 (Fourier slice theorem). *If $f \in C_c^\infty(\mathbb{R}^2)$, then*

$$(Rf)\tilde{}(\sigma, \omega) = \hat{f}(\sigma\omega).$$

PROOF. Parametrizing $\mathbb{R}^2$ by $y = s\omega + t\omega^\perp$, we have

$$\begin{aligned}(Rf)\tilde{}(\sigma, \omega) &= \int_{-\infty}^{\infty} e^{-i\sigma s}\left[\int_{-\infty}^{\infty} f(s\omega + t\omega^\perp)\,dt\right] ds = \int_{\mathbb{R}^2} e^{-i\sigma y\cdot\omega} f(y)\,dy \\ &= \hat{f}(\sigma\omega). \end{aligned}$$
□

This result gives uniqueness in the inverse problem for the Radon transform:

ollary_radon_injectivity

THEOREM 1.6 (Uniqueness). *If $f_1, f_2 \in C_c^\infty(\mathbb{R}^2)$ are such that $Rf_1 = Rf_2$, then $f_1 = f_2$.*

PROOF. Since R is linear, it is enough to write $f = f_1 - f_2$ and to show that $Rf \equiv 0$ implies $f \equiv 0$. But if $Rf \equiv 0$, then $\hat{f} \equiv 0$ by Theorem 1.5 and consequently $f \equiv 0$ by Fourier inversion. □

In fact, it is easy to prove a quantitative version of the above uniqueness result stating that if $Rf_1 \approx Rf_2$, then $f_1 \approx f_2$ (in suitable norms). Given any $s \in \mathbb{R}$, we will employ the Sobolev norms

$$\|f\|_{H^s(\mathbb{R}^2)} := \|(1+|\xi|^2)^{s/2}\hat{f}(\xi)\|_{L^2(\mathbb{R}^2)},$$
$$\|Rf\|_{H^s_T(\mathbb{R}\times S^1)} := \|(1+\sigma^2)^{s/2}(Rf)\tilde{\ }(\sigma,\omega)\|_{L^2(\mathbb{R}\times S^1)}.$$

EXERCISE 1.7. If $m \geq 0$ is an integer, use the Plancherel theorem for the Fourier transform to show that

$$\|f\|_{H^{2m}(\mathbb{R}^2)} \sim \sum_{|\alpha|\leq 2m} \|\partial^\alpha f\|_{L^2(\mathbb{R}^2)},$$
$$\|Rf\|_{H^{2m}_T(\mathbb{R}\times S^1)} \sim \sum_{j=0}^{2m} \|\partial_s^j Rf\|_{L^2(\mathbb{R}\times S^1)}$$

where $A \sim B$ means that $cA \leq B \leq CA$ for some constants $c, C > 0$ which are independent of f.

Thus, roughly, the $H^s(\mathbb{R}^2)$ norm of f measures the size of the first s derivatives of f in L^2. A similar statement holds for the H^t_T norm of Rf, with the difference that the H^t_T norm only involves derivatives in the s variable but not in ω.

thm_radon_stability

THEOREM 1.8 (Stability). *If $s \in \mathbb{R}$, then for any $f_1, f_2 \in C_c^\infty(\mathbb{R}^2)$ one has*

$$\|f_1 - f_2\|_{H^s(\mathbb{R}^2)} \leq \frac{1}{\sqrt{2}}\|Rf_1 - Rf_2\|_{H^{s+1/2}_T(\mathbb{R}\times S^1)}.$$

PROOF. Let $f = f_1 - f_2$. Using polar coordinates, we obtain that

f_hs_squared

$$\begin{aligned}\|f\|^2_{H^s(\mathbb{R}^2)} = \|(1+|\xi|^2)^{s/2}\hat{f}\|^2_{L^2(\mathbb{R}^2)} &= \int_0^\infty \int_{S^1} (1+\sigma^2)^s |\hat{f}(\sigma\omega)|^2 \sigma \, d\omega \, d\sigma \\ &= \frac{1}{2}\int_{-\infty}^\infty \int_{S^1} (1+\sigma^2)^s |\hat{f}(\sigma\omega)|^2 |\sigma| \, d\omega \, d\sigma \\ &= \frac{1}{2}\int_{-\infty}^\infty \int_{S^1} (1+\sigma^2)^s |(Rf)\tilde{\ }(\sigma,\omega)|^2 |\sigma| \, d\omega \, d\sigma. \qquad (1.2)\end{aligned}$$

In particular, since $|\sigma| \leq (1+\sigma^2)^{1/2}$, this implies the stability estimate

$$\|f\|^2_{H^s(\mathbb{R}^2)} \leq \frac{1}{2}\|Rf\|^2_{H^{s+1/2}_T(\mathbb{R}\times S^1)}.$$ □

If f is supported in a fixed compact set, the previous inequality can be reversed.

thm_radon_continuity

THEOREM 1.9 (Continuity). *Let $s \in \mathbb{R}$ and let $K \subset \mathbb{R}^n$ be compact. There is a constant $C_K > 0$ so that for any $f \in C_c^\infty(\mathbb{R}^2)$ with $\mathrm{supp}(f) \subset K$ one has*

$$\|Rf\|_{H^{s+1/2}_T(\mathbb{R}\times S^1)} \leq C_K \|f\|_{H^s(\mathbb{R}^2)}.$$

EXERCISE 1.10. Prove Theorem 1.9 when $s \geq 0$ by splitting the last integral in (1.2) in two parts, one over $\{|\sigma| \leq 1\}$ and the other over $\{|\sigma| > 1\}$.

EXERCISE 1.11. Prove Theorem 1.9 for all $s \in \mathbb{R}$. This requires the Sobolev duality assertion $|\int_{\mathbb{R}^n} fh\,dx| \leq \|f\|_{H^s}\|h\|_{H^{-s}}$.

remark_radon_continuity

REMARK 1.12. Theorem 1.9 implies that the Radon transform extends as a bounded map

$$R : H^s_K(\mathbb{R}^2) \to H^{s+1/2}_T(\mathbb{R} \times S^1)$$

where $H^s_K(\mathbb{R}^2) = \{f \in H^s(\mathbb{R}^2)\,;\, \mathrm{supp}(f) \subset K\}$. In fact one may replace the $H^{s+1/2}_T$ norm on the right by the $H^{s+1/2}$ norm (see for instance [**Nat01**, Theorem II.5.2]). Thus, in a sense, the Radon transform in the plane is smoothing of order 1/2 (it adds 1/2 derivatives). We also observe that Theorems 1.8 and 1.9 yield the two-sided inequality

$$\sqrt{2}\|f\|_{H^s} \leq \|Rf\|_{H^{s+1/2}_T(\mathbb{R}\times S^1)} \leq C_K\|f\|_{H^s}, \qquad f \in H^s_K(\mathbb{R}^2).$$

1.2. Range and support theorems

We will next consider the range characterization problem: which functions in $\mathbb{R} \times S^1$ are of the form Rf for some $f \in C^\infty_c(\mathbb{R}^2)$? There is an obvious restriction: one has

hl_range_condition_first

$$Rf(-s,-\omega) = Rf(s,\omega), \tag{1.3}$$

i.e. Rf is always *even*. Another restriction comes from studying the *moments*

$$\mu_k(Rf)(\omega) = \int_{-\infty}^{\infty} s^k (Rf)(s,\omega)\,ds, \qquad k \geq 0,\ \omega \in S^1.$$

It is easy to see that

l_range_condition_second

(1.4) for any $k \geq 0$, $\mu_k(Rf)$ is a homogeneous polynomial of degree k in ω.

This means that $\mu_k(Rf)(\omega) = \sum_{j_1,\ldots,j_k=1}^{2} a_{j_1\cdots j_k}\omega_{j_1}\cdots\omega_{j_k}$ for some constants $a_{j_1\cdots j_k}$.

EXERCISE 1.13. Prove that Rf always satisfies (1.3) and (1.4).

It turns out that these conditions (called *Helgason-Ludwig range conditions*) are essentially the only restrictions. We will first consider range characterization on $\mathscr{S}(\mathbb{R}^2)$. To do this, we need to define a Schwartz space on $\mathbb{R} \times S^1$.

DEFINITION 1.14. The space $\mathscr{S}(\mathbb{R} \times S^1)$ is the set of all $\varphi \in C^\infty(\mathbb{R} \times S^1)$ so that $(1+s^2)^k\partial_s^l\varphi \in L^\infty(\mathbb{R} \times S^1)$ for all $k, l \geq 0$. We write $\mathscr{S}_H(\mathbb{R} \times S^1)$ for the set of all functions $\varphi \in \mathscr{S}(\mathbb{R} \times S^1)$ that satisfy the Helgason-Ludwig conditions, i.e. (1.3) and (1.4).

The following result is a Radon transform analogue of the fact that the Fourier transform is bijective $\mathscr{S}(\mathbb{R}^2) \to \mathscr{S}(\mathbb{R}^2)$.

thm_range_schwartz

THEOREM 1.15 (Range characterization on Schwartz space). *The Radon transform is bijective $\mathscr{S}(\mathbb{R}^2) \to \mathscr{S}_H(\mathbb{R} \times S^1)$.*

The proof of Theorem 1.15 is outlined in the following exercises (the proof may be also be found in [**Hel99**]).

EXERCISE 1.16. Show that R maps $\mathscr{S}(\mathbb{R}^2)$ into $\mathscr{S}_H(\mathbb{R} \times S^1)$.

EXERCISE 1.17. Show that R is injective on $\mathscr{S}(\mathbb{R}^2)$. (It is enough to verify that the Fourier slice theorem holds for Schwartz functions.)

EXERCISE 1.18. Given $\varphi \in \mathscr{S}_H(\mathbb{R} \times S^1)$, show that there exists $f \in \mathscr{S}(\mathbb{R}^2)$ with $Rf = \varphi$ as follows:

(i) By the Fourier slice theorem one should have $\hat{f}(\sigma\omega) = \tilde{\varphi}(\sigma, \omega)$. Motivated by this, define the function F on $\mathbb{R}^2 \setminus \{0\}$ by
$$F(\xi) := \tilde{\varphi}(|\xi|, \xi/|\xi|), \qquad \xi \in \mathbb{R}^2 \setminus \{0\}.$$
(One wants to eventually show that $F = \hat{f}$ for the required function f.) Show that F is C^∞ in $\mathbb{R}^2 \setminus \{0\}$.

(ii) Show that F is Schwartz near infinity, i.e. $\xi^\alpha \partial^\beta F \in L^\infty(\mathbb{R}^2 \setminus B(0,1))$ for $\alpha, \beta \in \mathbb{N}_0^n$.

(iii) Show that F can be extended continuously near 0, by using the fact that $\mu_0\varphi(\omega)$ is homogeneous of degree 0 (i.e. a constant).

(iv) Use the fact that each $\mu_k\varphi$ is homogeneous of degree k to show that F can be extended as a C^∞ function near 0.

(v) Now that F is known to be in $\mathscr{S}(\mathbb{R}^2)$, let f be the inverse Fourier transform of F and show that $Rf = \varphi$.

There is a similar range characterization for the Radon transform when rapid decay is replaced by compact support conditions.

thm_range_compact

THEOREM 1.19 (Range characterization on $C_c^\infty(\mathbb{R}^2)$). *The map R is bijective $C_c^\infty(\mathbb{R}^2) \to \mathscr{D}_H(\mathbb{R} \times S^1)$, where*
$$\mathscr{D}_H(\mathbb{R} \times S^1) = \mathscr{S}_H(\mathbb{R} \times S^1) \cap C_c^\infty(\mathbb{R} \times S^1).$$

In fact, Theorem 1.19 is an immediate consequence of Theorem 1.15 and the following fundamental result:

THEOREM 1.20 (Helgason support theorem). *Let f be a continuous function on $\mathbb{R}^2$ such that $|x|^k f \in L^\infty(\mathbb{R}^2)$ for any $k \geq 0$. If $A > 0$ and if $Rf(s,\omega) = 0$ whenever $|s| > A$ and $\omega \in S^1$, then $f(x) = 0$ whenever $|x| > A$.*

The above result will not be needed later, and we refer to [**Hel99**] for its proof. However, we will prove a closely related result.

m_radon_local_uniqueness

THEOREM 1.21 (Local uniqueness). *Let B be a ball in $\mathbb{R}^2$, and let $f \in C_c(\mathbb{R}^2)$ be supported in $\overline{B}$. Let $x_0 \in \partial B$ and let L_0 be the tangent line to ∂B through x_0. If f integrates to zero along any line L in a small neighborhood of L_0, then $f = 0$ near x_0.*

PROOF. We will prove the result assuming that $f \in C_c^\infty(\mathbb{R}^2)$ and that f is supported in $\overline{B}$ (the general case is given as an exercise). After a translation and rotation we may assume that $x_0 = 0$, $B \subset \{x_n \geq 0\}$, and L_0 is the x-axis. It is convenient to use a slightly different parametrization of lines and to consider the operator
$$Pf(\xi, \eta) = \int_{-\infty}^{\infty} f(t, \xi t + \eta)\, dt, \qquad \xi, \eta \in \mathbb{R}.$$
The assumption implies that $Pf(\xi, \eta) = 0$ for (ξ, η) in some neighborhood V of $(0,0)$. Since $f \in C_c^\infty(\mathbb{R}^2)$, we may take derivatives in ξ so that
$$\partial_\xi Pf(\xi, \eta) = \int_{-\infty}^{\infty} t \partial_{x_2} f(t, \xi t + \eta)\, dt = \partial_\eta P(x_1 f)(\xi, \eta).$$

Since $Pf(\xi,\eta)=0$ for $(\xi,\eta)\in V$, we have $P(x_1 f)(\xi,\eta)=c(\xi)$ in V. But taking η negative and using the support condition for f gives $c(\xi)=0$ for ξ close to 0, i.e. $P(x_1 f)(\xi,\eta)=0$. Repeating this argument gives

$$P(x_1^k f)(\xi,\eta)=0 \text{ near } (0,0) \text{ for any } k\geq 0.$$

In particular, choosing $\xi=0$ gives

$$\int_{-\infty}^{\infty} t^k f(t,\eta)\,dt = 0 \text{ for } \eta \text{ near } 0 \text{ whenever } k\geq 0.$$

This means that all moments of $f(\,\cdot\,,\eta)$ vanish, and it follows that $f(\,\cdot\,,\eta)=0$ for η near 0 (see the following exercise). Thus f vanishes in a neighborhood of 0. $\square$

EXERCISE 1.22. If $f\in C_c(\mathbb{R})$ and $\int_{-\infty}^{\infty} t^k f(t)\,dt=0$ for any $k\geq 0$, show that $f=0$. (You may use the Weierstrass approximation theorem).

EXERCISE 1.23. Prove Theorem 1.21 for functions $f\in C_c(\mathbb{R}^2)$ supported in $\overline{B}$. Hint: consider mollifications $f_\varepsilon(x)=\int_{\mathbb{R}^2} f(x-y)\varphi_\varepsilon(y)\,dy$ where $\varphi_\varepsilon(x)=\varepsilon^{-n}\varphi(x/\varepsilon)$ is a standard mollifier, and show that the Radon transform of f_ε vanishes along certain lines when ε is small.

1.3. The normal operator and singularities

1.3.1. Normal operator. We will now proceed to studying the *normal operator* R^*R of the Radon transform, where the formal adjoint R^* is defined with respect to the natural L^2 inner products on $\mathbb{R}^2$ and $\mathbb{R}\times S^1$. A computation shows that R^* is the *backprojection operator*[1]

$$R^*: C^\infty(\mathbb{R}\times S^1)\to C^\infty(\mathbb{R}^2), \quad R^*h(y)=\int_{S^1} h(y\cdot\omega,\omega)\,d\omega.$$

The following result shows that the normal operator R^*R corresponds to multiplication by $\frac{4\pi}{|\xi|}$ on the Fourier side, and gives an inversion formula for reconstructing f from Rf.

em_normal_operator_radon

THEOREM 1.24 (Normal operator). *One has*

$$R^*R = 4\pi|D|^{-1} = \mathscr{F}^{-1}\left\{\frac{4\pi}{|\xi|}\mathscr{F}(\,\cdot\,)\right\},$$

and f can be recovered from Rf by the formula

$$f=\frac{1}{4\pi}|D|R^*Rf.$$

REMARK 1.25. Above we have written, for $\alpha\in\mathbb{R}$,

$$|D|^\alpha f := \mathscr{F}^{-1}\{|\xi|^\alpha \hat{f}(\xi)\}.$$

The notation $(-\Delta)^{\alpha/2}=|D|^\alpha$ is also used.

[1]The formula for R^* is obtained as follows: if $f\in C_c^\infty(\mathbb{R}^2)$, $h\in C^\infty(\mathbb{R}\times S^1)$ one has

$$\begin{aligned}(Rf,h)_{L^2(\mathbb{R}\times S^1)} &= \int_{-\infty}^{\infty}\int_{S^1} Rf(s,\omega)\overline{h(s,\omega)}\,d\omega\,ds \\ &= \int_{-\infty}^{\infty}\int_{S^1}\int_{-\infty}^{\infty} f(s\omega+t\omega^\perp)\overline{h(s,\omega)}\,dt\,d\omega\,ds \\ &= \int_{\mathbb{R}^2} f(y)\left(\int_{S^1}\overline{h(y\cdot\omega,\omega)}\,d\omega\right)\,dy.\end{aligned}$$

PROOF. The proof is based on computing $(Rf, Rg)_{L^2(\mathbb{R}\times S^1)}$ using the Parseval identity, Fourier slice theorem, symmetry and polar coordinates:

$$\begin{aligned}
(R^*Rf, g)_{L^2(\mathbb{R}^2)} &= (Rf, Rg)_{L^2(\mathbb{R}\times S^1)} \\
&= \int_{S^1} \left[\int_{-\infty}^{\infty} (Rf)(s,\omega)\overline{(Rg)(s,\omega)}\, ds \right] d\omega \\
&= \frac{1}{2\pi} \int_{S^1} \left[\int_{-\infty}^{\infty} (Rf)\tilde{\ }(\sigma,\omega)\overline{(Rg)\tilde{\ }(\sigma,\omega)} \right] d\sigma\, d\omega \\
&= \frac{1}{2\pi} \int_{S^1} \left[\int_{-\infty}^{\infty} \hat{f}(\sigma\omega)\overline{\hat{g}(\sigma\omega)} \right] d\sigma\, d\omega \\
&= \frac{2}{2\pi} \int_{S^1} \left[\int_{0}^{\infty} \hat{f}(\sigma\omega)\overline{\hat{g}(\sigma\omega)} \right] d\sigma\, d\omega \\
&= \frac{2}{2\pi} \int_{\mathbb{R}^2} \frac{1}{|\xi|} \hat{f}(\xi)\overline{\hat{g}(\xi)}\, d\xi \\
&= (4\pi \mathscr{F}^{-1}\left\{ \frac{1}{|\xi|}\hat{f}(\xi) \right\}, g). \qquad \square
\end{aligned}$$

The same argument, based on computing $(|D_s|^{1/2}Rf, |D_s|^{1/2}Rg)_{L^2(\mathbb{R}\times S^1)}$ instead of $(Rf, Rg)_{L^2(\mathbb{R}\times S^1)}$, leads to the famous *filtered backprojection* (FBP) inversion formula:

THEOREM 1.26 (Filtered backprojection). *If $f \in C_c^\infty(\mathbb{R}^2)$, then*

$$f = \frac{1}{4\pi} R^*|D_s|Rf$$

where $|D_s|Rf = \mathscr{F}^{-1}\{|\sigma|(Rf)\tilde{\ }\}$.

The FBP formula is efficient to implement and gives accurate reconstructions when one has complete X-ray data and relatively small noise, and hence FBP (together with its variants) has been commonly used in X-ray CT scanners.

1.3.2. Recovery of singularities. We will later study X-ray transforms in more general geometries. In such cases exact reconstruction formulas such as FBP are often not available. However, it will be important that some structural properties of the normal operator may still be valid. In particular, Theorem 1.24 implies that the normal operator is an *elliptic pseudodifferential operator* of order -1 in $\mathbb{R}^2$. The theory of pseudodifferential operators (i.e. *microlocal analysis*) then immediately yields that the *singularities* of f are uniquely determined from the knowledge of Rf. For the benefit of those readers who are not familiar with these notions, we will give a short presentation partly without proofs.

For a reference to distribution theory see [**Hör85**, vol. I], and for wave front sets see [**Hör85**, Chapter 8]. Sobolev wave front sets are considered in [**Hör85**, Section 18.1].

We first define compactly supported distributions.

DEFINITION 1.27. Define the set of *compactly supported distributions* in $\mathbb{R}^n$ as

$$\mathscr{E}'(\mathbb{R}^n) = \bigcup_{s\in\mathbb{R}} H^s_{\mathrm{comp}}(\mathbb{R}^n).$$

This definition coincides with the more standard ones defining $\mathscr{E}'(\mathbb{R}^n)$ as the dual of $C^\infty(\mathbb{R}^n)$ with a suitable topology, or as the compactly supported distributions in $\mathscr{D}'(\mathbb{R}^n)$. By Remark 1.12, the Radon transform R is well defined in $\mathscr{E}'(\mathbb{R}^2)$. We also recall that the Fourier transform maps $\mathscr{E}'(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$.

We next discuss the singular support of u, which consists of those points x_0 such that u is not a smooth function in any neighborhood of x_0. We also consider the Sobolev singular support, which also measures the "strength" of the singularity (in the L^2 Sobolev scale).

Definition 1.28 (Singular support). We say that a function or distribution u in $\mathbb{R}^n$ is C^∞ *(resp. H^α) near* x_0 if there is $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\varphi = 1$ near x_0 such that φu is in $C^\infty(\mathbb{R}^n)$ (resp. in $H^\alpha(\mathbb{R}^n)$). We define

$$\operatorname{sing\,supp}(u) = \mathbb{R}^n \setminus \{x_0 \in \mathbb{R}^n\,;\, u \text{ is } C^\infty \text{ near } x_0\},$$
$$\operatorname{sing\,supp}^\alpha(u) = \mathbb{R}^n \setminus \{x_0 \in \mathbb{R}^n\,;\, u \text{ is } H^\alpha \text{ near } x_0\}.$$

ample_piecewise_constant

Example 1.29. Let $D_1, \ldots, D_N$ be bounded domains with C^∞ boundary in $\mathbb{R}^n$ so that $\overline{D}_j \cap \overline{D}_k = \emptyset$ for $j \neq k$, and define

$$u = \sum_{j=1}^N c_j \chi_{D_j}$$

where $c_j \neq 0$ are constants, and χ_{D_j} is the characteristic function of D_j. Then

$$\operatorname{sing\,supp}^\alpha(u) = \emptyset \text{ for } \alpha < 1/2$$

since $u \in H^\alpha$ for $\alpha < 1/2$, but

$$\operatorname{sing\,supp}^\alpha(u) = \bigcup_{j=1}^N \partial D_j \text{ for } \alpha \geq 1/2$$

since u is not $H^{1/2}$ near any boundary point. Thus in this case the singularities of u are exactly at the points where u has a jump discontinuity, and their strength is precisely $H^{1/2}$. Knowing the singularities of u can already be useful in applications. For instance, if u represents some internal medium properties in medical imaging, the singularities of u could determine the location of interfaces between different tissues. On the other hand, if u represents an image, then the singularities in some sense determine the "sharp features" of the image.

Next we discuss the *wave front set* which is a more refined notion of a singularity. For example, if $f = \chi_D$ is the characteristic function of a bounded strictly convex C^∞ domain D and if $x_0 \in \partial D$, one could think that f is in some sense smooth in tangential directions at x_0 (since f restricted to a tangent hyperplane is identically zero, except possibly at x_0), but that f is not smooth in normal directions at x_0 since in these directions there is a jump. The wave front set is a subset of $T^*\mathbb{R}^n \setminus 0$, the cotangent space with the zero section removed:

$$T^*\mathbb{R}^n \setminus 0 := \{(x,\xi)\,;\, x, \xi \in \mathbb{R}^n, \xi \neq 0\}.$$

Definition 1.30 (Wave front set). Let u be a distribution in $\mathbb{R}^n$. We say that u is (microlocally) C^∞ *(resp. H^α) near* (x_0, ξ_0) if there exist $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\varphi = 1$ near x_0 and $\psi \in C^\infty(\mathbb{R}^n \setminus \{0\})$ so that $\psi = 1$ near ξ_0 and ψ is homogeneous of degree 0, such that

$$\text{for any } N \text{ there is } C_N > 0 \text{ so that } \psi(\xi)(\varphi u)\hat{\ }(\xi) \leq C_N(1+|\xi|)^{-N}$$

(resp. $\mathscr{F}^{-1}\{\psi(\xi)(\varphi u)\hat{}(\xi)\} \in H^\alpha(\mathbb{R}^n)$). The *wave front set* $WF(u)$ (resp. H^α *wave front set* $WF^\alpha(u)$) consists of those points (x_0, ξ_0) where u is not microlocally C^∞ (resp. H^α).

EXAMPLE 1.31. The wave front set of the function u in Example 1.29 is

$$WF(u) = \bigcup_{j=1}^{N} N^*(D_j)$$

where $N^*(D_j)$ is the conormal bundle of D_j,

$$N^*(D_j) := \{(x, \xi)\,;\, x \in \partial D_j \text{ and } \xi \text{ is normal to } \partial D_j \text{ at } x\}.$$

The wave front set describes singularities more precisely than the singular support, since one always has

ave_front_set_projection

$$\pi(WF(u)) = \operatorname{sing\,supp}(u) \tag{1.5}$$

where $\pi : (x, \xi) \mapsto x$ is the projection to x-space.

We now go back to the Radon transform. If one is mainly interested in the singularities of the image function f, then instead of using FBP to reconstruct the whole function f from Rf it is possible to use the even simpler *backprojection method*: just apply the backprojection operator R^* to the data Rf. Since R^*R is an elliptic pseudodifferential operator, the singularities are completely recovered:

l_operator_singularities

THEOREM 1.32. *If $f \in \mathscr{E}'(\mathbb{R}^2)$, then*

$$\operatorname{sing\,supp}(R^*Rf) = \operatorname{sing\,supp}(f),$$
$$\mathrm{WF}(R^*Rf) = \mathrm{WF}(f).$$

Moreover, for any $\alpha \in \mathbb{R}$ one has

$$\operatorname{sing\,supp}^{\alpha+1}(R^*Rf) = \operatorname{sing\,supp}^\alpha(f),$$
$$\mathrm{WF}^{\alpha+1}(R^*Rf) = \mathrm{WF}^\alpha(f).$$

REMARK 1.33. Since R^*R is a pseudodifferential operator of order -1, hence smoothing of order 1, one expects that R^*Rf gives a slightly blurred version of f where the main singularities are be visible. The previous theorem makes this precise and shows the singularities in R^*Rf are one Sobolev degree smoother than those in f.

1.3.3. Pseudodifferential operators. For the proof of Theorem 1.32 we recall quickly some relevant definitions, based on the following example. We refer to [**Hör85**, Chapter 18] for a detailed account on pseudodifferential operators.

EXAMPLE 1.34 (Differential operators). Let $A = a(x, D)$ be a differential operator of order m, acting on functions $f \in \mathscr{S}(\mathbb{R}^n)$ by

$$Af(x) = a(x, D)f(x) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha f(x)$$

where $a_\alpha \in C^\infty(\mathbb{R}^n)$. Here $D = \frac{1}{i}\nabla$, so that $D^\alpha = (\frac{1}{i}\partial_{x_1})^{\alpha_1} \cdots (\frac{1}{i}\partial_{x_n})^{\alpha_n}$.

If each a_α is a constant, i.e. $a_\alpha(x) = a_\alpha$ and $A = a(D) = \sum_{|\alpha| \le m} a_\alpha D^\alpha$, we may use (1.1) to compute the Fourier transform of Af:

$$(Af)\hat{}(\xi) = \sum_{|\alpha| \le m} a_\alpha \xi^\alpha \hat{f}(\xi).$$

The Fourier inversion formula gives that

a_constant_symbol

$$Af(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(\xi)\hat{f}(\xi)\, d\xi \tag{1.6}$$

where $a(\xi) = \sum_{|\alpha|\leq m} a_\alpha \xi^\alpha$ is the *symbol* of $A(D)$.

More generally, if each a_α is a C^∞ function with $\partial^\beta a_\alpha \in L^\infty(\mathbb{R}^n)$ for all $\beta \in \mathbb{N}_0^n$, we may use the Fourier inversion formula to compute

rential_operator_formula

$$\begin{aligned} Af(x) &= A\left[\mathscr{F}^{-1}\{\hat{f}(\xi)\}\right] \\ &= \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha \left[(2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi\right] \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \left[\sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha\right] \hat{f}(\xi)\, d\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)\hat{f}(\xi)\, d\xi. \qquad (1.7) \end{aligned}$$

where

a_differential_symbol

$$a(x,\xi) := \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha \tag{1.8}$$

is the (full) *symbol* of $A = a(x, D)$.

The above example shows that any differential operator of order m has the Fourier representation (1.7), where the symbol $a(x,\xi)$ in (1.8) is a polynomial of degree m in ξ. The following definition generalizes this setup.

Definition 1.35 (Pseudodifferential operators). For any $m \in \mathbb{R}$, denote by S^m (the set of *symbols* of order m) the set of all $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ so that for any multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ there is $C_{\alpha\beta} > 0$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta}(1+|\xi|)^{m-|\beta|}, \qquad x, \xi \in \mathbb{R}^n.$$

For any $a \in S^m$, define an operator $A = \mathrm{Op}(a)$ acting on functions $f \in \mathscr{S}(\mathbb{R}^n)$ by

$$Af(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)\hat{f}(\xi)\, d\xi, \qquad x \in \mathbb{R}^n.$$

Let $\Psi^m = \{\mathrm{Op}(a)\,;\, a \in S^m\}$ be the set of *pseudodifferential operators* of order m. We say that an operator $\mathrm{Op}(a)$ with $a \in S^m$ is *elliptic* if there are $c, R > 0$ such that

$$a(x,\xi) \geq c(1+|\xi|)^m, \qquad x \in \mathbb{R}^n, |\xi| \geq R.$$

It is a basic fact that any $A \in \Psi^m$ is a continuous map $\mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$, when $\mathscr{S}(\mathbb{R}^n)$ is given the natural topology induced by the seminorms $f \mapsto \|x^\alpha \partial^\beta f\|_{L^\infty}$ where $\alpha, \beta \in \mathbb{N}_0^n$. By duality, any $A \in \Psi^m$ gives a continuous map $\mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$, where $\mathscr{S}'(\mathbb{R}^n)$ is the weak* dual space of $\mathscr{S}(\mathbb{R}^n)$ (the space of *tempered distributions*). In particular, any $A \in \Psi^m$ is well defined on $\mathscr{E}'(\mathbb{R}^n)$.

It is an important fact that applying a pseudodifferential operator to a function or distribution never creates new singularities:

psdo_microlocal_property

THEOREM 1.36 (Pseudolocal/microlocal property). *Any $A \in \Psi^m$ has the pseudolocal property*

$$\operatorname{sing\,supp}(Au) \subset \operatorname{sing\,supp}(u),$$
$$\operatorname{sing\,supp}^{\alpha-m}(Au) \subset \operatorname{sing\,supp}^{\alpha}(u)$$

and the microlocal property

$$WF(Au) \subset WF(u),$$
$$WF^{\alpha-m}(Au) \subset WF^{\alpha}(u).$$

PROOF. We sketch the proof for the inclusion $\operatorname{sing\,supp}(Au) \subset \operatorname{sing\,supp}(u)$. For more details see [**Hör85**, Chapter 18]. Suppose that $x_0 \notin \operatorname{sing\,supp}(u)$, so we need to show that $x_0 \notin \operatorname{sing\,supp}(Au)$. By definition, there is $\psi \in C_c^\infty(\mathbb{R}^n)$ with $\psi = 1$ near x_0 so that $\psi u \in C_c^\infty(\mathbb{R}^n)$. We write

$$Au = A(\psi u) + A((1-\psi)u).$$

Since A maps Schwartz space to itself, one always has $A(\psi u) \in C^\infty$. Thus it is enough to show that $A((1-\psi)u)$ is C^∞ near x_0. To do this, choose $\varphi \in C_c^\infty(\mathbb{R}^n)$ so that $\varphi = 1$ near x_0 and $\operatorname{supp}(\varphi)$ is contained in the set where $\psi = 1$. Define

$$Bu = \varphi A((1-\psi)u).$$

It is enough to show that B is a smoothing operator, i.e. maps $\mathscr{E}'(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$.

We compute the integral kernel of B:

$$\begin{aligned} Bu(x) &= (2\pi)^{-n}\varphi(x)\int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)((1-\psi)u)\hat{\ }(\xi)\,d\xi \\ &= \int_{\mathbb{R}^n} K(x,y)u(y)\,dy \end{aligned}$$

where

$$K(x,y) = (2\pi)^{-n}\int_{\mathbb{R}^n} \varphi(x)e^{i(x-y)\cdot\xi}a(x,\xi)(1-\psi(y))\,d\xi.$$

Recall that a satisfies $|a(x,\xi)| \leq C(1+|\xi|)^m$. Thus if $m < -n$, the integral is absolutely convergent and one gets that $K \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. In the general case the main point is that $|x-y| \geq c > 0$ on the support of $K(x,y)$, due to the support conditions on φ and ψ. It follows that we may write, for any $N \geq 0$,

$$e^{i(x-y)\cdot\xi} = |x-y|^{-2N}(-\Delta_\xi)^N(e^{i(x-y)\cdot\xi})$$

and integrate by parts in ξ to obtain that

$$K(x,y) = (2\pi)^{-n}|x-y|^{-2N}\int_{\mathbb{R}^n} \varphi(x)e^{i(x-y)\cdot\xi}((-\Delta_\xi)^N a(x,\xi))(1-\psi(y))\,d\xi.$$

If N is chosen large enough (it is enough that $m - 2N < -n-1$), one has $|(-\Delta_\xi)^N a(x,\xi)| \leq C(1+|\xi|)^{-n-1}$. Thus the integral defining $K(x,y)$ is absolutely convergent, and in particular $K \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Taking derivatives gives that $\partial_x^\alpha \partial_y^\beta K$ is also bounded for any α and β, showing that $K \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. It follows from the next exercise that the operator B maps into $C^\infty(\mathbb{R}^n)$. □

EXERCISE 1.37. Show that an operator $Bu(x) = \int_{\mathbb{R}^n} K(x,y)u(y)\,dy$, where $K \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, induces a well defined map from $\mathscr{E}'(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$.

We now go back to the normal operator R^*R and the proof of Theorem 1.32. Theorem 1.24 states that R^*R has symbol $\frac{4\pi}{|\xi|}$, which would be in the symbol class S^{-1} except that the symbol is not smooth when $\xi = 0$. This can be dealt with in the following standard way.

thm_normal_operator_psdo

THEOREM 1.38. *The normal operator satisfies*

$$R^*R = Q + S$$

where $Q \in \Psi^{-1}$ is elliptic, and S is a smoothing operator *which maps $\mathscr{E}'(\mathbb{R}^2)$ to $C^\infty(\mathbb{R}^2)$.*

PROOF. Let $\psi \in C_c^\infty(\mathbb{R}^2)$ satisfy $\psi(\xi) = 1$ for $|\xi| \leq 1/2$ and $\psi(\xi) = 0$ for $|\xi| \geq 1$. Write

$$Qf = 4\pi\mathscr{F}^{-1}\left\{\frac{1-\psi(\xi)}{|\xi|}\hat{f}\right\}, \qquad Sf = 4\pi\mathscr{F}^{-1}\left\{\frac{\psi(\xi)}{|\xi|}\hat{f}\right\}.$$

Then Q is a pseudodifferential operator in Ψ^{-1} with symbol $q(x,\xi) = \frac{1-\psi(\xi)}{|\xi|}$, hence Q is elliptic. The operator S has the required property by Lemma 1.39 below since $\frac{\psi(\xi)}{|\xi|}$ is in $L^1_{\mathrm{comp}}(\mathbb{R}^2)$ (the function $\xi \mapsto \frac{1}{|\xi|}$ is locally integrable in $\mathbb{R}^2$). □

mma_smoothing_elementary

LEMMA 1.39. *If $m \in L^1_{\mathrm{comp}}(\mathbb{R}^n)$, then the operator*

$$S : f \mapsto \mathscr{F}^{-1}\{m(\xi)\hat{f}\}$$

is smoothing in the sense that it maps $\mathscr{E}'(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$.

PROOF. If $f \in \mathscr{E}'(\mathbb{R}^n)$ then $\hat{f} \in C^\infty(\mathbb{R}^n)$. Consequently $F(\xi) := m(\xi)\hat{f}(\xi)$ is in $L^1(\mathbb{R}^n)$ by the assumption on m. Moreover, F is compactly supported, which implies that $Sf = \mathscr{F}^{-1}F$ is C^∞. □

We can finally prove the recovery of singularities result.

PROOF OF THEOREM 1.32. We prove the claim for the singular support (the other parts are analogous). By Theorem 1.38, one has

$$R^*Rf = Qf + C^\infty.$$

Hence it is enough to show that $\operatorname{sing\,supp}(Qf) = \operatorname{sing\,supp}(f)$. It follows from Theorem 1.36 that $\operatorname{sing\,supp}(Qf) \subset \operatorname{sing\,supp}(f)$. The converse inclusion is a standard argument, which follows from the construction of an approximate inverse, or *parametrix*, for the elliptic pseudodifferential operator Q. Define

$$Ef = \mathscr{F}^{-1}\left\{(1-\chi(\xi))|\xi|\hat{f}\right\}$$

where $\chi \in C_c^\infty(\mathbb{R}^2)$ satisfies $\chi(\xi) = 1$ for $|\xi| \leq 2$. Note that $E \in \Psi^1$. Since $\psi(\xi) = 0$ for $|\xi| \geq 1$, it follows that

$$EQf = \mathscr{F}^{-1}\left\{(1-\chi(\xi))|\xi|\frac{1-\psi(\xi)}{|\xi|}\hat{f}\right\} = f - \mathscr{F}^{-1}\left\{\chi(\xi)\hat{f}\right\}.$$

Thus $EQf = f + S_1 f$, where S_1 is smoothing and maps $\mathscr{E}'(\mathbb{R}^2)$ to $C^\infty(\mathbb{R}^2)$ by Lemma 1.39. Hence Theorem 1.36 applied to E gives that

$$\operatorname{sing\,supp}(f) = \operatorname{sing\,supp}(EQf) \subset \operatorname{sing\,supp}(Qf).$$

□

1.3.4. Visible singularities. We conclude with a short discussion on more precise recovery of singularities results from limited X-ray data. This follows the microlocal approach to Radon transforms introduced in [**Gui75**]. For more detailed treatments we refer to the survey articles [**Qui06**], [**KQ15**].

There are various imaging situations where complete X-ray data (i.e. the function $Rf(s, \omega)$ for all s and ω) is not available. This is the case for limited angle tomography (e.g. in luggage scanners at airports, or dental applications), region of interest tomography, or exterior data tomography. In such cases explicit inversion formulas such as FBP are usually not available, but the analysis of singularities still provides a powerful paradigm for predicting which sharp features can be recovered stably from the measurements.

We will try to explain this paradigm a little bit more, starting with an example:

EXAMPLE 1.40. Let f be the characteristic function of the unit disc $\mathbb{D}$, i.e. $f(x) = 1$ if $|x| \leq 1$ and $f(x) = 0$ for $|x| > 1$. Then f is singular precisely on the unit circle (in normal directions). We have

$$Rf(s, \omega) = \begin{cases} 2\sqrt{1 - s^2}, & |s| \leq 1, \\ 0, & |s| > 1. \end{cases}$$

Thus Rf is singular precisely at those points (s, ω) with $|s| = 1$, which correspond to those lines that are tangent to the unit circle.

There is a similar relation between the singularities of f and Rf in general, and this is explained by microlocal analysis and the interpretation of R as a Fourier integral operator (see [**Hör85**, Chapter 25]):

THEOREM 1.41. *The operator R is an elliptic Fourier integral operator of order $-1/2$. There is a precise relationship between the singularities of f and singularities of Rf.*

We will not spell out the precise relationship here, but only give some consequences. It will be useful to think of the Radon transform as defined on the set of (non-oriented) lines in $\mathbb{R}^2$. If $\mathcal{A}$ is an open subset of lines in $\mathbb{R}^2$, we consider the Radon transform $Rf|_{\mathcal{A}}$ restricted to lines in $\mathcal{A}$. Recovering f (or some properties of f) from $Rf|_{\mathcal{A}}$ is a *limited data* tomography problem. Examples:

- If $\mathcal{A} = \{\text{lines not meeting } \overline{\mathbb{D}}\}$, then $Rf|_{\mathcal{A}}$ is called *exterior data*.
- If $0 < a < \pi/2$ and $\mathcal{A} = \{\text{lines whose angle with } x\text{-axis is} < a\}$ then $Rf|_{\mathcal{A}}$ is called *limited angle data*.

It is known that any $f \in C_c^\infty(\mathbb{R}^2 \setminus \overline{D})$ is uniquely determined by exterior data (Helgason support theorem), and any $f \in C_c^\infty(\mathbb{R}^2)$ is uniquely determined by limited angle data (Fourier slice and Paley-Wiener theorems). However, both inverse problems are very unstable (inversion is not Lipschitz continuous in any Sobolev norms, but one has conditional logarithmic stability).

DEFINITION 1.42. A singularity at (x_0, ξ_0) is called *visible from $\mathcal{A}$* if the line through x_0 in direction $\xi_0^\perp$ is in $\mathcal{A}$.

One has the following dichotomy:

- If (x_0, ξ_0) is visible from $\mathcal{A}$, then from the singularities of $Rf|_{\mathcal{A}}$ one can determine for any α whether or not $(x_0, \xi_0) \in WF^\alpha(f)$. If $Rf|_{\mathcal{A}}$ uniquely determines f, one expects the reconstruction of visible singularities to be stable.

- If (x_0, ξ_0) is not visible from $\mathcal{A}$, then this singularity is smoothed out in the measurement $Rf|_{\mathcal{A}}$. Even if $Rf|_{\mathcal{A}}$ would determine f uniquely, the inversion is not Lipschitz stable in any Sobolev norms.

CHAPTER 2

Radial sound speeds

In this chapter we will discuss geometric inverse problems in a disk with radial sound speed. The fact that the sound speed is radial is a strong symmetry condition, which allows one to determine the geodesics and solve related inverse problems quite explicitly.

2.1. Geodesics

The fact that the geodesics of a radial sound speed can be explicitly determined is related to the existence of multiple conserved quantities in the Hamiltonian approach to geodesics. We first recall this approach.

2.1.1. Geodesics as a Hamilton flow. Let $M \subset \mathbb{R}^n$, let x be standard Cartesian coordinates, and let $g = (g_{jk}(x))_{j,k=1}^n$ be a Riemannian metric on M. A curve $x(t)$ is a geodesic iff it satisfies the geodesic equations

geodesic_equations_first

$$\ddot{x}^l(t) + \Gamma_{jk}^l(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0, \tag{2.1}$$

where Γ_{jk}^l are the Christoffel symbols given by

$$\Gamma_{jk}^l = \frac{1}{2}g^{lm}(\partial_j g_{km} + \partial_k g_{jm} - \partial_m g_{jk}).$$

Writing

$$\xi_j(t) := g_{jk}(x(t))\dot{x}^k(t), \qquad f(x,\xi) := \frac{1}{2}g^{jk}(x)\xi_j\xi_k,$$

a short computation shows that the geodesic equations are equivalent with the Hamilton equations

eodesic_equations_second

$$\begin{cases} \dot{x}(t) &= \nabla_\xi f(x(t),\xi(t)), \\ \dot{\xi}(t) &= -\nabla_x f(x(t),\xi(t)). \end{cases} \tag{2.2}$$

Here $f(x,\xi)$ (kinetic energy) is called the *Hamilton function*, and it is defined on the cotangent space $T^*M = \{(x,\xi)\,;\, x \in M,\, \xi \in \mathbb{R}^n\} = M \times \mathbb{R}^n$.

Exercise 2.1. Show that (2.1) is equivalent with (2.2).

Writing $\gamma(t) = (x(t),\xi(t))$ and using the Hamilton vector field H_f on T^*M, defined by

$$H_f := \nabla_\xi f \cdot \nabla_x - \nabla_x f \cdot \nabla_\xi = (\nabla_\xi f, -\nabla_x f)$$

we may write the Hamilton equations as

$$\dot{\gamma}(t) = H_f(\gamma(t)).$$

Definition 2.2. A function $u = u(x,\xi)$ is a *conserved quantity* if it is constant along the Hamilton flow, i.e. $t \mapsto u(x(t),\xi(t))$ is constant for any curve $(x(t),\xi(t))$ solving (2.2).

Now (2.2) implies that

$$u \text{ is conserved} \quad \iff \quad \frac{d}{dt}u(x(t),\xi(t)) = 0 \quad \iff \quad H_f u(x(t),\xi(t)) = 0.$$

Since

$$H_f f = (\nabla_\xi f, -\nabla_x f)\cdot(\nabla_x f, \nabla_\xi f) = 0,$$

the Hamilton function f (kinetic energy) is always conserved.

Let now $M \subset \mathbb{R}^2$, and consider a metric of the form

$$g_{jk}(x) = c(x)\delta_{jk}$$

where $c \in C^\infty(M)$ is positive. Then $f(x,\xi) = \frac{1}{2}c(x)|\xi|^2$ and

$$H_f = c(x)\xi\cdot\nabla_x - \frac{1}{2}|\xi|^2\nabla_x c(x)\cdot\nabla_\xi.$$

Define the angular momentum

$$L(x,\xi) = \xi\cdot x^\perp, \qquad x^\perp = (-x_2, x_1).$$

When is L conserved? We compute

$$H_f L = c(x)\xi\cdot(-\xi^\perp) - \frac{1}{2}|\xi|^2\nabla_x c(x)\cdot x^\perp = -\frac{1}{2}|\xi|^2\nabla_x c(x)\cdot x^\perp.$$

Thus $H_f L = 0$ iff $\nabla c(x)\cdot x^\perp = 0$, which is equivalent with the fact that c is radial:

lemma_angular_momentum

LEMMA 2.3. *The angular momentum L is conserved iff*

$$c = c(r), \qquad r = |x|.$$

If $M \subset \mathbb{R}^2$ and $c(x)$ is radial, then the Hamilton flow has two independent conserved quantities (the kinetic energy f and angular momentum L). One says that the flow is *completely integrable*, which implies that the geodesic equations can be solved quite explicitly by quadrature using f and h. See e.g. [**Tay11**, Chapter 1] for more details on these facts.

2.1.2. Geodesics of a radial sound speed. We will now begin to analyze geodesics in this setting. Let $M = \overline{\mathbb{D}}\setminus\{0\}$ where $\mathbb{D}$ is the unit disk in $\mathbb{R}^2$. Assume that

$$g_{jk}(x) = c(r)\delta_{jk}, \qquad r = |x|,$$

where $c \in C^\infty((0,1])$. Note that the origin is a special point and $g_{jk}(x)$ is not necessarily smooth there, hence we will consider geodesics only away from the origin.

We write

$$r(t) = |x(t)|, \qquad \hat{x} = \frac{x}{|x|}.$$

The Hamilton equations (2.2) become

geodesic_equations_third

$$\left\{\begin{array}{ll} \dot{x}(t) & = c(r(t))\xi(t), \\ \dot{\xi}(t) & = -\frac{1}{2}|\xi(t)|^2 c'(r(t))\hat{x}(t). \end{array}\right. \tag{2.3}$$

Decompose $\xi = (\xi\cdot\hat{x})\hat{x} + (\xi\cdot\hat{x}^\perp)\hat{x}^\perp$. Computing the derivative of $r(t)$ gives

$$\dot{r} = \frac{x\cdot\dot{x}}{|x|} = c(\xi\cdot\hat{x}) = \pm c\sqrt{|\xi|^2 - (\xi\cdot\hat{x}^\perp)^2}, \qquad \pm\xi\cdot\hat{x} \geq 0.$$

Consider geodesics starting on $\partial\mathbb{D}$, i.e. $r(0) = 1$, and let also $|\xi(0)| = 1$. Now

$$f \text{ conserved} \implies c(r)|\xi|^2 = c(1) \implies |\xi|^2 = \frac{c(1)}{c(r)},$$
$$L \text{ conserved} \implies \xi \cdot x^\perp = \xi(0) \cdot x(0)^\perp.$$

We write

$$\xi(0) = -\sqrt{1-p^2}x(0) + px(0)^\perp, \qquad 0 < p < 1.$$

Note that $\xi(0)$ points inward. Noting that $\sqrt{|\xi|^2 - (\xi \cdot \hat{x}^\perp)} = \sqrt{\frac{c(1)}{c(r)} - \frac{p}{r}}$, we see that $r(t)$ solves the equation

$$\dot{r} = \pm\sqrt{c(r)(c(1) - pc(r))}. \tag{2.4}$$

This is an autonomous ODE for $r(t)$ (all other dependence on t has been eliminated).

CHAPTER 3

Geometric Preliminaries

chapter:geoprelim

3.1. Non-trapping and Strict Convexity

Let (M, g) be a compact, connected and oriented Riemannian manifold with boundary ∂M and dimension $n \geq 2$.

Geodesics travel at constant speed, so we fix the speed to be one. We pack positions and velocities together in what we call the *unit sphere bundle* SM. This consists of pairs (x, v), where $x \in M$ and $v \in T_xM$ with norm $|v|_g = 1$, where g is the inner product in the tangent space at x (i.e. the Riemannian metric). Given $(x, v) \in SM$, let $\gamma_{x,v}$ denote the unique geodesic determined by (x, v) and let $\tau(x, v) \in [0, \infty]$ denote the first forward time where the geodesic $\gamma_{x,v}$ hits ∂M.

DEFINITION 3.1. We say that (M, g) is *non-trapping* if $\tau(x, v) < \infty$ for all $(x, v) \in SM$. Equivalently, there are no geodesics in M with infinite length.

Unit tangent vectors at the boundary of M constitute the boundary ∂SM of SM and will play a special role. Specifically

$$\partial SM := \{(x, v) \in SM : \ x \in \partial M\}.$$

We will need to distinguish those tangent vectors pointing inside ("influx boundary") and those pointing outside ("outflux boundary"), so we define two subsets of ∂SM

$$\partial_\pm SM := \{(x, v) \in \partial SM : \pm\langle v, \nu(x)\rangle_g \geq 0\}$$

where ν denotes the **inward** unit normal to the boundary.

G: convention, **inward** unit normal. Makes volume form $d\mu$ and other things a bit more natural

Let us denote

$$\partial_0 SM := \partial_+ SM \cap \partial_- SM.$$

This coincides with $S\partial M$.

Recall that the second fundamental form of ∂M is given by

$$\Pi_x(u, v) := -\langle \nabla_u \nu, v\rangle_g,$$

where $x \in \partial M$ and $u, v \in T_x\partial M$ (∇ is the Levi-Civita connection of g).

DEFINITION 3.2. We shall say that ∂M is *strictly convex* if Π_x is positive definite for all $x \in \partial M$.

The combination of non-trapping with strict convexity of the boundary will produce several desirable properties. We will begin discussing the regularity of the fundamental function τ. Note that by definition $\tau|_{\partial_- SM} = 0$.

Let $\rho \in C^\infty(M)$ be a function that coincides with $M \ni x \mapsto d(x, \partial M)$ in a neighbourhood of ∂M and such that $\rho \geq 0$ and $\partial M = \rho^{-1}(0)$.

EXERCISE 3.3. Show that such a function exists. Moreover, show that $\nabla\rho(x) = \nu(x)$ for all $x \in \partial M$.

3.1.1. Extensions. It turns out that it is quite convenient to consider (M, g) isometrically embedded into a complete connected boundaryless manifold (N, g) of the same dimension as M. There are two extensions which are particularly helpful.

(1) (N, g) is closed; this extension always exists for any compact (M, g).
(2) (N, g) is complete and geodesics leaving M never return to M. Moreover M is a deformation retract of N. This extension exists if ∂M is strictly convex. We shall refer this extension as the *no return extension*.

EXERCISE 3.4. Prove that these two extensions exist.

G: I learned about the no return extension from Jan Bohr, who also wrote a proof of its existence in his first year research project.

Given a complete extension (N, g) we may extend ρ smoothly to N and define a function $h : SN \times \mathbb{R} \to \mathbb{R}$ by setting $h(x, v, t) := \rho(\gamma_{x,v}(t))$. Then

$$\frac{\partial h}{\partial t}(x, v, t) = d\rho(\dot\gamma_{x,v}(t)) = \langle \nabla\rho(\gamma_{x,v}(t)), \dot\gamma_{x,v}(t))\rangle.$$

Set $y := \gamma_{x,v}(\tau(x, v)) \in \partial M$ for $(x, v) \in SM$. Since $\nabla\rho$ agrees with ν on ∂M, we see that

$$\left.\frac{\partial h}{\partial t}\right|_{t=\tau(x,v)} h(x, v, t) \neq 0$$

as long as $\gamma_{x,v}$ is transversal to ∂M at y. Since $h(x, v, \tau(x, v)) = 0$ and h is smooth, the implicit function theorem ensures that τ is smooth for those $(x, v) \in SM$ such that $\gamma_{x,v}$ is transversal to ∂M at y (here it is useful to use an extension (N, g) so that geodesics do not return to M after leaving, why?). Hence, by strict convexity, τ is smooth on $SM \setminus \partial_0 SM$.

EXERCISE 3.5. Show that τ is not smooth at the glancing region $\partial_0 SM$.

LEMMA 3.6. *Let (M, g) be a non-trapping manifold with strictly convex boundary and let*

$$\tilde\tau(x, v) := \begin{cases} \tau(x, v), & (x, v) \in \partial_+ SM, \\ -\tau(x, -v), & (x, v) \in \partial_- SM. \end{cases}$$

lemma:tautilde *Then $\tilde\tau \in C^\infty(\partial SM)$; in particular $\tau : \partial_+ SM \to \mathbb{R}$ is smooth.*

PROOF. As before we let $h(x, v, t) := \rho(\gamma_{x,v}(t))$ for $(x, v) \in \partial SM$ and $t \in \mathbb{R}$. Note

- $h(x, v, 0) = 0$;
- $\frac{\partial}{\partial t}\big|_{t=0} h(x, v, t) = \langle \nu(x), v\rangle$;
- $\frac{\partial^2}{\partial t^2}\Big|_{t=0} h(x, v, t) = \mathrm{Hess}_x\rho(v, v)$.

Hence for a smooth function $R(x, v, t)$ we can write

$$h(x, v, t) = \langle \nu(x), v\rangle t + \frac{1}{2}\mathrm{Hess}_x\rho(v, v)t^2 + R(x, v, t)t^3.$$

Since $h(x, v, \tilde\tau(x, v)) = 0$, it follows that

eq:tautilde
$$\langle \nu(x), v\rangle + \frac{1}{2}\mathrm{Hess}_x\rho(v, v)\tilde\tau + R(x, v, \tilde\tau)\tilde\tau^2 = 0. \tag{3.1}$$

Note that $\tilde{\tau}(x,v) = 0$ iff $(x,v) \in \partial_0 SM$. Hence if we let

$$F(x,v,t) := \langle \nu(x), v\rangle + \frac{1}{2}\mathrm{Hess}_x\rho(v,v)t + R(x,v,t)t^2$$

we see that F is smooth, $F(x,v,\tilde{\tau}(x,v)) = 0$ and

$$\frac{\partial}{\partial t}\bigg|_{t=0} F(x,v,t) = \frac{1}{2}\mathrm{Hess}_x\rho(v,v).$$

But for $(x,v) \in \partial_0 SM$, $\mathrm{Hess}_x\rho(v,v) = -\Pi_x(v,v) < 0$ and thus by the implicit function theorem, $\tilde{\tau}$ is smooth in a neighbourhood of $\partial_0 SM$. Since $\tilde{\tau}$ is smooth in $\partial SM \setminus \partial_0 SM$ the lemma follows.

□

EXERCISE 3.7. Check that for $(x,v) \in \partial_0 SM$, $\mathrm{Hess}_x\rho(v,v) = -\Pi_x(v,v)$. Show that if ∂M is strictly convex then any geodesic in N starting from a point $x \in \partial M$ in a direction tangent to ∂M stays outside M for small positive and negative times. This implies that any maximal geodesic going from ∂M into M stays inside M except for its endpoints.

G: Maybe we should prove this exercise and make it part of the text a bit earlier?

Let $\mu(x,v) := \langle \nu(x), v\rangle$ for $(x,v) \in \partial SM$.

lemma:tauextension

LEMMA 3.8. *Let (M,g) be a non-trapping manifold with strictly convex boundary. The function $\mu/\tilde{\tau}$ extends to a smooth positive function on ∂SM whose value at $(x,v) \in \partial_0 SM$ is*

$$\frac{\Pi_x(v,v)}{2}.$$

PROOF. Using (3.1) we can write

$$\mu(x,v) = -\frac{1}{2}\mathrm{Hess}_x\rho(v,v)\tilde{\tau} - R(x,v,\tilde{\tau})\tilde{\tau}^2$$

and hence for $(x,v) \in \partial SM \setminus \partial_0 SM$ near $\partial_0 SM$ we can write

$$\mu/\tilde{\tau} = -\frac{1}{2}\mathrm{Hess}_x\rho(v,v) - R(x,v,\tilde{\tau})\tilde{\tau}.$$

But the right hand side of the last equation is a smooth function near $\partial_0 SM$ since R and $\tilde{\tau}$ are; its value at $(x,v) \in \partial_0 SM$ is $\Pi_x(v,v)/2$. Finally, observe that μ and $\tilde{\tau}$ are both positive for $(x,v) \in \partial_+ SM \setminus \partial_0 SM$ and both negative for $(x,v) \in \partial_- SM \setminus \partial_0 SM$.

□

remark:tautilde

REMARK 3.9. Note that we can define $\tilde{\tau}$ on all SM by setting $\tilde{\tau}(x,v) := \tau(x,v) - \tau(x,-v)$. The restriction of this function to ∂SM coincides with the definition of $\tilde{\tau}$ given by Lemma 3.6. It turns out that in fact $\tilde{\tau} \in C^\infty(SM)$; see Lemma 3.12 below.

The next lemma will be very helpful when studying regularity properties of solutions to transport equations.

tau_glancing_square_root

Lemma 3.10. *Let (M,g) be compact with smooth boundary, let $(x_0,v_0) \in \partial_0 SM$, and let ∂M be strictly convex near x_0. Assume that M is embedded in a compact manifold N without boundary. Then, near (x_0,v_0) in SM, one has*

$$\begin{aligned} \tau(x,v) &= Q(\sqrt{a(x,v)},x,v), \\ -\tau(x,-v) &= Q(-\sqrt{a(x,v)},x,v), \end{aligned}$$

where Q is a smooth function near $(0,x_0,v_0)$ in $\mathbb{R}\times SN$, a is a smooth function near (x_0,v_0) in SN, and $a \geq 0$ in SM.

Proof. This follows directly by applying Lemma 3.11 below to $h(t,x,v) = \rho(\gamma_{x,v}(t))$ near $(0,x_0,v_0)$, where ρ is a boundary defining function for M. $\square$

lemma_h_square_root

Lemma 3.11. *Let $h(t,y)$ be smooth near $(0,y_0)$ in $\mathbb{R}\times\mathbb{R}^N$. If*

$$h(0,y_0)=0, \qquad \partial_t h(0,y_0)=0, \qquad \partial_t^2 h(0,y_0)<0,$$

then one has

$$h(t,y)=0 \text{ near } (0,y_0) \text{ when } h(0,y)\geq 0 \quad\Longleftrightarrow\quad t=Q(\pm\sqrt{a(y)},y)$$

where Q is a smooth function near $(0,y_0)$ in $\mathbb{R}\times\mathbb{R}^N$, a is a smooth function near y_0 in $\mathbb{R}^N$, and $a(y)\geq 0$ when $h(0,y)\geq 0$. Moreover, $Q(\sqrt{a(y)},y)\geq Q(-\sqrt{a(y)},y)$ when $h(0,y)\geq 0$.

Proof. We use the same argument as in [**Hör85**, Theorem C.4.2]. Using that $\partial_t^2 h(0,y_0)<0$, the implicit function theorem gives that

$$\partial_t h(t,y)=0 \text{ near } (0,y_0) \quad\Longleftrightarrow\quad t=g(y)$$

where g is smooth near y_0 and $g(y_0)=0$. Write

$$h_1(s,y) := h(s+g(y),y).$$

Then $\partial_s h_1(0,y)=0$ and $\partial_s^2 h_1(0,y_0)<0$. Thus by the Taylor formula we have

$$h_1(s,y) = h_1(0,y) - s^2 F(s,y)$$

where F is smooth near $(0,y_0)$ and $F(0,y_0)>0$. We define

$$r(s,y) := sF(s,y)^{1/2}$$

and note that $r(0,y_0)=0$, $\partial_s r(0,y_0)>0$. Thus the map $(s,y)\mapsto(r(s,y),y)$ is a local diffeomorphism near $(0,y_0)$, and there is a smooth function S near $(0,y_0)$ so that

$$r(s,y)=\bar r \quad\Longleftrightarrow\quad s=S(\bar r,y).$$

Moreover, $\partial_r S(0,y_0)>0$. Define the function

$$h_2(r,y) := h_1(0,y)-r^2.$$

Now

$$\begin{aligned} h(t,y) &= h_1(t-g(y),y) = h_1(0,y)-(t-g(y))^2F(t-g(y),y) \\ &= h_2(r(t-g(y),y),y). \end{aligned}$$

Thus $h(t,y)=0$ is equivalent with

r_gy_equation

$$r(t-g(y),y)^2 = h_1(0,y) = h(g(y),y). \tag{3.2}$$

We claim that

h_gy_y_nonnegative (3.3) $$h(g(y), y) \geq 0 \text{ near } y_0 \text{ when } h(0, y) \geq 0.$$

If (3.3) holds, then we may solve (3.2) to obtain

$$h(t, y) = 0 \text{ near } (0, y_0) \text{ when } h(0, y) \geq 0 \iff r(t - g(y), y) = \pm\sqrt{h(g(y), y)}.$$

The last condition is equivalent with

$$t - g(y) = S(\pm\sqrt{h(g(y), y)}, y).$$

This proves the lemma upon taking $Q(r, y) = g(y) + S(r, y)$ and $a(y) = h(g(y), y)$ (note that $r \mapsto Q(r, y)$ is increasing since $\partial_r S(0, y_0) > 0$). To prove (3.3), we use the Taylor formula

$$h(g(y) + s, y) = h(g(y), y) + \partial_t h(g(y), y)s + G(s, y)s^2$$

where $G(0, y_0) < 0$. Choosing $s = -g(y)$ and using that $\partial_t h(g(y), y) = 0$ shows that $h(g(y), y) \geq h(0, y)$ near $y = y_0$, and thus (3.3) indeed holds. □

LEMMA 3.12. *Let (M, g) be a non-trapping manifold with strictly convex boundary. Then the functions*

$$\tilde{\tau}(x, v) := \tau(x, v) - \tau(x, -v), \quad \textit{and} \quad T(x, v) := \tau(x, v)\tau(x - v)$$

lemma:tauT *are smooth in SM.*

PROOF. Given the properties of τ we just have to prove smoothness near a glancing point $(x_0, v_0) \in \partial_0 SM$. By Lemma 3.10 given $(x, v) \in SM$ near $(x_0, v_0) \in \partial_0 SM$ we have:

$$\tilde{\tau}(x, v) = Q(\sqrt{a(x, v)}, x, v) + Q(-\sqrt{a(x, v)}, x, v).$$

Since we can write $H(r^2, x, v) = Q(r, x, v) + Q(-r, x, v)$, where H is smooth near $(0, x_0, v_0)$, we deduce that

$$\tilde{\tau}(x, v) = H(a(x, v), x, v)$$

thus showing smoothness of $\tilde{\tau}$. The statement for T follows by taking products, rather than sums.

□

REMARK 3.13. Using this lemma, it is possible to write the functions Q and a from Lemma 3.10 in terms of $\tilde{\tau}$ and T. Indeed, since τ satisfies the quadratic equation

$$\tau(\tau - \tilde{\tau}) = T$$

we have

$$\tau = \frac{\tilde{\tau} + \sqrt{\tilde{\tau}^2 + 4T}}{2}$$

with $\tilde{\tau}, T \in C^\infty(SM)$. Thus $Q(t, x, v) = (\tilde{\tau}(x, v) + t)/2$ and $a = \tilde{\tau}^2 + 4T$.

3.2. The geodesic flow and the scattering relation

Let (M,g) be a compact, connected and oriented Riemannian manifold with boundary ∂M and dimension $n \geq 2$. Without loss of generality we may assume that (M,g) is isometrically embedded into a closed manifold (N,g) of the same dimension.

The geodesics of (N,g) are defined for all times in $\mathbb{R}$. We pack them into what is called the *geodesic flow*. For each $t \in \mathbb{R}$ this is a diffeomorphism

$$\varphi_t : SN \to SN$$

defined by

$$\varphi_t(x,v) := (\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t)).$$

This is a *flow*, i.e. $\varphi_{t+s} = \varphi_t \circ \varphi_s$ for all $s,t \in \mathbb{R}$. The flow has an infinitesimal generator called the *geodesic vector field* and denoted by X. This is a smooth section of TSN that can be regarded as the first order differential operator $X : C^\infty(SN) \to C^\infty(SN)$ given by

$$(Xu)(x,v) := \frac{d}{dt}\Big|_{t=0} u(\varphi_t(x,v)),$$

where $u \in C^\infty(SN)$. Observe that $X : C^\infty(SM) \to C^\infty(SM)$. The non-trapping property can be characterized using the operator X as follows:

prop:dh

PROPOSITION 3.14. Let (M,g) be a compact manifold with strictly convex boundary. The following are equivalent:

(1) (M,g) is non-trapping;
(2) $X : C^\infty(SM) \to C^\infty(SM)$ is surjective;
(3) there is $f \in C^\infty(SM)$ such that $Xf > 0$.

PROOF. By Exercise 3.16 below $f = -\tilde{\tau}$ is smooth and satisfies $Xf > 0$, thus (1) $\implies$ (3). Clearly (3) $\implies$ (1): if there is a geodesic in M with infinite length, since $Xf > c > 0$, integrating along it we would find $f(\varphi_t(x,v)) - f(x,v) > tc$ for all $t > 0$ which is absurd since f is bounded. The implication (2) $\implies$ (3) is obvious, so it remains to prove that (1) $\implies$ (2).

Consider (M,g) embedded in a closed manifold (N,g). Since strict convexity and $Xf > 0$ are open conditions there is a slightly larger compact manifold M_0 with $M \subset M_0^{int} \subset N$ and such that ∂M_0 is strictly convex and (M_0,g) is non-trapping. Let τ_0 denote the exit time of M_0 and given $h \in C^\infty(SM)$ extend it smoothly to SM_0. For $(x,v) \in SM$, set

$$u(x,v) := -\int_0^{\tau_0(x,v)} h(\varphi_t(x,v))\,dt.$$

Since $\tau_0|_{SM}$ is smooth, $u \in C^\infty(SM)$. A calculation shows that $Xu = h$ and thus $X : C^\infty(SM) \to C^\infty(SM)$ is surjective.

□

REMARK 3.15. The assumption of ∂M being strictly convex is not necessary, see [**DH72**, Theorem 6.4.1] for a proof of the same result for arbitrary vector fields.

EXERCISE 3.16. Let (M,g) be a non-trapping manifold with strictly convex boundary. Show that

$$X\tilde{\tau} = -2,$$

xca:tautilde where $\tilde{\tau}$ is the function from Remark 3.9.

DEFINITION 3.17. Let (M,g) be a non-trapping manifold with strictly convex boundary. We define the *scattering relation* as the map $\alpha : \partial SM \to \partial SM$ given by

$$\alpha(x,v) := \varphi_{\tilde{\tau}(x,v)}(x,v).$$

By Lemma 3.6, the map α is smooth. By definition of $\tilde{\tau}$ we see that $\alpha : \partial_+ SM \to \partial_- SM$ and $\alpha : \partial_- SM \to \partial_+ SM$. Moreover, α is an involution:

EXERCISE 3.18. Prove that $\tilde{\tau} \circ \alpha = -\tilde{\tau}$. Deduce that $\alpha^2 = \mathrm{Id}$ and conclude that α is a diffeomorphism whose fixed point set is $\partial_0 SM$.

3.2.1. The geodesic vector field and strict convexity. In this subsection we would like to understand how the strict convexity of ∂M reflects at level of the geodesic vector field and the unit tangent bundle.

Let (M,g) be a compact Riemannian manifold with unit sphere bundle $\pi : SM \to M$. For details of what follows see for example [**Kni02, Pat99**]. It is well known that SM carries a canonical metric called the Sasaki metric. If we let $\mathcal{V}$ denote the vertical subbundle given by $\mathcal{V} = \operatorname{Ker} d\pi$, then there is an orthogonal splitting with respect to the Sasaki metric:

$$TSM = \mathbb{R}X \oplus \mathcal{H} \oplus \mathcal{V}.$$

The subbundle $\mathcal{H}$ is called the horizontal subbundle. Elements in $\mathcal{H}(x,v)$ and $\mathcal{V}(x,v)$ are canonically identified with elements in the codimension one subspace $\{v\}^\perp \subset T_x M$. A vector in $\mathbb{R}X \oplus \mathcal{H}$ is canonically identified with the whole $T_x M$. The identifications are described as follows. Given $\xi \in T_{(x,v)}SM$, write it as $\xi = (\xi_H, \xi_V)$, where $\xi_H \in \mathbb{R}X \oplus \mathcal{H}$ and $\xi_V \in \mathcal{V}$. Then $\xi_H = d\pi(\xi)$ and $\xi_V = K\xi$, where K is the *connection map*. Consider any curve $Z : (-\varepsilon, \varepsilon) \to SM$ such that $Z(0) = (x,v)$ and $\dot{Z}(0) = \xi$ and write $Z(t) = (\alpha(t), W(t))$. Then

$$K\xi := \left.\frac{DW}{dt}\right|_{t=0},$$

where D stands for the covariant derivative of the vector field W along α given by the Levi-Civita connection. In this splitting, the geodesic vector field has a very simple form

eq:X
$$X(x,v) = (v,0). \tag{3.4}$$

Using the splitting, one can also define the *Sasaki metric* G of SM as

$$\langle \xi, \eta \rangle_G := \langle \xi_H, \eta_H \rangle_g + \langle \xi_V, \eta_V \rangle_g.$$

The next lemma identifies the tangent spaces to ∂SM and $S\partial M = \partial_0 SM$ using this splitting.

LEMMA 3.19.

$$T_{(x,v)}\partial SM = \{(\xi_H, \xi_V) : \ \xi_H \in T_x \partial M, \ \ \xi_V \in \{v\}^\perp\};$$

$$T_{(x,v)}\partial_0 SM = \{(\xi_H, \xi_V) : \ \xi_H \in T_x \partial M, \ \ \xi_V \in \{v\}^\perp, \ \ \langle \xi_V, \nu(x) \rangle - \Pi_x(v, \xi_H) = 0\}.$$

lemma:tangent

PROOF. To prove the first statement consider a curve $Z : (-\varepsilon, \varepsilon) \to \partial SM$ with $Z(0) = (x, v)$ and $\xi = \dot{Z}(0)$. Then if we write $Z(t) = (\alpha(t), W(t))$ with $\alpha : (-\varepsilon, \varepsilon) \to \partial M$, we see that $\xi_H = d\pi(\xi) = \dot{\alpha}(0) \in T_x\partial M$. Differentiating $\langle W(t), W(t)\rangle = 1$ at $t = 0$ we get that $\langle \xi_V, v\rangle = 0$ and counting dimensions the first statement follows.

To prove the second statement we need to take a curve $Z : (-\varepsilon, \varepsilon) \to \partial_0 SM$ which gives the additional equation $\langle W(t), \nu(\alpha(t))\rangle = 0$. Differentiate this at $t = 0$, to get using the definition of the connection map K:

$$\langle \xi_V, \nu(x)\rangle + \langle v, \nabla_{\xi_H}\nu\rangle = 0.$$

This is equivalent to $\langle \xi_V, \nu(x)\rangle - \Pi_x(v, \xi_H) = 0$ and the result follows. $\square$

When does X fail to be transversal to ∂SM? Using Lemma 3.19 and (3.4) we see that this happens iff $(x, v) \in \partial_0 SM$. In addition, the characterization of $T_{(x,v)}\partial_0 SM$, tell us that X is always transversal to $\partial_0 SM$ under the assumption that the boundary ∂M is strictly convex.

We summarize this in the following lemma:

LEMMA 3.20. *The geodesic vector field X is transverse to $\partial SM \setminus \partial_0 SM$. If ∂M is strictly convex, then X is transversal to $\partial_0 SM$. We always have $X(x, v) \in$*
lemma:summary *$T_{(x,v)}\partial SM$ for $(x, v) \in \partial_0 SM$.*

The picture described by the lemma will be helpful later on when discussing regularity results for the transport equation.

EXERCISE 3.21. Show that the horizontal vector $(\nu(x), 0)$ is a unit normal vector to ∂SM in the Sasaki metric. Moreover, show that the inner product of this vector with X is precisely the function μ introduced before.

3.3. The unit circle bundle of a surface

Consider now the case $\dim M = 2$. In this instance there is a very convenient frame of TSM that will be used throughout this book.

Since M is assumed oriented there is a circle action on the fibers of SM with infinitesimal generator V called the *vertical vector field*. It is possible to complete the pair X, V to a global frame of $T(SM)$ by considering the vector field $X_\perp$ defined as the commutator

eq:XV (3.5)
$$X_\perp := [X, V].$$

There are two additional structure equations given by

eq:VXperp (3.6)
$$X = [V, X_\perp],$$

and

eq:XXperp (3.7)
$$[X, X_\perp] = -KV,$$

where K is the Gaussian curvature of the surface. Using this frame we can define a Riemannian metric on SM by declaring $\{X, X_\perp, V\}$ to be an orthonormal basis.

EXERCISE 3.22. Show that this metric coincides with the Sasaki metric G on SM defined above.

The volume form of the metric G will be denoted by $d\Sigma^3$. The fact that $\{X, X_\perp, V\}$ are orthonormal together with the commutator formulas implies that the Lie derivative of $d\Sigma^3$ along the three vector fields vanishes, in other words, the three vector fields preserve the volume form $d\Sigma^3$.

EXERCISE 3.23. Show that each element in the frame $\{X, X_\perp, V\}$ preserves $d\Sigma^3$.

See [**ST76**] for more details on these facts.

G: We can easily expand and complement this part adding material from my notes with Will.

It will be useful to have explicit forms of the three vector fields in local coordinates. Since (M, g) is two dimensional, we can always choose isothermal coordinates (x_1, x_2) so that the metric can be written as $ds^2 = e^{2\lambda}(dx_1^2 + dx_2^2)$ where λ is a smooth real-valued function of $x = (x_1, x_2)$. This gives coordinates (x_1, x_2, θ) on SM where θ is the angle between a unit vector v and $\partial/\partial x_1$. In these coordinates the vertical vector field is just

$$V = \frac{\partial}{\partial\theta},$$

and the other vector fields are given by

eq:eqX

$$X = e^{-\lambda}\left(\cos\theta\frac{\partial}{\partial x_1} + \sin\theta\frac{\partial}{\partial x_2} + \left(-\frac{\partial\lambda}{\partial x_1}\sin\theta + \frac{\partial\lambda}{\partial x_2}\cos\theta\right)\frac{\partial}{\partial\theta}\right), \tag{3.8}$$

eq:Xperp

$$X_\perp = -e^{-\lambda}\left(-\sin\theta\frac{\partial}{\partial x_1} + \cos\theta\frac{\partial}{\partial x_2} - \left(\frac{\partial\lambda}{\partial x_1}\cos\theta + \frac{\partial\lambda}{\partial x_2}\sin\theta\right)\frac{\partial}{\partial\theta}\right). \tag{3.9}$$

G: Motto: try not to leave the unit sphere bundle SM if you can! It is the natural habitat where everything takes place. This is similar to other people's mottos; for instance András will say "do not project to M and stay in phase space". Michel Herman told me something similar but more amusing in 97: "Riemannian geometers get a bit confused with some of this stuff since they are not used to working in phase space".

G: We should give a reference for isothermal coordinates and mention that non-trapping surfaces with strictly convex boundary are diffeomorphic to discs and hence admit *global* isothermal coordinates.

3.3.1. Herglotz condition and rotationally symmetric cases. TODO. Include the fact that a rotationally symmetric surface is non-trapping iff it satisfies the Herglotz condition.

3.3.2. Additional facts.

PROPOSITION 3.24. Let (M, g) be a 2D disc with strictly convex boundary. Suppose $\tau(x, v) < \infty$ for all $(x, v) \in \partial_+SM$. Then (M, g) is non-trapping.

SKETCH. Run the mean curvature flow inwards starting from the boundary. We know that two things can happen: either it converges to a simple closed geodesic or it produces a strictly convex function. In the latter case (M, g) is non-trapping, so let us assume we have a closed geodesic at the end of the flow. Hence we have produced an annulus A where one boundary component is ∂M and the other is the closed geodesic γ. The annulus admits a function f that is strictly convex except at γ. Now consider the universal cover of A; this is a strip bounded by the lifts of ∂M and γ. Let us denote those lifts by $\widetilde{\partial M}$ and $\tilde{\gamma}$. Consider a sequence of points $p_n \in \tilde{\gamma}$ with $p_n \to \infty$ and fix $x \in \widetilde{\partial M}$. The geometry of the strip is so that for each n there is a minimizing unit speed geodesic σ_n connecting x to p_n. By compactness and strict convexity of the boundary the unit vectors $\dot{\sigma}_n(0) \in \partial_+ S\widetilde{A}$ must converge to a unit vector $w \in T_x\widetilde{A}$ pointing strictly inside $\widetilde{A}$. But $\tau(x, w) = \infty$ contradicting the assumption that $\tau(x, v) < \infty$ for every $(x, v) \in \partial_+ SM$.

□

3.4. Morse Theory approach

The classical Morse theory of the energy functional on loop spaces provides several relevant results. These results are pretty standard on complete manifolds without boundary or closed manifolds. Given a compact manifold (M, g) with strictly convex boundary, throughout this subsection, (N, g) is a no return extension. Recall that this is a complete extension of (M, g) with the property that geodesics leaving M do not return to M. Moreover, since $N \setminus M$ can be taken as to be diffeomorphic to $(0, \infty) \times \partial M$, M is a deformation retract of N.

G: all these results are proved with the same tools. It makes sense to pack them in one section and then use them as needed. This should make the 10+ section a bit easier.

prop:HR

Proposition 3.25. Let (M, g) be a compact Riemannian manifold with strictly convex boundary. Then given given two points $x, y \in M$ there is a minimizing geodesic in M connecting x to y.

Proof. Since (N, g) is complete, the Hopf-Rinow theorem ensures that there is a minimizing geodesic γ in N connecting x and y. Since $x, y \in M$ and geodesics leaving M do not return to M we must have that γ is a geodesic entirely contained in M.

□

prop:contractible

Proposition 3.26. Let (M, g) be a non-trapping manifold with strictly convex boundary. Then M is contractible.

Proof. Since M is a deformation retract of N, it follows that M is contractible iff N is. A classical result of Serre [**Ser51**] (proved using Morse theory) asserts that if x and y are not conjugate and if N is not contractible, there are geodesics connecting x to y with arbitrarily large length. Since N is a no return extension, if we pick x and y in M, then M itself admits geodesics of arbitrarily large length connecting x to y thus violating the non-trapping property. Note that by Sard's theorem if we fix x almost every $y \in N$ is not conjugate to x. It follows that M is contractible.

□

Remark 3.27. The proposition also follows from another well-known fact in Riemannian geometry: a compact connected and non-contractible Riemannian manifold with strictly convex boundary must have a closed geodesic in its interior [**Tay11**, Theorem 4.2]. This is also proved with Morse theory, but using the space of free loops.

prop:homotopy

Proposition 3.28. Let (M, g) be a compact Riemannian manifold without conjugate points and with strictly convex boundary. Let γ be a geodesic with endpoints $x, y \in M$. If α is any other smooth curve in M connecting x to y that is homotopic to γ with a homotopy fixing the end points, then the length of α is larger than the length of γ. In other words, there is a unique geodesic connecting x to y in a given homotopy class and this geodesic must be minimizing.

Proof. We follow [**GM18**, Lemma 2.2] where this very same proposition is proved. We let $\Omega(x, y)$ denote the Hilbert manifold of absolutely continuous curves $c : [0, 1] \to N$ with $c(0) = x$, $c(1) = y$ and finite energy

$$E(c) := \frac{1}{2}\int_0^1 |\dot{c}|^2 \, dt.$$

It is well known that $E : \Omega(x, y) \to \mathbb{R}$ is C^2 and satisfies the Palais-Smale condition. The critical points of E are precisely the geodesics connecting x to y. Moreover, since there are no conjugate points, the Morse index theorem [**Mil63**] guarantees that the Hessian of E at a critical point is positive definite (recall that N is a no return extension, so it suffices to assume that M has no conjugate points). Thus all critical points of E are local minimizers of E and are isolated. We now argue with E restricted to the connected component of $\Omega(x, y)$ containing γ, which we denote by $\Omega_{[\gamma]}(x, y)$. This coincides with the set of paths connecting x to y and homotopic to γ. We claim that γ is the unique minimizer of $E|_{\Omega_{[\gamma]}(x,y)}$. Indeed a mountain pass argument, shows that if this is not the case, then there is a geodesic $\sigma \in \Omega_{[\gamma]}(x, y)$ that is not a local minimum of $E|_{\Omega_{[\gamma]}(x,y)}$ (cf. [**Str96**, Theorem 10.3] and [**Hof85**]). Again by the Morse index theorem σ must contain conjugate points, and since it must be entirely contained in M we get a contradiction.

G: check smoothness claim and give reference

$\square$

3.5. 10+ Definitions of simple manifold

Definition 3.29. A compact connected manifold (M, g) is said to be *simple* if (M, g) is non-trapping, it has strictly convex boundary and has no conjugate points.

Proposition 3.30. Let (M, g) be a simple manifold. Given $x, y \in M$, there is a unique geodesic connecting x to y and this geodesic is minimizing.

Proof. Since ∂M is strictly convex, Proposition 3.25 ensures that there is a minimizing geodesic connecting x to y. Since M is non-trapping, it must be simply connected by Proposition 3.26. Thus Proposition 3.28 implies that there is only one geodesic connecting x to y and this geodesic must be minimizing. $\square$

Given $x \in M$, let $D_x \subset T_x M$ be the set given by

$$D_x := \{tv : \ v \in S_x M, \ t \in [0, \tau(x, v)]\}.$$

The previous proposition asserts that if M is simple, then

$$\exp_x : D_x \to M$$

is a bijection. Since there are no conjugate points, $\exp_x$ is a local diffeomorphism at any $tv \in D_x$ and hence $\exp_x : D_x \to M$ is a diffeomorphism. This implies in particular that M is diffeomorphic to a closed ball in Euclidean space: if x is in the interior of M, then D_x is a closed star-shaped domain around zero with smooth boundary and hence diffeomorphic to a closed ball.

Proposition 3.31. Let (M, g) be a compact manifold with strictly convex boundary. The following are equivalent:

(1) (M, g) is simple;
(2) M is simply connected and has no conjugate points.

Any of these two properties implies:

- Given two points in M, there is a unique geodesic connecting them and this geodesic is minimizing.

Proof. (1) $\implies$ (2): If M is simple, then it has no conjugate points by definition. It is simply connected due to Proposition 3.26.

(2) $\implies$ (1): Suppose M has strictly convex boundary, is simply connected and has no conjugate points. Proposition 3.28 implies that between two points in M there is a unique geodesic and this geodesic must be minimizing. It follows that all geodesics have length less than or equal to the diameter of M, hence the manifold is non-trapping and (M, g) is simple.

$\square$

G: The condition of having a unique geodesic between two points does not seem to be quite enough to imply that (M, g) is simple. It is enough to guarantee that the interior of M has no conjugate points, but in principle there could be conjugate points between boundary points. It would be good to find an example. The "original" definition of simple manifold seems to be lost in time. I could not find it as we use it today in Michel's original paper. Also the definition of strictly convex boundary could vary a bit. We are using the strongest form. If you demand the exponential map to be a diffeomorphism or equivalently ask for the unique geodesic to depend smoothly on end points you remove this issue, as you ensure the no conjugate point condition also at the boundary. Who coined the word "simple"?

Proposition 3.32. Let (M, g) be simple manifold. Any sufficiently small neighbourhood U of M in N has the property that $\overline{U}$ is simple.

Proof. Clearly any sufficiently small neighbourhood U has the property that its closure $\overline{U}$ has strictly convex boundary and is simply connected. To see that the property of having no conjugate points persists when we go to $\overline{U}$, for U sufficiently close to M, assume that this is not the case. Then there exists a sequence $(x_n, v_n) \in SN \setminus SM$ converging to $(x, v) \in \partial_+ SM$ and a sequence $(y_n, w_n) \in SN$ converging to $(y, w) \in SM$ such that $\varphi_{t_n}(x_n, v_n) = (y_n, v_n)$ and $d_{(x_n,v_n)}\varphi_{t_n}(\mathcal{V}(x_n, v_n)) \cap \mathcal{V}(y_n, w_n) \neq \{0\}$ (conjugate point condition).

If the sequence t_n is bounded, by passing to a subsequence, we deduce that there is $t_0 > 0$ such that $d_{(x,v)}\varphi_{t_0}(\mathcal{V}(x, v)) \cap \mathcal{V}(y, w) \neq \{0\}$ and thus M has conjugate points. Indeed, we have unit vectors (in the Sasaki metric) $\xi_n \in \mathcal{V}(x_n, v_n)$ such that

$$d_{(x_n,y_n)}\pi \circ \varphi_{t_n}(\xi_n) = 0$$

and passing to subsequences if necessary we find a unit norm $\xi \in \mathcal{V}(y, w)$ for which

$$d\pi \circ d_{(y,w)}\varphi_{t_0}(\xi) = 0.$$

If t_n is unbounded, we may assume by passing to a subsequence that $t_n \to \infty$ and thus the geodesic determined by (x, v) has infinite length thus violating the non-trapping property.

□

G: I think there is another proof of this using the continuity of the cut time function $t_c : SN \to (0, \infty)$. A classical result in Riemannian geometry (cf. Sakai's book, Proposition 4.1) asserts that t_c is locally Lipschitz around a point (x, v) for which is finite. Hence if N is closed, t_c is continuous in all SN. If geodesics on M have no conjugate points and between two points there is only one, then cut points do **not** occur in M (again cf. Sakai's book, Proposition 4.1), i.e. for all $(x, v) \in SM$, $\tau(x, v) < t_c(x, v)$. This means that one can go a bit further along any geodesic and by a uniform amount.

REMARK 3.33. Using this proposition we can now see that (M, g) is simple iff M has a neighbourhood U such that any two points in U are joined by a unique geodesic.

Then there is also the equivalence of simplicity with smoothness of the boundary distance function. There is the following folklore result:

PROPOSITION 3.34. Let (N, g) be a complete Riemannian manifold. Take $x \neq y \in N$. Then the distance function d_g is smooth in a neighbourhood of (x, y) iff x and y are connected by a unique geodesic that is minimizing and free of conjugate points.

prop:dsmooth

SKETCH. If the condition on geodesics hold, write $d(x, y) = |\exp_x^{-1}(y)|$ and smoothness of d follows. For the converse fix x and set $f(y) := d(x, y)$. Then if f is differentiable at y and there is a unit speed minimizing geodesic γ connecting x to y, then $\nabla f(y)$ is the velocity vector of γ at y. So if we have more than one minimizing geodesic the gradient wants to be two different things at the same time; absurd. For the conjugate points we have to go to the second derivatives of d and see that if x and y are conjugate along the unique minimizing geodesic joining them, then the Hessian blows up.

□

Next we would like to prove the claim:

PROPOSITION 3.35. Let (M, g) be a compact manifold with strictly convex boundary. Then M is simple iff the boundary distance function $d_g|_{\partial M \times \partial M}$ is smooth away from the diagonal.

This is another folklore claim that appears often in the papers of Burago and Ivanov. At first glance it looks as if it follows from Proposition 3.34 but some additional work is needed. This is because we have to go from smoothness of the restriction of d to $\partial M \times \partial M \setminus \Delta$ to smoothness of d in the ambient manifold M (or an extension N). Now there is this observation used later in the proof of Proposition 11.7 that says that if $f(y) := d(x, y)$ and $h = f|_{\partial M}$, then the gradient of h at $y \in \partial M$ determines the gradient of f at y. This goes in the right direction and uses of course that distance functions solve the Hamilton-Jacobi equation $|\nabla f| = 1$.

G: This seems OK but it may take some effort to write down in full detail. We do not really use it, so we need to discuss if it is worth including it.
One last point to discuss has to do with convexity: positive definite second fundamental form implies the interiors of geodesics lie in the interior of M, but the converse is not necessarily true, right?

CHAPTER 4

The geodesic X-ray transform

chapter3

This chapter should set basic properties of the X-ray transform on a non-trapping manifold with strictly convex boundary.

4.1. Volume forms and Santaló's formula

Let (M, g) be a compact, connected and oriented Riemannian manifold of dimension $n \geq 2$. The unit tangent bundle SM carries a natural volume form called the *Liouville form*. We shall denote it by $d\Sigma^{2n-1}$. This form can also be interpreted as the volume form of the Sasaki metric on SM or the volume form associated with the contact form of the geodesic flow. At a point $(x, v) \in SM$ it can be written as

$$d\Sigma^{2n-1} = dV^n \wedge dS_x$$

where dV^n is the volume form of (M, g) and dS_x is the volume form on the fibre S_xM induced by the metric g at x. Loiuville's theorem in classical mechanics asserts that the geodesic flow preserves $d\Sigma^{2n-1}$, in other words, the Lie derivative $L_X d\Sigma^{2n-1} = 0$. Similarly, ∂SM carries a natural volume

$$d\Sigma^{2n-2} := dV^{n-1} \wedge dS_x$$

where dV^{n-1} is the volume form of $(\partial M, g)$. This is just the volume form of the Sasaki metric restricted to ∂SM.

G: Elaborate on this? Prove some of the claims perhaps and/or give references. Should the volume forms be introduced in Chapter 2 instead?

Exercise 4.1. Show that $j^* i_{\overline{\nu}} d\Sigma^{2n-1} = -d\Sigma^{2n-2}$, where $\overline{\nu} = (\nu, 0)$ is the horizontal lift of the unit normal ν and $j : \partial SM \to SM$ is the inclusion map. Show that $j^* i_X d\Sigma^{2n-1} = -\mu d\Sigma^{2n-2}$.

G: check orientations

Proposition 4.2 (Santaló's formula). Let (M, g) be a non-trapping manifold with strictly convex boundary. Given $f \in C(SM, \mathbb{R})$ we have

$$\int_{SM} f \, d\Sigma^{2n-1} = \int_{\partial_+ SM} d\mu(x, v) \int_0^{\tau(x,v)} f(\varphi_t(x, v)) \, dt,$$

proposition:santalo

where $d\mu = \mu d\Sigma^{2n-2}$.

Proof. Consider the set

$$D := \{(x, v, t) : \ (x, v) \in \partial_+ SM, \ t \in [0, \tau(x, v)]\} \subset \partial_+ SM \times \mathbb{R}$$

and the map $\Psi : D \to SM$ given by $\Psi(x,v,t) = \varphi_t(x,v)$. The map Ψ is a diffeomorphism restricted to the interior of D and thus

$$\int_{SM} f\, d\Sigma^{2n-1} = \pm \int_D (f \circ \Psi)\Psi^*(d\Sigma^{2n-1}).$$

(the $\pm$ depends on whether Ψ preserves or reverts orientations.) By definition, the differential of Ψ maps $\partial/\partial t$ to X. Since by Liouville's theorem $L_X d\Sigma^{2n-1} = 0$ we see that $L_{\partial/\partial t}\Psi^* d\Sigma^{2n-1} = 0$. This means that we can write

$$\Psi^* d\Sigma^{2n-1} = a d\Sigma^{2n-2} \wedge dt$$

where the function a does not depend on t, i.e. $a \in C^\infty(\partial_+ SM)$. This implies

$$\int_{SM} f\, d\Sigma^{2n-1} = \pm \int_{\partial_+ SM} a(x,v) d\Sigma^{2n-2} \int_0^{\tau(x,v)} f(\varphi_t(x,v))\, dt.$$

To complete the proof, we will check that $a = -\mu$. To this end it is enough to compute $\Psi^* d\Sigma^{2n-1}(x,v,0)$. Since $\varphi_0(x,v) = (x,v)$, the map $d\Psi_{(x,v,0)}$ is given by the identity restricted to $T_{(x,v)}\partial_+ SM$ and it maps $\partial/\partial t$ to $X(x,v)$. Consider an oriented orthonormal basis $\{\xi_1, \dots, \xi_{2n-2}\}$ of $T_{(x,v)}\partial_+ SM$. By definition of $d\Sigma^{2n-1}$ and the boundary orientation

$$d\Sigma^{2n-1}(\overline{\nu}, \xi_1, \dots, \xi_{2n-2}) = -1.$$

($\overline{\nu}$ is inward unit normal.) On the other hand

$$\Psi^* d\Sigma^{2n-1}(x,v,0)(\xi_1, \dots, \xi_{2n-2}, \partial/\partial t) = d\Sigma^{2n-1}(\xi_1, \dots, \xi_{2n-2}, X(x,v)).$$

Writing $X = (X - \mu\overline{\nu}) + \mu\overline{\nu}$ we see that

$$a = \Psi^* d\Sigma^{2n-1}(x,v,0)(\xi_1, \dots, \xi_{2n-2}, \partial/\partial t) = -\mu$$

since $(X - \mu\overline{\nu})$ is tangent to ∂SM. And finally we observe that Ψ reverses orientation so all signs agree and the proof is completed.

□

G:Obviously signs and orientations need a careful checking... We can avoid discussing orientations if we only consider measures.

4.1.1. Alternative proof of Santaló's formula. We shall need the following lemma which is an easy consequence of Stokes' theorem.

Lemma 4.3. *Let N be a compact manifold with boundary, Θ a volume form, Y a vector field and $u \in C^\infty(N)$. Then*

$$\int_N Y(u)\Theta = -\int_N u L_Y \Theta + \int_{\partial N} u i_Y \Theta.$$

lemma:Yu

Proposition 4.4 (Alternative proof of Santaló's formula). Let (M,g) be a non-trapping manifold with strictly convex boundary. Given $f \in C_c^\infty(SM, \mathbb{R})$ we have

$$\int_{SM} f\, d\Sigma^{2n-1} = \int_{\partial_+ SM} d\mu(x,v) \int_0^{\tau(x,v)} f(\varphi_t(x,v))\, dt,$$

where $d\mu = \mu d\Sigma^{2n-2}$.

PROOF. Recall that $\tau \in C(SM, \mathbb{R})$. Given $f \in C_c^\infty(SM, \mathbb{R})$, define for $(x, v) \in SM$,

eq:uf
$$u^f(x,v) := \int_0^{\tau(x,v)} f(\varphi_t(x,v))\,dt. \tag{4.1}$$

Clearly $u^f \in C(SM, \mathbb{R})$ and $u^f|_{\partial_- SM} = 0$. But if f has compact support in the interior of M, then u^f is in fact smooth. A simple computation shows that

eq:transport
$$Xu^f = -f. \tag{4.2}$$

We now apply Lemma 4.3 for the case $N = SM$, $Y = X$ and $u = u^f$. Since $L_X d\Sigma^{2n-1} = 0$ and $u^f|_{\partial_- SM} = 0$ we deduce

$$\int_{SM} f\, d\Sigma^{2n-1} = -\int_{\partial_+ SM} u^f i_X d\Sigma^{2n-1}.$$

The proposition now follows from the fact that $j^* i_X d\Sigma^{2n-1} = -d\mu$.

□

EXERCISE 4.5. Use an approximation argument to show that Santaló's formula holds for $f \in L^1(SM)$.

Using Santaló's formula we can determine a natural volume form that is preserved by the scattering relation. Given $h \in C^\infty(\partial_+ SM, \mathbb{R})$ we can naturally associate to it a first integral of the geodesic flow by writing

$$h^\sharp(x,v) := h(\varphi_{-\tau(x,-v)}(x,v)), \qquad (x.v) \in SM.$$

PROPOSITION 4.6. Let (M, g) be a non-trapping manifold with strictly convex boundary. Then

$$\alpha^*(\mu d\Sigma^{2n-2}) = -\mu d\Sigma^{2n-2}.$$

Moreover

$$\alpha^*\left(\frac{\mu}{\tilde{\tau}} d\Sigma^{2n-2}\right) = \frac{\mu}{\tilde{\tau}} d\Sigma^{2n-2}.$$

PROOF. Let $w \in C^\infty(\partial_+ SM)$. Then using Santaló's formula

$$\int_{\partial_+ SM} w\tau\mu\, d\Sigma^{2n-2} = \int_{\partial_+ SM}\int_0^{\tau(x,v)} w^\sharp(\varphi_t(x,v))\mu\, dt d\Sigma^{2n-2} = \int_{SM} w^\sharp\, d\Sigma^{2n-1}$$

Set $\widehat{u}(x,v) = u(x,-v)$ for $u \in C(SM)$, one has

$$\begin{aligned}
\int_{SM} w^\sharp\, d\Sigma^{2n-1} &= \int_{SM} \widehat{w^\sharp}\, d\Sigma^{2n-1} \\
&= \int_{\partial_- SM}\int_0^{\tau(y,-\eta)} \widehat{w^\sharp}(\varphi_t(y,-\eta))(-\mu)\, dt d\Sigma^{2n-2} \\
&= \int_{\partial_- SM}\int_0^{\tau(y,-\eta)} w(\alpha(y,\eta))(-\mu)\, dt d\Sigma^{2n-2} \\
&= \int_{\partial_+ SM} w\tau\alpha^*(-\mu d\Sigma^{2n-2}),
\end{aligned}$$

where we used Santaló's formula again in the second line. Varying w shows that $\alpha^*(-\mu d\Sigma^{2n-2}) = \mu d\Sigma^{2n-2}$ on $\partial_+ SM \backslash \partial_0 SM$. Using that $\alpha^2 = \mathrm{id}$ we deduce that $\alpha^*(\mu d\Sigma^{2n-2}) = -\mu d\Sigma^{2n-2}$ in all ∂SM. The second identity in the proposition follows from $\tilde{\tau} \circ \alpha = -\tilde{\tau}$ and Lemma 3.19. □

Does α preserve or revert orientation? Check. This is implicitly used above

4.2. The geodesic X-ray transform

Motivated by the proof of Santaló's formula we make the following definition.

DEFINITION 4.7. Let (M, g) be a compact non-trapping manifold with strictly convex boundary. The *geodesic X-ray transform* is the operator

$$I : C^\infty(SM) \to C^\infty(\partial_+ SM)$$

given by

$$(If)(x, v) := \int_0^{\tau(x,v)} f(\varphi_t(x, v))\, dt.$$

Note that since $\tau \in C^\infty(\partial_+ SM)$, $If \in C^\infty(\partial_+ SM)$. We shall denote by $L^2(SM)$ the space of L^2-functions on SM with respect to the volume form $d\Sigma^{2n-1}$ and by $L^2_\mu(\partial_+ SM)$ the space of L^2-functions on $\partial_+ SM$ with respect to the measure $d\mu$. If we drop we subscript μ it means that we are considering the L^2-space with respect to the volume form $d\Sigma^{2n-2}$. In general, if p is a weight, we denote by L^2_p the L^2 space with respect to the measure $p d\Sigma^{2n-2}$.

PROPOSITION 4.8. The operator I extends to a bounded operator

$$I : L^2(SM) \to L^2_\mu(\partial_+ SM).$$

Moreover, the following stronger result holds: I extends to a bounded operator

$$I : L^2(SM) \to L^2(\partial_+ SM).$$

prop:L2bound

PROOF. Since $p := \mu/\tilde{\tau} \in C^\infty(\partial SM)$ and is strictly positive by Lemma 3.8, it suffices to prove the lemma using the measure $p d\Sigma^{2n-2}$ in the target space. Take $f \in C^\infty(SM)$ and write using Cauchy-Schwarz

$$\begin{aligned}
\|If\|^2_{L^2_p(\partial_+ SM)} &= \int_{\partial_+ SM} \left(\int_0^{\tau(x,v)} f(\varphi_t(x, v))\, dt \right)^2 p d\Sigma^{2n-2} \\
&\leq \int_{\partial_+ SM} \left(\int_0^{\tau(x,v)} f^2(\varphi_t(x, v))\, dt \right) \tau p d\Sigma^{2n-2} \\
&= \int_{\partial_+ SM} \left(\int_0^{\tau(x,v)} f^2(\varphi_t(x, v))\, dt \right) \mu d\Sigma^{2n-2} \\
&= \int_{SM} f^2\, d\Sigma^{2n-1} = \|f\|^2_{L^2(SM)},
\end{aligned}$$

where in the last line we have used Santaló's formula in Proposition 4.2.

□

The next result characterizes functions in the kernel of the geodesic X-ray transform in terms of solutions to the transport equation $Xu = f$.

PROPOSITION 4.9. A function $f \in C^\infty(SM)$ satisfies $If = 0$ iff there is $u \in C^\infty(SM)$ such that $u|_{\partial SM} = 0$ and $Xu = f$.

proposition:redtrans

PROOF. Given any smooth function u with $Xu = f$, if we integrate along the geodesic flow we obtain

$$u \circ \alpha(x,v) - u(x,v) = \int_0^{\tau(x,v)} f(\varphi_t(x,v))\,dt = (If)(x,v)$$

for $(x,v) \in \partial_+ SM$. Hence if $u|_{\partial SM} = 0$, the above equality implies $If = 0$.

For the converse we would like to use the function

$$u^f(x,v) = \int_0^{\tau(x,v)} f(\varphi_t(x,v))\,dt$$

introduced before. If $If = 0$, then $u^f|_{\partial SM} = 0$. One issue is that in principle, u^f is not smooth at the glancing $\partial_0 SM$ (since τ suffers the same condition). However, it turns out that if $u^f|_{\partial SM} = 0$, then actually $u^f \in C^\infty(SM)$. This will follow from a general regularity result to be proved in Chapter 5, namely Theorem 5.10. Since $Xu^f = -f$, the proposition is proved.

□

4.3. The adjoint I^*

To compute the adjoint of $I : L^2(SM) \to L^2_\mu(\partial_+ SM)$ with respect to the L^2 inner products consider $f \in C^\infty(SM, \mathbb{R})$ and $h \in C^\infty(\partial_+ SM, \mathbb{R})$ and write

$$(If, h) = \int_{\partial_+ SM} If\, h\, d\mu = \int_{\partial + SM} d\mu(x,v) \left(\int_0^{\tau(x,v)} f(\varphi_t(x,v)) h(x,v)\,dt \right).$$

Recall that given $h \in C^\infty(\partial_+ SM, \mathbb{R})$ we denote

$$h^\sharp(x,v) = h(\varphi_{-\tau(x,-v)}(x,v)), \qquad (x.v) \in SM.$$

We can write the above expression as

$$(If, h) = \int_{\partial_+ SM} d\mu(x,v) \left(\int_0^{\tau(x,v)} f(\varphi_t(x,v)) h^\sharp(\varphi_t(x,v))\,dt \right).$$

Using Santaló's formula we derive

$$(If, h) = \int_{SM} f h^\sharp d\Sigma^{2n-1} = (f, h^\sharp)$$

and hence

$$I^* h = h^\sharp.$$

EXERCISE 4.10. Let $\ell_0 : C^\infty(M) \to C^\infty(SM)$ be the map given by $\ell_0 f = f \circ \pi$, where $\pi : SM \to M$ is the canonical projection. Show that the adjoint ℓ_0^* is given by

$$(\ell_0^* g)(x) = \int_{S_x M} g(x,v)\,dS_x(v).$$

4.4. Pestov identity

In this section we consider the Pestov identity in 2D, which is the basic energy identity that has been used since the work of Mukhometov [**Muh77**] in most injectivity proofs of ray transforms in the absence of real-analyticity or special symmetries. Pestov type identities were also used in [**AR97**] to prove solenoidal injectivity for I acting on 1-forms on simple manifolds and in [**PS87**] to prove solenoidal injectivity in any dimensions and for tensors of any order, if the sectional curvatures are negative. These identities have often appeared in a somewhat ad hoc way, but here we give a point of view which makes its derivation more transparent.

The easiest way to motivate the Pestov identity is to consider the injectivity of the ray transform on functions. We let $I_0 : C^\infty(M) \to C^\infty(\partial_+ SM)$ be defined by $I_0 := I \circ \ell_0$, where ℓ_0 is the pull-back of functions from M to SM.

The first step is to recast the injectivity problem for I_0 as a uniqueness question for the partial differential operator P on SM where

$$P := VX.$$

This involves a standard reduction to the transport equation as we have done already in Proposition 4.9.

proposition:redI0

Proposition 4.11. Let (M, g) be a compact oriented nontrapping surface with strictly convex smooth boundary. The following statements are equivalent.

(a) The ray transform $I_0 : C^\infty(M) \to C^\infty(\partial_+ SM)$ is injective.
(b) Any smooth solution of $Pu = 0$ in SM with $u|_{\partial SM} = 0$ is identically zero.

Proof. Assume that the ray transform is injective, and let $u \in C^\infty(SM)$ solve $Pu = 0$ in SM with $u|_{\partial SM} = 0$. This implies that $Xu = -f$ in SM for some smooth f only depending on x, and we have $0 = u|_{\partial_+ SM} = I_0 f$. Since I_0 is injective one has $f = 0$ and thus $Xu = 0$, which implies $u = 0$ by the boundary condition.

Conversely, assume that the only smooth solution of $Pu = 0$ in SM which vanishes on ∂SM is zero. Let $f \in C^\infty(M)$ be a function with I_0. Proposition proposition:redtrans gives a $u \in C^\infty(SM)$ such that $Xu = f$ and $u|_{\partial SM} = 0$. Since f only depends on x we have $Vf = 0$, and consequently $Pu = 0$ in SM and $u|_{\partial(SM)} = 0$. It follows that $u = 0$ and also $f = -Xu = 0$. □

We now focus on proving a uniqueness statement for solutions of $Pu = 0$ in SM. For this it is convenient to express P in terms of its self-adjoint and skew-adjoint parts in the $L^2(SM)$ inner product as

$$P = A + iB, \quad A := \frac{P + P^*}{2}, \quad B := \frac{P - P^*}{2i}.$$

Here the formal adjoint P^* of P is given by

$$P^* := XV.$$

In fact, if $u \in C^\infty(SM)$ with $u|_{\partial SM} = 0$, then

p_ab_computation

(4.3)

$$\begin{aligned} \|Pu\|^2 &= ((A + iB)u, (A + iB)u) = \|Au\|^2 + \|Bu\|^2 + i(Bu, Au) - i(Au, Bu) \\ &= \|Au\|^2 + \|Bu\|^2 + (i[A, B]u, u). \end{aligned}$$

This computation suggests to study the commutator $i[A,B]$. We note that the argument just presented is typical in the proof of L^2 Carleman estimates [**HÖ9**].

By the definition of A and B it easily follows that $i[A,B] = \frac{1}{2}[P^*,P]$. By the commutation formulas for X, $X_\perp$ and V, this commutator may be expressed as

$$\begin{aligned}[P^*,P] &= XVVX - VXXV = VXVX + X_\perp VX - VXVX - VXX_\perp \\ &= V[X_\perp, X] - X^2 = -X^2 + VKV.\end{aligned}$$

Consequently

$$([P^*,P]u,u) = \|Xu\|^2 - (KVu,Vu).$$

If the curvature K is nonpositive, then $[P^*,P]$ is positive semidefinite. More generally, one can try to use the other positive terms in (4.3). Note that

$$\|Au\|^2 + \|Bu\|^2 = \frac{1}{2}(\|Pu\|^2 + \|P^*u\|^2).$$

The identity (4.3) may then be expressed as

$$\|Pu\|^2 = \|P^*u\|^2 + ([P^*,P]u,u).$$

Moving the term $\|Pu\|^2$ to the other side, we have proved the version of the Pestov identity which is most suited for our purposes. The main point in this proof was that the Pestov identity boils down to a standard L^2 estimate based on separating the self-adjoint and skew-adjoint parts of P and on computing one commutator, $[P^*,P]$.

prop_pestov_standard

Proposition 4.12 (Pestov Identity). If (M,g) is a compact oriented surface with smooth boundary, then

$$\|XVu\|^2 - (KVu,Vu) + \|Xu\|^2 - \|VXu\|^2 = 0$$

for any $u \in C^\infty(SM)$ with $u|_{\partial SM} = 0$.

We now show:

Proposition 4.13. Let (M,g) be a simple surface. Then given $\psi \in C^\infty(SM)$ with $\psi|_{\partial SM} = 0$ we have

$$\|X\psi\|^2 - (K\psi,\psi) \geq 0,$$

proposition:nonconj

with equality iff $\psi = 0$.

Proof. Using Santaló's formula, we may write

eq:psisan

(4.4)
$$\int_{SM} ((X\psi)^2 - K\psi^2)\, d\Sigma^3 = \int_{\partial_+ SM} d\mu(x,v) \int_0^{\tau(x,v)} (\dot\psi^2(t) - K(\gamma_{x,v}(t))\psi^2(t))\, dt,$$

where $\psi(t) = \psi(\varphi_t(x,v))$. Observe that $\psi(0) = \psi(\tau(x,v)) = 0$. The argument that follows is done on a fixed geodesic $\gamma_{x,v}$ with $(x,v) \in \partial_+ SM \setminus \partial_0 SM$. Since (M,g) has no conjugate points, the unique solution y to the Jacobi equation $\ddot y + K(\gamma_{x,v}(t))y = 0$ with $y(0) = 0$ and $\dot y(0) = 1$, does **not** vanish for $t \in (0,\tau]$. Hence we may define a function q by writing

$$\psi(t) = q(t)y(t), \text{ for } t \in (0,\tau].$$

Since $\psi(0) = y(0) = 0$ and $\dot y(0) = 1$, it is immediate that q extends smoothly to $t = 0$. Using the Jacobi equation we compute

$$(\ddot\psi + K\psi)\psi = q\frac{d}{dt}(\dot q y^2).$$

Thus integrating by parts we derive

$$\int_0^\tau (\dot\psi^2 - K\psi^2)\,dt = -\int_0^\tau q\frac{d}{dt}(\dot q y^2)\,dt = -[q\dot q y^2]_0^\tau + \int_0^\tau \dot q^2 y^2\,dt$$
$$= \int_0^\tau \dot q^2 y^2\,dt \geq 0,$$

since $\psi(0) = \psi(\tau) = 0$. Equality in the last line holds iff q is constant. Since $\psi(\tau) = qy(\tau) = 0$ with $y(\tau) \neq 0$, it follows that equality holds iff $\psi \equiv 0$. Going back to (4.4), we see that the inequality in the proposition holds with equality iff ψ vanishes.

□

We can now combine these results to prove

thm:mukh

THEOREM 4.14. *Let (M, g) be a simple surface. Then I_0 is injective.*

PROOF. By Proposition 4.11 it suffices to show a vanishing result for $Pu = 0$ with $u|_{\partial SM} = 0$. Proposition 4.12 gives

$$\|XVu\|^2 - (KVu, Vu) + \|Xu\|^2 = 0$$

and combining this with Proposition 4.13 (note that $Vu|_{\partial SM} = 0$) we derive $Vu = Xu = 0$ and hence $u = 0$ as desired.

□

EXERCISE 4.15. Prove the Pestov identity with boundary terms. More precisely, given any $u \in C^\infty(SM)$ show that

$$\|XVu\|^2 - (KVu, Vu) + \|Xu\|^2 - \|VXu\|^2 = (Tu, Vu)_{\partial SM},$$

where T is the vector field on ∂SM given by $T := (V\mu)X + \mu X_\perp$. Check that T is
xca:T
tangent to ∂SM.

G: Check the sign of T given the inward unit normal choice.

We are now going to prove some additional useful properties of the function τ. As in Chapter 3, consider a function $\rho \in C^\infty(M)$ such that it coincides with $M \ni x \mapsto d(x, \partial M)$ in a neighbourhood of ∂M and such that $\rho \geq 0$ and $\partial M = \rho^{-1}(0)$. Clearly $\rho(x) = \nu(x)$ for $x \in \partial M$. Using ρ, we extend ν to the interior of M as $\nu(x) = \nabla\rho(x)$ for $x \in M$.

As before we let $\mu(x, v) := \langle v, \nu(x)\rangle$ and

$$T := V(\mu)X + \mu X_\perp.$$

Note that T is now defined on all SM and agrees with the vector field T defined in Exercise 4.15 on ∂SM. In fact T and V are tangent to every $\partial SM_\varepsilon = \{(x, v) \in SM : \ x \in \rho^{-1}(\varepsilon)\}$, where $M_\varepsilon = \rho^{-1}(-\infty, \varepsilon]$.

EXERCISE 4.16. Prove that $[V, T] = 0$ in SM.

lemma:tau

LEMMA 4.17. *The functions $T\tau$ and $V\tau$ are bounded on $SM \setminus \partial_0 SM$.*

PROOF. We set $h(x, v, t) := \rho(\gamma_{x,v}(t))$ and compute

$$T(h(x, v, 0)) = T(\rho) = V(\mu)X(\rho) + \mu X_\perp(\rho) = V(\mu)d\rho - \mu V(d\rho) = 0$$

since $d\rho(x,v) = \mu(x,v)$. Therefore, there exists a smooth function $a(x,v,t)$ such that

$$T(h(x,v,t)) = ta(x,v,t).$$

Next we apply T to the equality $h(x,v,\tau(x,v)) = 0$ to get

$$T(h(x,v,t))|_{t=\tau(x,v)} + \frac{\partial h}{\partial t}(x,v,\tau(x,v))T\tau = 0.$$

If we write $(y,w) = (\gamma_{x,v}(\tau(x,v)), \dot{\gamma}_{x,v}(\tau(x,v)) \in \partial_- SM$, then the identity above can be re-written as

$$\tau(x,v)a(x,v,\tau(x,v)) + \mu(y,w)T\tau = 0.$$

If $(x,v) \in SM \setminus \partial_0 SM$, then $\mu(y,w) \neq 0$ and we may write

$$T\tau = \frac{-\tau(x,v)a(x,v,\tau(x,v))}{\mu(y,w)}$$

and since

$$0 < \frac{-\tau(x,v)}{\mu(y,w)} \leq \frac{\tau(y,-w)}{\mu(y,-w)}$$

it follows that $T\tau$ is bounded by Lemma 3.8. Since $V(\rho) = 0$, the proof for $V\tau$ is entirely analogous.

□

The following corollary is immediate.

Corollary 4.18. Let (M,g) a non-trapping surface with strictly convex boundary. Given $f \in C^\infty(SM)$, the function

$$u^f(x,v) = \int_0^{\tau(x,v)} f(\varphi_t(x,v))\,dt$$

corollary:uf has Tu^f and Vu^f bounded in $SM \setminus \partial_0 SM$.

4.5. Stability estimate in non-positive curvature

In this section we show how the Pestov identity can be used to derive a basic stability estimate for I_0 when the Gaussian curvature $K \leq 0$. Later on we shall improve this estimate and extend it to include tensors. Given $w \in C^\infty(\partial_+ SM)$ we define its H^1-norm as

$$\|w\|_{H^1}^2 := \|Tw\|^2 + \|Vw\|^2 + \|w\|^2.$$

Theorem 4.19. *Let (M,g) be a non-trapping surface with strictly convex boundary and $K \leq 0$. Then*

$$\|f\| \leq \frac{1}{\sqrt{2}}\|I_0 f\|_{H^1}$$

for any $f \in C^\infty(M)$.

G: We know how to do this in the simple case as well, we need to introduce solutions to the Riccati equations, etc. Maybe this can be done later. At this point it is better to keep things as elementary as possible.

Proof. We wish to use identity from Exercise 4.15; since this identity was derived for smooth functions and u^f fails to be smooth at the glancing we apply it to M_ε (as defined above) and $u = u^f|_{SM_\varepsilon}$ for ε small. Since $K \leq 0$, $Xu^f = -f$ and $Vf = 0$, we derive

$$\|f\|^2_{L^2(SM_\varepsilon)} \leq (Tu^f, Vu^f)_{\partial SM_\varepsilon}.$$

Let $\varepsilon \to 0$ and using Corollary 4.18 we deduce (cf. Exercise 4.20 below)

$$\|f\|^2_{L^2(SM)} \leq (Tu^f, Vu^f)_{\partial SM}.$$

Since $u^f|_{\partial_- SM} = 0$ and $I_0 f = u^f|_{\partial_+ SM}$ we deduce

$$\|f\|^2_{L^2(SM)} \leq (TI_0 f, VI_0 f)_{\partial_+ SM} \leq \frac{1}{2}(\|TI_0 f\|^2 + \|VI_0 f\|^2) \leq \frac{1}{2}\|I_0 f\|^2_{H^1}.$$

and the theorem is proved.

□

Exercise 4.20. Consider the vector field $N := \mu X - V(\mu)X_\perp$ and let F_t be its flow. Show that for ε small enough $F_\varepsilon : \partial SM \to \partial SM_\varepsilon$. Write $F_\varepsilon^* d\Sigma_\varepsilon^2 = q_\varepsilon d\Sigma^2$, where q_ε is smooth and $q_0 = 1$ since F_0 is the identity. Show that

$$(Tu^f, Vu^f)_{\partial SM_\varepsilon} = (q_\varepsilon(Tu^f \circ F_\varepsilon), Vu^f \circ F_\varepsilon)_{\partial SM}.$$

Use Corollary 4.18 and the dominated convergence theorem to conclude that as $\varepsilon \to 0$

xca:justi

$$(q_\varepsilon(Tu^f \circ F_\varepsilon), Vu^f \circ F_\varepsilon)_{\partial SM} \to (Tu^f, Vu^f)_{\partial SM}.$$

Exercise 4.21. Let (M, g) be a non-trapping surface with strictly convex boundary and let $f \in C^\infty(SM)$. Using the Pestov identity with boundary term and Corollary 4.18 show that $XVu^f \in L^2(SM)$. Using that $X_\perp = [X, V]$ conclude that $X_\perp u^f \in L^2(SM)$ and thus $u^f \in H^1(SM)$.

4.6. Examples showing that injectivity fails if conditions are not imposed

CHAPTER 5

Regularity results for the transport equation

chapter4

Here we show all the necessary regularity results for the various transport equations we will be using (including systems and general attenuations).

5.1. Smooth first integrals

Let (M, g) be a non-trapping manifold with strictly convex boundary. Recall that given $w \in C^\infty(\partial_+ SM)$ we set

$$w^\sharp(x, v) = w(\varphi_{-\tau(x,-v)}(x, v)).$$

The function $w^\sharp$ is a first integral of the geodesic flow, i.e. it is constant along its orbits. From the properties of τ we know that $w^\sharp$ is smooth on $SM \setminus \partial_0 SM$, but it may not be smooth at the glancing. In the section we will characterize when smoothness holds. We can easily guess a necessary condition. Indeed, since $w^\sharp(x, v) = w \circ \alpha(x, v)$ for $(x, v) \in \partial_- SM$, we see that if $w^\sharp \in C^\infty(SM)$, then

$$w^\sharp|_{\partial SM} = \begin{cases} w(x, v), & (x, v) \in \partial_+ SM, \\ w \circ \alpha(x, v), & (x, v) \in \partial_- SM \end{cases}$$

must be smooth in ∂SM. We shall show that this condition is also sufficient.

Following [**PU05**] we introduce the operator of even continuation with respect to α:

$$A_+ w(x, v) := \begin{cases} w(x, v), & (x, v) \in \partial_+ SM, \\ w \circ \alpha(x, v), & (x, v) \in \partial_- SM \end{cases}$$

for $w \in C^\infty(\partial_+ SM)$. Clearly $A_+ : C^\infty(\partial_+ SM) \to C(\partial SM)$. We also introduce the space

$$C^\infty_\alpha(\partial_+ SM) := \{w \in C^\infty(\partial_+ SM) : \ A_+ w \in C^\infty(\partial SM)\}.$$

The main result of this section is the following characterization.

Theorem 5.1 ([**PU05**]). *Let (M, g) be a non-trapping manifold with strictly convex boundary. Then*

$$C^\infty_\alpha(\partial_+ SM) = \{w \in C^\infty(\partial_+ SM) : \ w^\sharp \in C^\infty(SM)\}.$$

thm:keyregularity

Proof. We assume (M, g) isometrically embedded in a closed manifold (N, g) of the same dimension as M. Consider some smooth extension W of $A_+ w = w^\sharp|_{\partial SM}$ into SN. Writing $F(t, x, v) = \frac{1}{2} W(\varphi_t(x, v))$, it follows that

$$\begin{aligned} w^\sharp(x, v) &= \frac{1}{2}\left[W(\varphi_{\tau(x,v)}(x, v)) + W(\varphi_{-\tau(x,-v)}(x, v))\right] \\ &= F(\tau(x, v), x, v) + F(-\tau(x, -v), x, v). \end{aligned}$$

Recall that we already know that $w^\sharp$ is smooth in $SM \setminus \partial_0 SM$, so let us discuss what happens at the glancing. Fix some $(x_0, v_0) \in \partial_0 SM$ and use Lemma 3.10 to write

$$w^\sharp(x,v) = F(Q(\sqrt{a(x,v)}, x, v), x, v) + F(Q(-\sqrt{a(x,v)}, x, v), x, v)$$

near (x_0, v_0) in SM. Setting $G(r,x,v) := F(Q(r,x,v),x,v)$, we have

$$w^\sharp(x,v) = G(\sqrt{a(x,v)}, x, v) + G(-\sqrt{a(x,v)}, x, v)$$

near (x_0, v_0) in SM, where G is smooth near $(0, x_0, v_0)$ in $\mathbb{R} \times SN$. Now

$$G(r,x,v) + G(-r,x,v) = H(r^2, x, v)$$

M: this is the fact that an even function $f(t)$ can be written as $g(t^2)$, probably we should give a short proof

where H is smooth near $(0, x_0, v_0)$. This finally shows that

$$w^\sharp(x,v) = H(a(x,v), x, v)$$

near (x_0, v_0) in SM, proving that $w^\sharp$ is smooth near (x_0, v_0) in SM. Since $(x_0, v_0) \in \partial_0 SM$ was arbitrary, we have $w^\sharp \in C^\infty(SM)$. □

5.2. Folds and the scattering relation

G:rephrase?

The original proof of Theorem 5.1 was based on a result in [**Hör85**, Theorem C.4.4] which is in turned underpinned by a result like Lemma 3.11.In this section we explain the original approach in [**PU05**] as it is geometrically quite illuminating.

We start with a general definition from Differential Topology; for what follows we refer to [**Hör85**, Appendix C] for details.

Definition 5.2. Let $f : M \to N$ be a smooth map between manifolds of the same dimension. We say that f is a *Whitney fold* (with fold L) at $m \in M$ if $df_m : T_m M \to T_{f(m)} N$ drops rank simply by one, so $\{x :\ df_x \text{ is singular}\}$ is a smooth hypersurface L near m and $\ker df_m$ is transversal to $T_m L$.

G: This is the definition in [**PU05**], but I am not sure I am happy with the terminology "drops rank simply by one". Hörmander phrases this using the Hessian of the map at the point.

If f has a fold at $m \in M$, there exists an involution $\sigma : M \to M$ (locally defined) such that $\sigma^2 = \mathrm{Id}$, $\sigma \neq \mathrm{Id}$, $f \circ \sigma = f$ and the set of fixed points of σ coincide with L near m. In fact, f has a very simple normal form near m, that is, in suitable coordinates f has a local expression at zero:

$$f(y_1, \dots, y_n) = (y_1, \dots, y_{n-1}, y_n^2).$$

Moreover, the involution is just $\sigma(y', y_n) = (y', -y_n)$, where $y' = (y_1, \dots, y_{n-1})$ and L is determined by $y_n = 0$. Using this normal form it is not hard to show following result holds:

thm:Horfold

Theorem 5.3. [**Hör85**, Theorem C.4.4] *Suppose f has a fold at m and let u be C^∞ in a neighbourhood of $m \in M$. Then, there exists $v \in C^\infty$ in a neighbourhood of $f(m) \in N$ with $v \circ f = u$ iff $u \circ \sigma = u$.*

One implication in the theorem is straightforward: if v exists with $v \circ f = u$, then $u \circ \sigma = v \circ f \circ \sigma = v \circ f = u$, so the content of the theorem is the converse statement.

Let us return now to the situation we are interested in, namely, let (M, g) be a non-trapping manifold with strictly convex boundary. As usual we consider M

isometrically embedded in a closed manifold of the same dimension and we let M_0 be a slightly larger manifold so that it is also non-trapping and with strictly convex boundary. We let its exit time function be τ_0. We define map $\phi : \partial SM \to \partial_- SM_0$ by

$$\phi(x,v) := \varphi_{\tau_0(x,v)}(x,v).$$

This map is C^∞ since $\tau_0|_{SM}$ is C^∞. Here is the main claim about ϕ:

proposition:wf

PROPOSITION 5.4. The map ϕ is a Whitney fold at every point of the glancing $\partial_0 SM$. Moreover, the relevant involution is the scattering relation α.

PROOF. Let us first check that $\phi \circ \alpha = \phi$. Indeed

$$\phi(\alpha(x,v)) = \varphi_{\tau_0(\varphi_{\tilde{\tau}(x,v)}(x,v))}(\varphi_{\tilde{\tau}(x,v)}(x,v)) = \varphi_{\tau_0(\varphi_{\tilde{\tau}(x,v)}(x,v))+\tilde{\tau}(x,v)}(x,v)$$

and since $\tau_0(\varphi_{\tilde{\tau}(x,v)}(x,v)) = \tau_0(x,v) - \tilde{\tau}(x,v)$ the claim follows.

To prove that ϕ is a Whitney fold with fold $\partial_0 SM$ we must now show that given $(x,v) \in \partial_0 SM$ we have

eq:fold

$$\ker d\phi_{(x,v)} \oplus T_{(x,v)}\partial_0 SM = T_{(x,v)}\partial SM. \tag{5.1}$$

To this end, we consider $\xi \in T_{(x,v)}\partial SM$ and we compute using the chain rule

$$d\phi_{(x,v)}(\xi) = d\tau_0(\xi) X(\phi(x,v)) + d\varphi_{\tau_0(x,v)}(\xi)$$

and from this it follows that $\ker d\phi_{(x,v)} = \mathbb{R}X(x,v)$ since $d\tau_0(X(x,v)) = -1$ and $d\varphi_{\tau_0(x,v)}(X(x,v)) = X(\phi(x,v))$. Since we are assuming that ∂M is strictly convex, (5.1) follows directly from Lemma 3.20.

G: This seems incomplete to me, as one also needs to check the condition on the Hessian. An ammeded slightly different proof appears in the paper with Nurlan, Plamen and Gunther; this follows.

Let Σ be a manifold and F a smooth function with 0 as regular value. Let $\mathbb{M} = F^{-1}(0)$ and consider a non-vanishing vector field X on Σ such that $XF(m) = 0$ and $X^2F(m) \neq 0$ for a point $m \in \mathbb{M}$. Let $\mathbb{N}$ be a hypersurface in Σ transversal to X such that $f : \mathbb{M} \to \mathbb{N}$, the projection along integral curves of X is well defined. We claim that f is Whitney fold at m with fold $L = \mathbb{M} \cap (XF)^{-1}(0)$. Indeed, this claim can be checked by looking at the picture in $\Sigma = \mathbb{R}^n$ with $X = \frac{\partial}{\partial x_n}$ and $\mathbb{N} = \{x_n = 0\}$.

Let us apply this observation to the following situation of interest to us. We take $\Sigma = SN$ where N is a closed manifold containing M. Let $\rho : N \to \mathbb{R}$ be a boundary defining function for ∂M as in Chapter 3 so that $\rho^{-1}(0) = \partial M$. If $\pi : SN \to N$ is the canonical projection we set $F := \rho \circ \pi$. We now take as X the geodesic vector field and as m a point $(x,v) \in \partial_0 SM$. We have already computed $XF(m)$ and $X^2F(m)$. Indeed from the proof of Lemma 3.6 we see that $XF(m) = 0$ and $X^2F(m) = \mathrm{Hess}_x \rho(v,v) = -\Pi_x(v,v) < 0$. Note that $\mathbb{M} = F^{-1}(0) = \partial SM$.

We next take as $\mathbb{N} := \partial_- SM_0$ and we see that $\phi : \partial SM \to \partial_- SM_0$ is precisely projection along the geodesic flow. Thus Proposition 5.4 follows.

□

We now explain how to use Theorem 5.3 to give a proof of Theorem 5.1. Consider a function $w \in C^\infty(\partial_+ SM)$ such that $A_+ w \in C^\infty(\partial SM)$. Clearly $A_+ w$ is invariant under α and thus by Theorem 5.3, there is a smooth function v defined in a neighbourhood of $\phi(\partial SM)$ such that $v \circ \phi = w$.

Consider the map $\Psi : SM \to \partial_- SM$ given by $\Psi(x,v) = \varphi_{\tau(x,v)}(x,v)$ and the analogous one $\Psi_0 : M_0 \to \partial_- SM_0$ using τ_0. Note that $w^\sharp = w \circ \alpha \circ \Psi$ and that $\phi \circ \alpha \circ \Psi = \Psi_0|_{SM}$. Hence

$$w^\sharp = w \circ \alpha \circ \Psi = v \circ \phi \circ \alpha \circ \Psi = v \circ \Psi_0|_{SM}$$

and since v and $\Psi_0|_{SM}$ are C^∞ it follows that $w^\sharp$ is C^∞ as desired.

ction:attenuationgeneral

5.3. A general regularity result

Let (M,g) be a non-trapping manifold with strictly convex boundary and let $\mathcal{A} : SM \to \mathbb{C}^{m\times m}$ be a smooth function.

We would like to study regularity results for solutions $u : SM \to \mathbb{C}^m$ to equations of the form

$$Xu + \mathcal{A}u = f$$

where $f \in C^\infty(SM, \mathbb{C}^m)$ and $u|_{\partial SM} = 0$. We shall show that under these conditions u must be C^∞.

As we have done before, consider (M,g) isometrically embedded in a closed manifold (N,g) and we extend $\mathcal{A}$ smoothly to N. Under these assumptions $\mathcal{A}$ on N defines a *smooth* cocycle over the geodesic flow φ_t of (N,g). The cocycle takes values in the group $GL(m,\mathbb{C})$ and is defined as follows: let $C : SN \times \mathbb{R} \to GL(m,\mathbb{C})$ be determined by the following matrix ODE along the orbits of the geodesic flow

$$\frac{d}{dt}C(x,v,t) + \mathcal{A}(\varphi_t(x,v))C(x,v,t) = 0, \qquad C(x,v,0) = \mathrm{Id}.$$

The function C is a *cocycle*:

$$C(x,v,t+s) = C(\varphi_t(x,v),s)\,C(x,v,t)$$

for all $(x,v) \in SN$ and $s,t \in \mathbb{R}$.

EXERCISE 5.5. Prove the cocycle property by using uniqueness for ODEs and the fact that φ_t is a flow.

G: Having this cocycle is just as convenient as having φ_t in all SN. We can then reduce smoothness questions to τ; a recurrent theme.

Consider a slightly larger manifold M_0 engulfing M so that (M_0,g) is still non-trapping with strictly convex boundary and let τ_0 be the exit time of M_0.

G: add in preliminaries that non-trapping with strictly convex boundary is C^2 open. Same for the simple manifolds. For non-trapping just use characterization $Xf > 0$ for some smooth f.

LEMMA 5.6. *The function $R : SM \to GL(m,\mathbb{C})$ defined by*

$$R(x,v) := [C(x,v,\tau_0(x,v))]^{-1},$$

is smooth and satisfies

$$XR + \mathcal{A}R = 0.$$

lemma:matrixintegrating

PROOF. Since $\tau_0|_{SM}$ is smooth and the cocycle C is smooth, the smoothness of R follows right away. To check that R satisfies the stated equation, we use that $\tau_0(\varphi_t(x,v)) = \tau_0(x,v) - t$ together with the cocycle property to obtain

$$R(\varphi_t(x,v)) = [C(\varphi_t(x,v),\tau_0(\varphi_t(x,v)))]^{-1} = C(x,v,t)[C(x,v,\tau_0(x,v))]^{-1}.$$

Diiferentiating at $t = 0$ yields

$$XR = -\mathcal{A}R$$

as desired.

$\square$

Recall that in the scalar case, the attenuated ray transform $I_a f$ of a function $f \in C^\infty(SM, \mathbb{C})$ with attenuation coefficient $a \in C^\infty(SM, \mathbb{C})$ can be defined as the integral

$$I_a f(x,v) := \int_0^{\tau(x,v)} f(\varphi_t(x,v)) \exp\left[\int_0^t a(\varphi_s(x,v))\,ds\right] dt, \quad (x,v) \in \partial_+ SM.$$

Alternatively, we may set $I_a f := u|_{\partial_+ SM}$ where u is the unique solution of the transport equation

$$Xu + au = -f \text{ in } SM, \quad u|_{\partial_- SM} = 0.$$

The last definition generalizes without difficulty to the case of a general attenuation $\mathcal{A}$. Let $f \in C^\infty(SM, \mathbb{C}^n)$ be a vector valued function and consider the following transport equation for $u : SM \to \mathbb{C}^n$,

$$Xu + \mathcal{A}u = -f \text{ in } SM, \quad u|_{\partial_- SM} = 0.$$

On a fixed geodesic the transport equation becomes a linear ODE with zero final condition, and therefore this equation has a unique solution denoted by u^f.

DEFINITION 5.7. The attenuated ray transform of $f \in C^\infty(SM, \mathbb{C}^n)$ is given by

$$I_{\mathcal{A}} f := u^f|_{\partial_+ SM}.$$

def:aXray

It is a simple task to write an integral formula for u^f using a matrix integrating factor as in Lemma 5.6.

LEMMA 5.8. *With R as in Lemma 5.6 we have*

$$u^f(x,v) = R(x,v) \int_0^{\tau(x,v)} (R^{-1} f)(\varphi_t(x,v))\,dt \text{ for } (x,v) \in SM.$$

lemma:formulau

PROOF. A computation using $XR^{-1} = R^{-1}\mathcal{A}$ (which follows easily from $XR + \mathcal{A}R = 0$) and $Xu^f + \mathcal{A}u^f = -f$ yields

$$X(R^{-1}u^f) = (XR^{-1})u^f + R^{-1}u^f = -R^{-1} f.$$

Since $R^{-1}u^f|_{\partial_- SM} = 0$, the lemma follows.

$\square$

REMARK 5.9. It is useful for future purposes, to understand how the formula in the lemma changes, if we consider a different integrating factor, i.e. another invertible matrix R_1 satisfying $XR_1 + \mathcal{A}R_1 = 0$. Since

$$X(R^{-1}R_1) = X(R^{-1})R_1 + R^{-1}X(R_1) = R^{-1}\mathcal{A}R_1 - R^{-1}\mathcal{A}R_1 = 0$$

we derive

$$R_1 = RW^\sharp$$

where $W = R^{-1}R_1|_{\partial_+ SM}$.

Lemma 5.8 shows that u^f is in general as smooth as τ, i.e. smooth everywhere except, perhaps at the glancing. However, the next result will show that if $I_{\mathcal{A}}f = 0$, then u^f is C^∞.

thm:regularity_general

THEOREM 5.10 ([**PSU12**]). *Let (M, g) be a non-trapping manifold with strictly convex boundary. Let $\mathcal{A} \in C^\infty(SM, \mathbb{C}^{m\times m})$ and $f \in C^\infty(SM, \mathbb{C}^m)$ be such that $I_{\mathcal{A}}f = 0$. Then $u^f \in C^\infty(SM, \mathbb{C}^m)$.*

PROOF. It is enough to show that the function $r := R^{-1}u^f$ smooth. According to Lemma 5.8, r satisfies

$$Xr = -R^{-1}f \text{ in } SM, \quad r|_{\partial SM} = 0.$$

Choose $h \in C^\infty(SM, \mathbb{C}^m)$ such that $Xh = -R^{-1}f$. We know such a function exists either by appealing to Proposition 3.14 or by using the enlargement M_0 of M, extending $R^{-1}f$ smoothly to N and setting

$$h(x, v) = \int_0^{\tau_0(x,v)} (R^{-1}f)(\varphi_t(x, v))\, dt \text{ for } (x, v) \in SM.$$

($\tau_0|_{SM}$ is smooth.) Thus the function $h - r$ satisfies $X(h - r) = 0$ and since $(h - r)|_{\partial SM} = h|_{\partial SM} \in C^\infty(\partial SM, \mathbb{C}^m)$, Theorem 5.1 gives that $h - r$ is smooth and thus r is smooth as desired.

□

5.4. The adjoint $I^*_{\mathcal{A}}$.

Let (M, g) be a non-trapping manifold with strictly convex boundary and let $\mathcal{A} : SM \to \mathbb{C}^{m\times m}$ be a smooth matrix attenuation. In this section we shall compute the adjoint $I^*_{\mathcal{A}}$ of

$$I_{\mathcal{A}} : L^2(SM, \mathbb{C}^m) \to L^2_\mu(\partial_+SM, \mathbb{C}^m).$$

We endow $\mathbb{C}^m$ with its standard Hermitian inner product, so the L^2 spaces are defined using this inner product and the usual volume forms $d\Sigma^{2n-1}$ and $d\mu$.

Using the same arguments as in Proposition 4.8 one shows:

PROPOSITION 5.11. The operator $I_{\mathcal{A}}$ extends to a bounded operator

$$I_{\mathcal{A}} : L^2(SM, \mathbb{C}^m) \to L^2_\mu(\partial_+SM, \mathbb{C}^m).$$

Moreover, the following stronger result holds: $I_{\mathcal{A}}$ extends to a bounded operator

$$I_{\mathcal{A}} : L^2(SM, \mathbb{C}^m) \to L^2(\partial_+SM, \mathbb{C}^m).$$

EXERCISE 5.12. Prove the proposition.

LEMMA 5.13. *If $R : SM \to GL(m, \mathbb{C})$ is and such that $XR + \mathcal{A}R = 0$, then*

$$I^*_{\mathcal{A}}h = (R^*)^{-1}(R^*h)^\sharp.$$

PROOF. Recall that given R we can write

$$I_{\mathcal{A}}f = u^f|_{\partial_+SM} = R(x, v)\int_0^{\tau(x,v)} (R^{-1}f)(\varphi_t(x, v))\, dt \text{ for } (x, v) \in \partial_+SM.$$

Let us compute using Santaló's formula:

$$\begin{aligned}(I_{\mathcal{A}}f,h) &= \int_{\partial_+SM}\langle I_{\mathcal{A}}f,h\rangle_{\mathbb{C}^m}\,d\mu\\ &= \int_{\partial_+SM} d\mu\left\langle \int_0^\tau (R^{-1}f)(\varphi_t(x,v))\,dt, R^*h\right\rangle_{\mathbb{C}^m}\\ &= \int_{\partial_+SM} d\mu \int_0^\tau \left\langle R^{-1}f,(R^*h)^\sharp\right\rangle_{\mathbb{C}^m}(\varphi_t(x,v))\,dt\\ &= \int_{SM}\left\langle R^{-1}f,(R^*h)^\sharp\right\rangle_{\mathbb{C}^m}\,d\Sigma^{2n-1}\\ &= (f,(R^*)^{-1}(R^*h)^\sharp)\end{aligned}$$

and thus $I_{\mathcal{A}}^*h = (R^*)^{-1}(R^*h)^\sharp$ as desired.

□

Remark 5.14. Observe that $U=(R^*)^{-1}$ solves the matrix transport equation $XU-\mathcal{A}^*U=0$ and since $(R^*h)^\sharp$ is a first integral of the geodesic flow, $f=I_{\mathcal{A}}^*h$ solves

$$\begin{cases} Xf-\mathcal{A}^*f=0,\\ f|_{\partial_+SM}=h.\end{cases}$$

Remark 5.15 (The matrix weighted X-ray transform). Given a smooth matrix weight $W:SM\to GL(m,\mathbb{C})$ we may also consider a closely related X-ray transform with a matrix weight:

$$I_Wf := \int_0^\tau (Wf)(\varphi_t(x,v))\,dt.$$

Clearly

$$I_{\mathcal{A}}f = RI_{R^{-1}}f$$

where R is any integrating factor so that $XR+\mathcal{A}R=0$.

The adjoint $I_W^*: L^2_\mu(\partial_+SM,\mathbb{C}^m)\to L^2(SM,\mathbb{C}^m)$ is easily computed as above to obtain

remark:equiv_inj

$$I_W^*h = W^*h^\sharp.$$

CHAPTER 6

Vertical Fourier Analysis

G: This needs to be expanded, but the core is here.

Given functions $u, v : SM \to \mathbb{C}$ we consider the L^2 inner product and norm

$$(u, v) = \int_{SM} u\bar{v}\, d\Sigma^3, \qquad \|u\| = (u, u)^{1/2}.$$

Since $X, X_\perp, V$ are volume preserving we have $(Vu, v) = -(u, Vv)$ for $u, v \in C^\infty(SM)$, and if additionally $u|_{\partial SM} = 0$ or $v|_{\partial SM} = 0$ then also $(Xu, v) = -(u, Xv)$ and $(X_\perp u, v) = -(u, X_\perp v)$.

The space $L^2(SM)$ decomposes orthogonally as a direct sum

$$L^2(SM) = \bigoplus_{k\in\mathbb{Z}} H_k$$

where H_k is the eigenspace of $-iV$ corresponding to the eigenvalue k. A function $u \in L^2(SM)$ has a Fourier series expansion

$$u = \sum_{k=-\infty}^{\infty} u_k,$$

where $u_k \in H_k$. Also $\|u\|^2 = \sum \|u_k\|^2$, where $\|u\|^2 = (u, u)^{1/2}$. The even and odd parts of u with respect to velocity are given by

$$u_+ := \sum_{k \text{ even}} u_k, \qquad u_- := \sum_{k \text{ odd}} u_k.$$

In the (x, θ)-coordinates previously introduced using isothermal coordinates we may write

$$u_k(x, \theta) = \left(\frac{1}{2\pi}\int_0^{2\pi} u(x, t)e^{-ikt}\, dt\right) e^{ik\theta} = \tilde{u}_k(x)e^{ik\theta}.$$

Observe that for $k \geq 0$, u_k may be identified with a section of the k-th tensor power of the canonical line bundle; the identification takes u_k into $\tilde{u}_k e^{k\lambda}(dz)^k$ where $z = x_1 + ix_2$.

The next definition introduces holomorphic and antiholomorphic functions with respect to the θ variable.

Definition 6.1. A function $u : SM \to \mathbb{C}$ is said to be (fibre-wise) holomorphic if $u_k = 0$ for all $k < 0$. Similarly, u is said to be (fibre-wise) antiholomorphic if $u_k = 0$ for all $k > 0$.

definition:fhol

Remark 6.2. Later on we will be dealing with situations where we have both types of holomorphicity, namely, the fibre-wise described above (vertical) and holomorphicity due to the underlying Riemann surface structure of (M, g) (horizontal, variable "z" above in isothermal coordinates). In most cases the type of holomorphicity is given by the context, but if necessary we might the use the word fibre-wise to indicate that we mean the one in Definition 6.1.

Let $\Omega_k := H_k \cap C^\infty(SM)$. As in [**GK80**] we introduce the following first order operators

$$\eta_+, \eta_- : C^\infty(SM, \mathbb{C}) \to C^\infty(SM, \mathbb{C})$$

given by

$$\eta_+ := (X + iX_\perp)/2, \qquad \eta_- := (X - iX_\perp)/2.$$

Clearly $X = \eta_+ + \eta_-$. From the structure equations for the frame $\{X, X_\perp, V\}$ one easily derives:

LEMMA 6.3. *The following bracket relations hold*

$$[\eta_\pm, iV] = \pm\eta_\pm, \quad [\eta_+, \eta_-] = \frac{iK}{2}V.$$

lemma:bracketseta

EXERCISE 6.4. Prove the lemma.

LEMMA 6.5. *We have*

$$\eta_+ : \Omega_k \to \Omega_{k+1}, \quad \eta_- : \Omega_k \to \Omega_{k-1}, \quad (\eta_+)^* = -\eta_-.$$

PROOF. We only prove the first statement, the others are left are simple exercises. Take $u \in \Omega_k$. Using the bracket relation $[\eta_+, iV] = \eta_+$ we derive

$$\eta_+(iVu) - iV\eta_+ u = \eta_+ u.$$

Since $iVu = -ku$ we derive

$$-iV\eta_+ u = (k+1)\eta_+ u$$

thus showing $\eta_+ u \in \Omega_{k+1}$.

□

EXERCISE 6.6. Show that X maps even functions to odd functions and odd functions to even functions.

Finally using the expressions (3.4), (3.9) we can derive analogous expression for $\eta_\pm$:

LEMMA 6.7. *In isothermal coordinates (x_1, x_2) we can write the operators $\eta_\pm$ as*

$$\eta_+ = e^{-\lambda} e^{i\theta} \left(\frac{\partial}{\partial z} + i \frac{\partial \lambda}{\partial z} \frac{\partial}{\partial \theta} \right), \qquad \eta_- = e^{-\lambda} e^{-i\theta} \left(\frac{\partial}{\partial \bar{z}} - i \frac{\partial \lambda}{\partial \bar{z}} \frac{\partial}{\partial \theta} \right).$$

In particular

eq:etaplus
$$\eta_+(he^{ik\theta}) = e^{(k-1)\lambda} \partial(he^{-k\lambda}) e^{i(k+1)\theta}, \tag{6.1}$$

eq:etaminus
$$\eta_-(he^{ik\theta}) = e^{-(1+k)\lambda} \overline{\partial}(he^{k\lambda}) e^{i(k-1)\theta}, \tag{6.2}$$

where $h = h(x_1, x_2)$ and

$$\bar{\partial} = \frac{1}{2} \left(\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right),$$

$$\partial = \frac{1}{2} \left(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right).$$

EXERCISE 6.8. Prove the lemma using (3.4), (3.9) and the definitions of $\eta_\pm$.

The Riemannian metric g makes M naturally into a Riemannan surface. The cotangent bundle T^*M of M turns into a complex line bundle over M denoted by κ and known as the *canonical line bundle*. The sections of this bundle consist of $(1,0)$-forms and locally have the form $w(z)dz$. The conjugate bundle $\overline{\kappa}$ is the complex line bundle obtained by letting the complex numbers act by multiplication by their conjugates. The sections of $\overline{\kappa}$ are the $(0,1)$-forms and locally have the form $w(z)d\bar{z}$.

LEMMA 6.9. *For $k \geq 0$, elements in Ω_k can be identified with smooth sections of the bundle $\kappa^{\otimes k}$. Similarly, for $k \leq 0$, elements in Ω_k can be identified with smooth sections of the bundle $\overline{\kappa}^{\otimes -k}$*

PROOF. We only consider the proof for $k \geq 0$, leaving the case $k \leq 0$ as an exercise. Let $\Gamma(M, \kappa^{\otimes k})$ denote the space of smooth sections of the k-th tensor power of the canonical line bundle κ. Given a metric g on M, there is map

$$\varphi_g : \Gamma(M, \kappa^{\otimes k}) \to \Omega_k$$

given by restriction to SM. In other words, an element $f \in \Gamma(M, \kappa^{\otimes k})$ gives rise to a function in SM simply by setting $f_x(\underbrace{v, \dots, v}_{k})$. Let us check what this map looks like in isothermal coordinates. An element of $\Gamma(M, \kappa^{\otimes k})$ is locally of the form $w(z)dz^k$. Consider a tangent vector $\dot{z} = \dot{x}_1 + i\dot{x}_2$. It has norm one in the metric g iff $e^{i\theta} = e^{\lambda}\dot{z}$. Hence the restriction of $w(z)dz^k$ to SM is

$$w(z)e^{-k\lambda}e^{ik\theta}.$$

Observe that φ_g is surjective because given $u \in \Omega_k$ we can write it locally as $u = he^{ik\theta}$ and the local sections $he^{k\lambda}(dz)^k$ glue together to define an element in $\Gamma(M, \kappa^{\otimes k})$. Since it is clearly injective, φ_g is a complex linear isomorphism. □

EXERCISE 6.10. Check that in the proof above, the local sections $he^{k\lambda}(dz)^k$ glue together to define an element in $\Gamma(M, \kappa^{\otimes k})$.

Using the identification from the lemma, we can explicitly conjugate η_- to a $\overline{\partial}$-operator. Observe that there is also a restriction map

$$\psi_g : \Gamma(M, \kappa^{\otimes k} \otimes \bar{\kappa}) \to \Omega_{k-1}$$

which is an isomorphism. The restriction of $w(z)dz^k \otimes d\bar{z}$ to SM is

$$w(z)e^{-(k+1)\lambda}e^{i(k-1)\theta},$$

because $e^{-i\theta} = e^{\lambda}\dot{\bar{z}}$.

Given any holomorphic line bundle ξ over M, there is a $\overline{\partial}$-operator defined on:

$$\overline{\partial} : \Gamma(M, \xi) \to \Gamma(M, \xi \otimes \bar{\kappa}).$$

In particular we can take $\xi = \kappa^{\otimes k}$. Combining this with (6.2) we derive the following commutative diagram:

$$\begin{array}{ccc} \Gamma(M,\kappa^{\otimes k}) & \xrightarrow{\varphi_g} & \Omega_k \\ \downarrow{\scriptstyle\overline{\partial}} & & \downarrow{\scriptstyle\eta_-} \\ \Gamma(M,\kappa^{\otimes k}\otimes\bar{\kappa}) & \xrightarrow{\psi_g} & \Omega_{k-1} \end{array}$$

In other words:

eq:formeta

$$\eta_- = \psi_g\,\overline{\partial}\,\varphi_g^{-1}. \tag{6.3}$$

G: what else should we include here? Since we are not going to discuss closed surfaces maybe there is no point discussing Riemann-Roch etc. Perhaps the anti-canonical line bundle $\kappa^{-1} = \kappa^*$? The case $k \leq 0$?

6.1. The fibrewise Hilbert transform

We will also employ the fiberwise Hilbert transform $H : C^\infty(SM) \to C^\infty(SM)$, defined in terms of Fourier coefficients as

$$Hu_k := -i\,\mathrm{sgn}(k)u_k.$$

Here $\mathrm{sgn}(k)$ is the sign of k, with the convention $\mathrm{sgn}(0) = 0$. Thus, u is holomorphic iff $(\mathrm{Id} - iH)u = u_0$ and antiholomorphic iff $(\mathrm{Id} + iH)u = u_0$.

The following commutator formula for the Hilbert transform and the geodesic vector field, proved in [**PU05**], has been a crucial component for many of the recent developments in 2D geometric inverse problems.

prop:hxcommutator

Proposition 6.11. Let (M, g) be a two dimensional Riemannian manifold. For any smooth function u on SM we have the identity

$$[H, X]u = X_\perp u_0 + (X_\perp u)_0$$

where

$$u_0(x) = \frac{1}{2\pi}\int_{S_x} u(x, v)\,dS_x$$

is the average value.

Proof. It suffices to show that

$$[\mathrm{Id} + iH, X]u = iX_\perp u_0 + i(X_\perp u)_0.$$

Since $X = \eta_+ + \eta_-$ we need to compute $[\mathrm{Id} + iH, \eta_\pm]$, so let us find $[\mathrm{Id} + iH, \eta_+]u$, where $u = \sum_k u_k$. Recall that $(\mathrm{Id} + iH)u = u_0 + 2\sum_{k\geq 1} u_k$. We find:

$$(\mathrm{Id} + iH)\eta_+ u = \eta_+ u_{-1} + 2\sum_{k\geq 0}\eta_+ u_k,$$

$$\eta_+(\mathrm{Id} + iH)u = \eta_+ u_0 + 2\sum_{k\geq 1}\eta_+ u_k.$$

Thus

$$[\mathrm{Id} + iH, \eta_+]u = \eta_+ u_{-1} + \eta_+ u_0.$$

Similarly we find

$$[\mathrm{Id} + iH, \eta_-]u = -\eta_- u_0 - \eta_- u_1.$$

Therefore using that $iX_\perp = \eta_+ - \eta_-$ we obtain

$$[\mathrm{Id} + iH, X]u = iX_\perp u_0 + i(X_\perp u)_0$$

as desired. $\square$

EXERCISE 6.12. Let S be the holomorphic projection operator, i.e. $Su = \sum_{k=0}^{\infty} u_k$. Show that

$$[X, S]u = \eta_- u_0 - \eta_+ u_{-1}.$$

6.2. Relationship between symmetric tensors and functions on SM

Let (M, g) be any compact Riemannian manifold. We denote by $C^\infty(S^m(T^*M))$ the set of smooth complex-valued covariant symmetric tensors of rank m. There is natural map

$$\ell_m : C^\infty(S^m(T^*M)) \to C^\infty(SM)$$

given by

$$\ell_m(h)(x, v) = h_x(\underbrace{v, \dots, v}_{m}).$$

The Levi-Civita connection ∇ acts on a tensor h of rank m as follows:

$$\nabla h(Z, Y_1, \dots, Y_m) = Zh(Y_1, \dots, Y_m) - \sum_{i=1}^{m} h(Y_1, \dots, \nabla_Z Y_i, \dots, Y_m).$$

However, if h is symmetric, ∇h is in general not symmetric. By composing with the symmetrization map σ of a tensor we obtain a map

$$d := \sigma \circ \nabla : C^\infty(S^m(T^*M)) \to C^\infty(S^{m+1}(T^*M)).$$

G: Perhaps use d_s so we can distinguish this operator from the exterior derivative d?

The next lemma shows that the maps ℓ_m intertwine d and X:

lemma:Xd

LEMMA 6.13. *For any $p \in C^\infty(S^{m-1}(T^*M))$ we have $X\ell_{m-1}p = \ell_m dp$.*

PROOF. By definition

$$\ell_m(dp)(x, v) = (dp)_x(v, \dots, v) = (\nabla p)_x(v, \dots, v)$$

since all entries in the tensor ∇p are the same and hence symmetrization is innocuous. Since $\nabla_{\dot\gamma_{x,v}} \dot\gamma_{x,v} = 0$, we have

$$\ell_m(dp)(x, v) = \left.\frac{d}{dt}\right|_{t=0} p_{\gamma_{x,v}(t)}(\dot\gamma_{x,v}(t), \dots, \dot\gamma_{x,v}(t)) = X\ell_m p.$$

$\square$

Suppose now that $\dim M = 2$. We would like to understand the relationship between the maps ℓ_m and the vertical Fourier decomposition introduced above.

We begin observing:

LEMMA 6.14. *Given $h \in C^\infty(S^m(T^*M))$, the function $\ell_m \in C^\infty(SM)$ has*

$$(\ell_m h)_k = 0, \quad \textit{for } |k| \geq m + 1.$$

lemma:finitedegree

Moreover, if m is even (resp. odd), $\ell_m h$ is an even (resp. odd) function of SM.

PROOF. Indeed, observe that $\ell_m h$ is a trigonometric polynomial of degree $\leq m$, hence all its Fourier coefficients are zero for $|k| \geq m+1$. The last claim is ovious. $\square$

PROPOSITION 6.15. Let $m = 2N$ be even. Then the map

$$\ell_m : C^\infty(S^m(T^*M)) \to \bigoplus_{i=-N}^{i=N} \Omega_{2i}$$

is a linear isomorphism. Similarly, if $m = 2N + 1$ is odd, the map

$$\ell_m : C^\infty(S^m(T^*M)) \to \bigoplus_{i=-N-1}^{i=N} \Omega_{2i+1}$$

proposition:elliso is a linear isomorphism.

PROOF. We do the proof for m even; the proof for m odd is analogous. Clearly ℓ_m is injective, since any covariant symmetric m-tensor is determined by its values on m-tuples of the form $(v, \dots, v)$. Hence we need to show it is also surjective.

Suppose that we are given a smooth real-valued function $f \in C^\infty(SM)$ such that $f_k = 0$ for $|k| \geq m + 1$. Since f is real-valued $\bar{f}_k = f_{-k}$. For each $k \geq 1$, the function $f_{-k} + f_k$ gives rise to a unique real-valued symmetric k-tensor F_k such that $\ell_k F_k = f_{-k} + f_k$. This can be seen as follows: recall that a smooth element f_k can be identified with a section of $\kappa^{\otimes k}$ hence, its real part defines a symmetric k-tensor. (For $k = 0$, $\bar{f}_0 = f_0$ is obviously a real-valued 0-tensor.) More explicitly, in the coordinates (x, θ), given $f_k = \tilde{f}_k e^{ik\theta}$ we define

$$F_k := 2\Re(\tilde{f}_k e^{k\lambda}(dz)^k).$$

It is straightforward to check that these local expressions glue together to give a real-valued symmetric k-tensor whose restriction to SM is $f_{-k} + f_k$.

By tensoring with the metric tensor g and symmetrizing it is possible to raise the degree of a symmetric tensor by two. Hence if σ denotes symmetrization, $\alpha F_k := \sigma(F_k \otimes g)$ will be a symmetric tensor of degree $k + 2$ such that $\ell_{k+2}\alpha F_k$ is again $f_k + f_{-k}$ since g restricts as the constant function 1 to SM. Now consider the symmetric m-tensor

$$F := \sum_{i=0}^{m/2} \alpha^i F_{m-2i}.$$

It is easy to check that $\ell_m F = f$ and thus ℓ_m is surjective.

□

6.3. Pestov and Guillemin-Kazhdan energy identities.

We conclude this chapter by discussing the relation between two basic energy identities. On the one hand, we have the Pestov identity from Proposition 4.12 and on the other hand we have the following simple lemma:

lemma_gk_energy_identity LEMMA 6.16. *Let (M, g) be a compact oriented Riemannian surface with possibly non-empty boundary. Then*

$$\|\eta_- u\|^2 = \|\eta_+ u\|^2 - \frac{i}{2}(KVu, u), \quad u \in C^\infty(SM) \textit{ with } u|_{\partial SM} = 0.$$

PROOF. Lemma 6.3 gives the commutator formula

$$[\eta_+, \eta_-] = \frac{i}{2}KV.$$

This implies that, for $u \in C^\infty(SM)$ with $u|_{\partial SM} = 0$

$$\|\eta_- u\|^2 = \|\eta_+ u\|^2 + ([\eta_-, \eta_+]u, u) = \|\eta_+ u\|^2 - \frac{i}{2}(KVu, u). \quad \Box$$

We now show that the Pestov identity applied to $u \in \Omega_k$ is just the Guillemin-Kazhdan identity in Lemma 6.16 for $u \in \Omega_k$ with $u|_{\partial SM} = 0$. Indeed, we compute

$$\|VXu\|^2 = \|V\eta_+ u\|^2 + \|V\eta_- u\|^2 = (k+1)^2\|\eta_+ u\|^2 + (k-1)^2\|\eta_- u\|^2$$

and

$$\|XVu\|^2 - (KVu, Vu) + \|Xu\|^2 = k^2(\|\eta_+ u\|^2 + \|\eta_- u\|^2) + ik(KVu, u) + \|\eta_+ u\|^2 + \|\eta_- u\|^2.$$

The Pestov identity and simple algebra show that

$$2k(\|\eta_+ u\|^2 - \|\eta_- u\|^2) = ik(KVu, u)$$

This is the Guillemin-Kazhdan identity if $k \neq 0$.

In the converse direction, assume that we know the Guillemin-Kazhdan identity for each Ω_k,

$$\|\eta_+ u_k\|^2 - \|\eta_- u_k\|^2 = \frac{i}{2}(KVu_k, u_k), \qquad u \in \Omega_k \text{ with } u|_{\partial SM} = 0.$$

Multiplying by $2k$ and summing gives

$$\sum 2k(\|\eta_+ u_k\|^2 - \|\eta_- u_k\|^2) = \sum ik(KVu_k, u_k).$$

On the other hand, the Pestov identity for $u = \sum_{k=-\infty}^{\infty} u_k$ reads

$$\begin{aligned}&\sum k^2\|\eta_+ u_{k-1} + \eta_- u_{k+1}\|^2 \\ &= \sum(\|\eta_+(Vu_{k-1}) + \eta_-(Vu_{k+1})\|^2 + ik(KVu_k, u_k) + \|\eta_+ u_{k-1} + \eta_- u_{k+1}\|^2).\end{aligned}$$

Notice that

$$k^2\|\eta_+ u_{k-1} + \eta_- u_{k+1}\|^2 = k^2(\|\eta_+ u_{k-1}\|^2 + \|\eta_- u_{k+1}\|^2) + 2k^2\mathrm{Re}(\eta_+ u_{k-1}, \eta_- u_{k+1})$$

and

$$\begin{aligned}&\|\eta_+(Vu_{k-1}) + \eta_-(Vu_{k+1})\|^2 + \|\eta_+ u_{k-1} + \eta_- u_{k+1}\|^2 \\ &= (k^2 - 2k + 2)\|\eta_+ u_{k-1}\|^2 + (k^2 + 2k + 2)\|\eta_- u_{k+1}\|^2 + 2k^2\mathrm{Re}(\eta_+ u_{k-1}, \eta_- u_{k+1}).\end{aligned}$$

Thus the Pestov identity is equivalent with

$$\sum \left[(2k-2)\|\eta_+ u_{k-1}\|^2 - (2k+2)\|\eta_- u_{k+1}\|^2\right] = \sum ik(KVu_k, u_k).$$

This becomes the summed Guillemin-Kazhdan identity after relabeling indices.

CHAPTER 7

Optimal stability

CHAPTER 8

Microlocal aspects, surjectivity of I_0^*

Here we include proofs of the ΨDO nature of $I_0^* I_0$. We might consider doing boundary behaviour (work with François and Richard), although the transmission condition might be too technical. Stability estimates.

8.1. The normal operator

Let (M, g) be a non-trapping manifold with strictly convex boundary. Recall from Chapter 4, that the adjoint of $I : L^2(SM) \to L^2_\mu(\partial_+ SM)$ is given by $I^* h = h^\sharp$. If we let $I_0 = I \circ \ell_0$, then

$$(I_0^* h)(x) = \int_{S_x M} h^\sharp(x, v)\, dS_x(v).$$

Using this we can easily derive an integral expression for the *normal operator*

$$\mathcal{N} := I_0^* I_0 : L^2(M) \to L^2(M).$$

Indeed from the definitions

$$\int_{S_x M} (I_0 f)^\sharp(x, v)\, dS_x(v) = \int_{S_x M} dS_x(v) \int_{-\tau(x,-v)}^{\tau(x,v)} f(\gamma_{x,v}(t))\, dt.$$

Thus

$$(\mathcal{N} f)(x) = \int_{S_x M} dS_x(v) \int_0^{\tau(x,v)} f(\gamma_{x,v}(t))\, dt + \int_{S_x M} dS_x(v) \int_{-\tau(x,-v)}^{0} f(\gamma_{x,v}(t))\, dt$$

and after performing the change of variables $(v, t) \mapsto (-v, -t)$ in the second integral we derive

eq:formulaN (8.1)
$$(\mathcal{N} f)(x) = 2 \int_{S_x M} dS_x(v) \int_0^{\tau(x,v)} f\left(\gamma_{x,v}(t)\right) dt.$$

thm:psidoN

Theorem 8.1. *Let (M, g) be a simple manifold. Then $\mathcal{N} = I_0^* I_0$ is an elliptic pseudo-differential operator (ΨDO) on M^{int} of order -1.*

Proof. From the Schwartz kernel theorem, we know that the operator given by (8.1) must have a Schwartz kernel $K(x, y)$ so that

eq:NSK (8.2)
$$(\mathcal{N} f)(x) = \int_M f(y) K(x, y)\, dV^n(y).$$

For general operators, K could be very singular, in general it is just a distribution on $M^{int} \times M^{int}$, but ΨDOs are characterized by having kernels of a very special type, namely K is what is called a *conormal distribution* with respect to the diagonal of $M^{int} \times M^{int}$. This means that it is smooth off the diagonal and at the diagonal, it has a singularity of a special type which is well understood.

G:amplify? refer to Hörmander Vol 3, or perhaps Melrose?

Our first task is then to try to find out what K looks like. Another glance at (8.1) suggests how to proceed: we would like to switch variables from $(v,t) \in D := S_xM \times (0, \tau(x,v))$ to $y \in M^{int}$ while keeping $x \in M^{int}$ fixed. Thus we introduce the map

$$\psi_x : D \to M^{int}$$

given by

$$\psi_x(v,t) := \gamma_{x,v}(t) = \pi \circ \varphi_t(x,v).$$

The manifold D carries the metric $g_x + dt^2$ and volume form $dS_x \wedge dt$, so we consider naturally the quantity

$$A_x(v,t) := |\det d_{(v,t)}\psi_x|$$

where $d_{(v,t)}\psi_x : T_vS_xM \times \mathbb{R} \to T_xM$ and the determinant is taken with respect to the relevant volume forms. This is an ubiquitous quantity in Riemannian Geometry as it dictates how to compute the volume of balls in M of radius r by integrating over $S_xM \times [0,r]$. It can be easily described in terms of Jacobi fields as follows: if we let $\{e_1 = v, e_2, \dots, e_n\}$ be an orthonormal basis of T_xM and we let J_i be the Jacobi field with initial conditions $J_i(0) = 0$ and $\dot{J}_i(0) = e_i$, then

$$A_x(v,t) := \sqrt{\det(\langle J_i(t), J_j(t)\rangle)_{2\le i,j\le n}}.$$

G: this needs proof given our definition of simple manifold, Chapter 2

If the manifold M is simple, the function $A_x > 0$ for all $(x,v) \in D$ and moreover, ψ_x is a diffeomorphism onto $M^{int} \setminus \{x\}$. Under these conditions, we can change variables in (8.1) and obtain

$$(\mathcal{N}f)(x) = \int_M \frac{f(y)}{A_x(\psi_x^{-1}(y))}\, dV^n(y),$$

and thus we can identify K with

$$K(x,y) = \frac{2}{A_x(\psi_x^{-1}(y))}.$$

Clearly for $x \neq y$ this is a smooth function since A_x and ψ_x are smooth, and both depend smoothly on x.

To gain further insight into the singularity of K at $x = y$ and the ΨDO nature of $\mathcal{N}$, let us suppose that (M,g) that is a strictly convex domain in Euclidean space $\mathbb{R}^n$. In this case $A_x(v,t) = t^{n-1}$ and $\psi_x(v,t) = x + vt$. Thus

$$K(x,y) = \frac{2}{|x-y|^{n-1}}.$$

It follows that

$$\mathcal{N}f = 2f * \frac{1}{|x|^{n-1}}.$$

where f is extended by zero outside M and $*$ stands for convolution in $\mathbb{R}^n$. If we let $\mathcal{F}$ be Fourier transform, standard properties give

$$\mathcal{N}f = 2\mathcal{F}^{-1}\mathcal{F}\left(f * \frac{1}{|x|^{n-1}}\right) = 2\mathcal{F}^{-1}\left(\mathcal{F}(f)\mathcal{F}\left(\frac{1}{|x|^{n-1}}\right)\right)$$

and it is well known that

$$\mathcal{F}\left(\frac{1}{|x|^{n-1}}\right) = c_n|\xi|^{-1},$$

where c_n is some constant. Hence we can write

$$\mathcal{N}f = 2c_n\mathcal{F}^{-1}(\mathcal{F}(f)|\xi|^{-1}) = \int\int e^{i\xi\cdot(x-y)}p(x,\xi)f(y)\,dy\,d\xi.$$

This is precisely the formula that describes $\mathcal{N}$ as a ΨDO with symbol $p(x,\xi) = 2c_n|\xi|^{-1}$. The fact that the symbol has this form means that $\mathcal{N}$ has order -1 and is *elliptic*.

For a general simple metric g, pretty much the same picture holds, but we need to work a little harder to derive it. First we elucidate the behaviour of $A_x(v,t)$ at $t=0$. This is easy if we introduce the *exponential map* given by $\exp_x : T_xM \to M$, $\exp_x(tv) = \psi_x(v,t)$. At first it seems we have not achieved much, but if we recall that the polar coordinate change $q : S_xM \times \mathbb{R} \to T_xM$, $q(v,t) = tv$ has Jacobian t^{n-1} then we see that

$$A_x(v,t) = |\det(d_{tv}\exp_x)|t^{n-1}.$$

Since $d_0\exp_x = \mathrm{id}$ and $t = d(x,y)$ we derive a Schwartz kernel of the form

eq:Ksing
$$K(x,y) = \frac{2}{[d(x,y)]^{n-1}|\det d_{q\psi_x^{-1}(y)}\exp_x|} \tag{8.3}$$

with singularity of type $1/[d(x,y)]^{n-1}$.

At this point we shall need the following lemma:

LEMMA 8.2. *In local coordinates, there are smooth functions $G_{ij}(x,y)$ such that $G_{ij}(x,x) = g_{ij}(x)$ and*

$$[d(x,y)]^2 = G_{ij}(x,y)(x-y)^i(x-y)^j.$$

EXERCISE 8.3. Prove the lemma. Hint: do a Taylor expansion at x of the function $f(y) = \|\exp_x^{-1}(y)\|^2$.

To show that we have a ΨDO we need to localize matters by considering two cut-off functions $\psi(x)$ and $\phi(y)$ supported in charts of M^{int} (since M is simple, M^{int} is in fact diffeomorphic to $\mathbb{R}^n$, so one chart will do). If we let

$$\tilde{K}(x,y) := \psi(x)K(x,y)\sqrt{\det g(y)}\phi(y)$$

we need to show that the operator defined by $\tilde{K}$ is a ΨDO in $\mathbb{R}^n$. (Recall that in local coordinates $dV^n = \sqrt{\det g(y)}dy$.)

The lemma, together with (8.3), shows that in local coordinates, the kernel $\tilde{K}$ satisfies an estimate of the form

$$|\partial_x^\alpha\partial_y^\beta\tilde{K}(x,x-y)| \le C_{\alpha,\beta}|y|^{-n+1-|\beta|}.$$

This is the property needed to show that the symbol

$$p(x,\xi) = \int \tilde{K}(x,x-y)e^{-i\xi\cdot y}\,dy$$

is a classical symbol of order -1. The last part of the proof consists in computing the *principal symbol* of $\mathcal{N}$. This is the leading term in a suitable expansion of $p(x,\xi)$. If we let

G: need to give reference or add a proof, perhaps as a separate lemma?

G: add details

$$\tilde{K}_0(x,y) := \frac{2\psi(x)\sqrt{\det g(x)}\phi(y)}{(g_{ij}(x)(x-y)^i(x-y)^j)^{(n-1)/2}},$$

then it is clear that $\tilde{K} - \tilde{K}_0$ will have a singularity of type $|x-y|^{-n+2}$. Hence the operators determined by $\tilde{K}$ and $\tilde{K}_0$ share the same principal symbol and we infer that the principal symbol is given (up to a constant) by

$$\int_{\mathbb{R}^n} \frac{e^{-i\xi\cdot y}\sqrt{\det g(x)}}{(g_{ij}(x)y^i y^j)^{(n-1)/2}}\, dy.$$

Using that $\mathcal{F}\left(\frac{1}{|x|^{n-1}}\right) = d_n|\xi|^{-1}$ and a substitution shows that the principal symbol of $\mathcal{N}$ is

$$c_n|\xi|_g^{-1}$$

and thus $\mathcal{N}$ is elliptic.

$\square$

8.2. Surjectivity of I_0^*

Let (M,g) be a compact simple manifold. In this section we prove a fundamental surjectivity result for I_0^* which underpins the successful solution of many geometric inverse problems in 2D.

As usual, we consider (M,g) isometrically embedded into a closed manifold (N,g). Since M is simple, there is an open neighborhood U_1 of M in N, such that its closure $M_1 := \overline{U}_1$ is a compact simple manifold. Let $I_{0,1}$ denote the geodesic ray transform associated to (M_1,g) and let $\mathcal{N}_1 = I_{0,1}^* I_{0,1}$.

Following [**PU05**] we may cover (N,g) with finitely many simple open sets U_k with $M \subset U_1$, $M \cap \overline{U}_j = \emptyset$ for $j \geq 2$, and consider a partition of unity $\{\varphi_k\}$ subordinate to $\{U_k\}$ so that $\varphi_k \geq 0$, $\operatorname{supp}\varphi_k \subset U_k$ and $\sum \varphi_k^2 = 1$. We pick φ_1 such that $\varphi_1 \equiv 1$ on a neighborhood of M compactly supported in U_1. Hence, for I_0, k the ray transform associated to $(\overline{U}_k, g)$, we can define

eq:P

$$Pf := \sum_k \varphi_k (I_{0,k}^* I_{0,k})(\varphi_k f), \qquad f \in C^\infty(N). \tag{8.4}$$

Each operator $I_{0,k}^* I_{0,k} : C_c^\infty(U_k) \to C^\infty(U_k)$ is an elliptic ΨDO of order -1 with principal symbol $c_n|\xi|^{-1}$, and hence so is P. Having P defined on a closed manifold is convenient, since one can use standard mapping properties for ΨDOs without having to worry about boundary behaviour. For instance for P defined by (8.4) we have

G:Perhaps make this into a lemma and give a proof? The squares in the partition of unity is to make the principal symbol come as claimed but it is not strictly necessary

$$P : H^s(N) \to H^{s+1}(N) \qquad \text{for all } s \in \mathbb{R},$$

where $H^s(N)$ denotes the standard L^2 Sobolev space of the closed manifold N (when s is a nonnegative integer, $H^s(N)$ can be identified with the set of $u \in L^2(N)$ such that $Du \in L^2(N)$ for all differential operators D of order $\leq s$ with coefficients in $C^\infty(N)$, see [**Tay11**] for the definition for arbitrary $s \in \mathbb{R}$).

Remark 8.4. There are other natural ways of producing an ambient operator P with the desired properties. Let ψ be a smooth function on N with support contained in U_1 and such that it is equal to 1 near M. Let Δ_g denote the Laplacian of (N,g). Define

$$P := \psi\mathcal{N}_1\psi + (1-\psi)(1+\Delta_g)^{-1/2}(1-\psi).$$

As we have already mentioned, $\mathcal{N}_1$ is an elliptic ΨDO of order -1 on U_1 and thus P is also an elliptic ΨDO of order -1 in N. Instead of $(1+\Delta_g)^{-1/2}$ we could have used any other invertible self-adjoint elliptic ΨDO of order -1.

lemma:Fred

LEMMA 8.5. *The operator P is injective. Moreover, $P : C^\infty(N) \to C^\infty(N)$ is a bijection.*

PROOF. Since P is elliptic, an element in the kernel of P must be smooth. Let f be such that $Pf = 0$ and write

$$\begin{aligned} 0 = (Pf, f)_{L^2(N)} &= \sum_k (\mathcal{N}_k(\varphi_k f), \varphi_k f)_{L^2(\overline{U}_k)} \\ &= \sum_k \|I_{0,k}(\varphi_k f)\|_{L^2_\mu(\partial_+ S\overline{U}_k)}. \end{aligned}$$

Hence $I_{0,k}(\varphi_k f) = 0$ for each k. Using injectivity of I_0 on simple manifolds it follows that $\varphi_k f = 0$ for each k and thus $f = 0$.

Since P is elliptic and self-adjoint, it has index zero. Thus injectivity implies surjectivity and P is a bijection. ∎

G: We might wish to give some background, certainly some references.

We are now ready to prove the main result of this section.

thm:I0*onto

THEOREM 8.6. *Let (M, g) be a simple manifold. Then the operator*

$$I_0^* : C_\alpha^\infty(\partial_+ SM) \to C^\infty(M)$$

is surjective.

PROOF. Let $h \in C^\infty(M)$ be given and extend it smoothly to a smooth function in N, still denoted by h. By Lemma 8.5 there is a unique $f \in C^\infty(N)$ such that $Pf = h$. Let $w_1 := I_{0,1}(\varphi_1 f)$. Clearly $w_1^\sharp|_{SM} \in C^\infty(SM)$ and we let $w := w_1^\sharp|_{\partial_+ SM}$. We must have

$$w^\sharp = w_1^\sharp|_{SM}$$

since both are first integrals of the geodesic flow and they agree on $\partial_+ SM$. Hence $w \in C_\alpha^\infty(\partial_+ SM)$. To complete the proof we must check that $I_0^* w = h$. To this end, we write for $x \in M$:

$$\begin{aligned} (I_0^* w)(x) &= \int_{S_x M} w^\sharp(x, v)\, dS_x(v) \\ &= \int_{S_x M} w_1^\sharp(x, v)\, dS_x(v) \\ &= (I_{0,1}^* w_1)(x) \\ &= I_{0,1}^* I_{0,1}(\varphi_1 f)(x) \\ &= P(f)(x) \\ &= h(x), \end{aligned}$$

where in the penultimate line we used (8.4) and that $x \in M$. ∎

8.3. Adding an invertible matrix weight

Virtually everything we have done in this section so far can be upgraded to include an invertible matrix weight. Let (M,g) be a non-trapping manifold with strictly convex boundary and let $W : SM \to GL(m,\mathbb{C})$ be a smooth invertible matrix weight. Consider the geodesic X-ray transform with a matrix weight:

$$I_W f := \int_0^\tau (Wf)(\varphi_t(x,v))\,dt.$$

The adjoint $I_W^* : L^2_\mu(\partial_+ SM, \mathbb{C}^m) \to L^2(SM,\mathbb{C}^m)$ is (cf. Chapter 5):

$$I_W^* h = W^* h^\sharp.$$

If we let $I_{0,W} = I_W \circ \ell_0$, then

$$(I_{0,W}^* h)(x) = \int_{S_xM} W^* h^\sharp(x,v)\,dS_x(v).$$

Using this we can derive as before an integral expression for the *normal operator*

$$\mathcal{N}_W := I_{0,W}^* I_{0,W} : L^2(M,\mathbb{C}^m) \to L^2(M,\mathbb{C}^m).$$

Indeed from the definitions

$$\int_{S_xM} (I_0 f)^\sharp(x,v)\,dS_x(v) = \int_{S_xM} W^*(x,v) dS_x(v) \int_{-\tau(x,-v)}^{\tau(x,v)} W(\varphi_t(x,v)) f(\gamma_{x,v}(t))\,dt.$$

Thus

eq:formulaNW

$$(\mathcal{N}_W f)(x) = \int_{S_xM} W^*(x,v) \left(\int_{-\tau(x,-v)}^{\tau(x,v)} W(\varphi_t(x,v)) f(\gamma_{x,v}(t))\,dt \right) dS_x(v). \tag{8.5}$$

thm:Welliptic

THEOREM 8.7. *Let (M,g) be a simple manifold. Then $\mathcal{N}_W = I_{0,W}^* I_{0,W}$ is an elliptic pseudo-differential operator (ΨDO) on M^{int} of order -1.*

PROOF. TODO. For matrix weights this non-where to be found in the literature in this form. Of course the ideas are the same but the weight creates additional work. The integral in (8.5) has to be splitted in two corresponding to positive times and negative times. For each one we compute the kernel. For the positive we get something like the old stuff times

$$W^*(x,h(x,y)) W(y, d_{h(x,y)} \exp_x(h(x,y))$$

where

$$h(x,y) := \frac{\exp_x^{-1}(y)}{|\exp_x^{-1}(y)|}.$$

Hence we need to work a little more analyzing the singularity at the diagonal. For the scalar case this has been done of course and there are several references. The book project by Gunther and Plamen has a nice section discussing this in the Euclidean case. For the matrix case we dodged the bullet in IMRN, but something similar appears in the AMJ paper with Hanming.

G: I think we should do this in detail once and for all

□

With this result in hand, Theorem 8.6 can be upgraded to:

THEOREM 8.8. *Let (M, g) be a simple manifold. Then $I_{0,W}$ is injective on $L^2(M, \mathbb{C}^m)$ if and only if*

$$I_{0,W}^* : C_\alpha^\infty(\partial_+ SM, \mathbb{C}^m) \to C^\infty(M, \mathbb{C}^m)$$

thm:IOW*onto *is surjective.*

PROOF. Let $f \in L^2(M, \mathbb{C}^m)$ be such that $I_{0,W} f = 0$. Consider a slightly larger simple manifold $\widetilde{M}$ engulfing M and extend W smoothly to it. Extending f by zero to $\widetilde{M}$ we see that

$$I_{0,\widetilde{W}} f = 0$$

and thus $\mathcal{N}_{\widetilde{W}} f = 0$. By Theorem 8.7, $\mathcal{N}_{\widetilde{W}}$ is elliptic and hence f is smooth in the interior of $\widetilde{M}$ and hence on M. Assume now that $I_{0,W}^*$ is surjective. Then there exists $h \in C_\alpha^\infty(\partial_+ SM, \mathbb{C}^m)$ such that $I_{0,W}^* h = f$. Now write

$$0 = (I_{0,W} f, h) = (f, I_{0,W}^* h) = (f, f)$$

and thus $f = 0$.

Assume now that $I_{0,W}$ is injective. We wish to show that $I_{0,W}^*$ is surjective. This part of the proof proceeds exactly as the proof of Theorem 8.6. We construct an elliptic operator $P : C^\infty(N, \mathbb{C}^m) \to C^\infty(N, \mathbb{C}^m)$ and we show it is a bijection by showing first that it has trivial kernel. The surjectivity of P implies the surjectivity of $I_{0,W}^*$ exactly as in the proof of Theorem 8.6.

$\square$

EXERCISE 8.9. Fill in the details in the proof of Theorem 8.8.

Let us state explicitly the following rephrasing of Theorem 8.8 that shall be very useful later on.

COROLLARY 8.10. Let (M, g) be a simple manifold with $I_{0,W}$ injective. Given $f \in C^\infty(M, \mathbb{C}^m)$ there exists $u \in C^\infty(SM, \mathbb{C}^m)$ such that

$$\begin{cases} Xu + \mathcal{A}u = 0, \\ u_0 = f \end{cases}$$

cor:transportA where $\mathcal{A} = -X(W^*)(W^*)^{-1}$ and $u_0 = \ell_0^* u = \int_{S_x M} u(x, v)\, dS_x(v)$.

PROOF. By Theorem 8.8 there is $h \in C_\alpha^\infty(\partial_+ SM, \mathbb{C}^m)$ such that $(W^* h^\sharp)_0 = f$. We let $u := W^* h^\sharp \in C^\infty(SM, \mathbb{C}^m)$. Since $Xh^\sharp = 0$, the function u satisfies

$$Xu = X(W^*) h^\sharp = -\mathcal{A}u$$

and the corollary follows.

$\square$

CHAPTER 9

Inversion formulas and range

This chapter summarizes the inversion formulas and range from Pestov-Uhlmann. Discussion of the operator W, open problems. Range for tensors as in IMRN paper. One could add a connection as we did with François recently. Here it might makes sense to discuss François numerical work briefly; having nice pictures will certainly enhance the book!

9.1. The derivative cocycle in 2D

Let (N,g) be a closed oriented Riemannian surface. The usual Jacobi equation $\ddot{y}+K(t)y=0$ determines the differential of the geodesic flow φ_t: if we fix $(x,v)\in SM$ and $T_{(x,v)}(SM)\ni\xi=-\xi_1X_\perp+\xi_2V$ then

$$d\varphi_t(\xi)=-y(t)X_\perp(\varphi_t(x,v))+\dot{y}(t)V(\varphi_t(x,v)),$$

where $y(t)$ is the unique solution to the Jacobi equation with initial conditions $y(0)=\xi_1$ and $\dot{y}(0)=\xi_2$ and $K(t)=K(\pi\circ\varphi_t(x,v))$. The differential of the geodesic flow determines an $SL(2,\mathbb{R})$-cocyle Ψ over φ_t with infinitesimal generator:

$$\mathcal{A}:=\left(\begin{array}{cc}0 & -1\\ K & 0\end{array}\right).$$

We may write Ψ as

$$\Psi(x,v,t)=\left(\begin{array}{cc}a & b\\ \dot{a} & \dot{b}\end{array}\right)$$

where the functions $a,b:SN\times\mathbb{R}\to\mathbb{R}$ satisfy the Jacobi equation in the t-variable and $a(x,v,0)=1$, $\dot{a}(x,v,0)=0$ and $b(x,v,0)=0$, $\dot{b}(x,v,0)=1$. Clearly the cocycle Ψ can be identified with $d\varphi_t$ acting on the kernel of the contact 1-form of the geodesic flow (i.e. the 2-plane spanned by $X_\perp$ and V).

G: Complement this section with 4.2 on the Jacobi equations in the notes with Will (that can be copied almost verbatim keeping in mind $H=-X_\perp$ or bring onto section 2 in preliminaries.

The functions a,b have the following expansions around $t=0$:

PROPOSITION 9.1. There exist smooth functions $R(x,v,t)$ and $P(x,v,t)$ such that

eq:aexp (9.1) $$a(x,v,t)=1-K(x)\frac{t^2}{2}-d_xK(v)\frac{t^3}{6}+t^4R(x,v,t),$$

eq:bexp (9.2) $$b(x,v,t)=t-K(x)\frac{t^3}{6}+t^4P(x,v,t).$$

Moreover, if we consider Taylor expansions at $t=0$ up to order N:

$$a(x,v,t)=\sum_{k=0}^{N} a_k(x,v)t^k+\mathcal{O}(t^{N+1}),$$

$$b(x,v,t)=\sum_{k=0}^{N} b_k(x,v)t^k+\mathcal{O}(t^{N+1}),$$

then $a_k(x,v)$ is the restriction to SM of a symmetric tensor of order k and $b_k(x,v)$ is the restriction to SM of a symmetric tensor of order $k-1$.

prop:ab

PROOF. Equations (9.1) and (9.2) follow right away from the Taylor expansion using the differential equation $\ddot{y}+Ky=0$ and the initial conditions for a and b.

By differentiating the equation $\ddot{y}+Ky=0$ repeatedly we obtain:

eq:kk+2

$$a_{k+2}(x,v)=-\sum_{i=0}^{k} d(i,k)X^i(K)(x,v)a_{k-i}(x,v) \tag{9.3}$$

for some coefficients $d(i,k)$ whose precise value is irrelevant for us. We can now show the claim about the functions a_k by induction on k. Indeed for $k=0$, $a_0=1$, so assume that a_{k-i} is the restriction to SM of a symmetric tensor of order $k-i$. Since K only depends on x, $X^i(K)$ is also the restriction to SM of a symmetric tensor of order i. Hence using (9.3) we see that a_{k+2} is the restriction to SM of a symmetric tensor of order $k+2$. The proof for b_k is analogous.

□

9.2. The smoothing operator W

G: We have too many W's scattered throughout the text, eventually we will need to polish the notation.

Let (M,g) be a non-trapping surface with strictly convex boundary. We consider as usual (M,g) sitting inside a closed oriented surface (N,g).

We shall define an operator $W: C_c^\infty(M^{int})\to C^\infty(M)$, where $C_c^\infty(M^{int})$ denotes the space of C^∞ functions with compact support inside the interior of M. This operator will have the property that it extends as a smoothing operator $W: L^2(M)\to C^\infty(M)$ when M is free of conjugate points, and it will play an important role in the Fredholm inversion formulas in the next section.

Given $f\in C_c^\infty(M^{int})$ define for $x\in M$.

$$(Wf)(x):=(X_\perp u^f)_0(x)=\ell_0(X_\perp u^f)(x).$$

Observe that since f has compact support contained in the interior of M, then function $u^f\in C^\infty(SM)$ and thus $Wf\in C^\infty(M)$. This is the way W was introduced in [**PU04**], however we note that we can just as well define $W: C^\infty(M)\to C^\infty(M)$. This is because while u^f might be smooth at the glancing, the odd part u_-^f of u^f actually belongs to $C^\infty(SM)$ so we could simply set

$$Wf:=(X_\perp u_-^f)_0.$$

and the two definitions clearly agree on $C_c^\infty(M^{int})$.

EXERCISE 9.2. Prove that $u_-^f\in C^\infty(SM)$.

G: Perhaps we should set this exercise as a lemma in the chapter on regularity?

EXERCISE 9.3. Show that $Wf = i(\eta_- u_1^f - \eta_+ u_{-1}^f)$.

We now give an integral representation for W when (M,g) is a simple surface. We will use the functions a, b introduced in the previous section in the context of the derivative cocycle. Note that (M,g) has no conjugate points iff $b(x,v,t) \neq 0$ for $t \in [-\tau(x,-v), \tau(x,v)]$, $t \neq 0$ and $(x,v) \in SM$.

PROPOSITION 9.4. Let (M,g) be a simple surface. The function $w(x,v,t) := V\left(\frac{a(x,v,t)}{b(x,v,t)}\right)$ is smooth on the set of (x,v,t) such that $(x,v) \in SM$ and $t \in [-\tau(x,-v), \tau(x,v)]$. Moreover,

$$(Wf)(x) = \frac{1}{2\pi}\int_{S_xM}\int_0^{\tau(x,v)} w(x,v,t) f(\gamma_{x,v}(t))\, dt dS_x(v).$$

PROOF. For the first part we just need to study the smoothness of w near $t = 0$. Using (9.1) and (9.2) we see that

$$w(x,v,t) = \frac{V(a)}{b} - \frac{aV(b)}{b^2} = \frac{t^2(-V(d_xK(v))/6 + tVR)}{1 - K(x)t^2/6 + t^3P} - \frac{at^2VR}{(1 - K(x)t^2/6 + t^3P)^2}$$

and thus $w(x,v,t)$ is smooth.

To derive the integral formula for W we just use its definition and write

eq:W1

$$(Wf)(x) = \frac{1}{2\pi}\int_{S_xM} X_\perp \int_0^{\tau(x,v)} f(\gamma_{x,v}(t))\, dt dS_x(v). \tag{9.4}$$

Since f has compact support contained in the interior of M:

$$X_\perp \int_0^{\tau(x,v)} f(\gamma_{x,v}(t))\, dt = \int_0^{\tau(x,v)} X_\perp(f(\gamma_{x,v}(t)))\, dt.$$

Now observe that

$$X_\perp(f(\gamma_{x,v}(t))) = df \circ d\pi \circ d\varphi_t(X_\perp(x,v))$$

and similarly

$$V(f(\gamma_{x,v}(t))) = df \circ d\pi \circ d\varphi_t(V(x,v)).$$

But

$$d\pi \circ d\varphi_t(X_\perp(x,v)) = -ai\dot\gamma_{x,v}(t)$$

and

$$d\pi \circ d\varphi_t(V(x,v)) = bi\dot\gamma_{x,v}(t),$$

therefore

$$X_\perp(f(\gamma_{x,v}(t)) = -\frac{a}{b}V(f(\gamma_{x,v}(t)).$$

Inserting the last expression into (9.4) we derive

$$(Wf)(x) = \frac{1}{2\pi}\int_{S_xM}\int_0^{\tau(x,v)} -\frac{a}{b}V(f(\gamma_{x,v}(t)))\, dt dS_x(v)$$

and since

$$\int_{S_xM} V\left(\int_0^{\tau(x,v)} \frac{a}{b} f(\gamma_{x,v}(t))\, dt\right) dS_x(v) = 0$$

we finally obtain

$$(Wf)(x) = \frac{1}{2\pi} \int_{S_x M} \int_0^{\tau(x,v)} V\left(\frac{a}{b}\right) f(\gamma_{x,v}(t))\, dt dS_x(v)$$

as desired.

□

REMARK 9.5. The proof above was done assuming that $f \in C_c^\infty(M^{int})$ but we could have carried out the same proof with $f \in C^\infty(M)$, i.e. smooth and supported all the way to the boundary. This would have produced two additional boundary terms $X_\perp(\tau) f(\gamma_{x,v}(\tau(x,v))$ and $V(\tau)\frac{a(x,v,\tau(x,v))}{b(x,v,\tau(x,v))} f(\gamma_{x,v}(\tau(x,v))$. However these two terms cancel out due to the following fact that is easily checked:

eq:abtau

$$a(x,v,\tau(x,v))V(\tau) + b(x,v,\tau(x,v))X_\perp(\tau) = 0. \tag{9.5}$$

Hence we get the same integral formula for $f \in C^\infty(M)$.

EXERCISE 9.6. Prove identity (9.5).

We now analyze the kernel w in more detail.

LEMMA 9.7. *Let (M, g) be simple surface. Then*

$$b(x,v,t) = t \det(d_{tv} \exp_x).$$

lemma:bQ

Moreover, there exists $Q \in C^\infty(TM)$ such that $w(x,v,t) = tQ(x,tv)$.

PROOF. The first claim follows from what explained in the proof of Theorem 8.1 since in the notation of that proof $A_x(v,t) = b(x,v,t)$.

The second claim is essentially a corollary of Proposition 9.1. Since a_k and b_k are restrictions of symmetric tensors of order k and $k-1$ respectively and $V(a_k)$ and $V(b_k)$ have the same property, each one of the functions $\frac{V(a)}{b}$ and $\frac{aV(b)}{b^2}$ can be written as $tQ(x,tv)$ with Q smooth in TM. Thus w can also be written in this form. □

G: Is there another proof of this?

PROPOSITION 9.8. Let (M, g) be a simple surface. The operator W extends to a smoothing operator $W : L^2(M) \to C^\infty(M)$.

PROOF. We will make a change of variables that transforms the integral expression for W into something of the form

$$(Wf)(x) = \int_M k(x,y) f(y)\, dV^2(y)$$

with k smooth. The change of variables is exactly the same we used in the proof of Theorem 8.1. Using the notation from that proof we set $y = \psi_x(v,t) = \exp_x(tv)$ and we see that $A_x(v,t) = b(x,v,t)$. Thus

$$(Wf)(x) = \int_M k(x,y) f(y)\, dV^2(y)$$

where

$$k(x,y) := \frac{w(x, \psi_x^{-1}(y))}{b(x, \psi_x^{-1}(y))}.$$

Using Lemma 9.7 we can re-write this as

$$k(x,y) = \frac{Q(x, \exp_x^{-1}(y))}{\det(d_{\exp_x^{-1}(y)} \exp_x)}$$

that clearly exhibits k as a smooth function.

□

9.2.1. The adjoint W^*. The adjoint of W with respect to the L^2-inner product of M can be easily computed:

LEMMA 9.9. *Given $h \in C_c^\infty(M^{int})$ we have*

$$W^* h = \left(u^{X_\perp h}\right)_0 .$$

PROOF. TODO

□

9.3. Fredholm formulas

G: the next bit is the only one so far needing the Hilbert transform H and the bracket $[H, X]$.

THEOREM 9.10. *Let (M, g) be a non-trapping surface with strictly convex boundary. Then given $f \in C^\infty(M)$ we have*

$$f + W^2 f = -(X_\perp w^\sharp)_0$$

where

$$w := \frac{1}{2}[H\,(I_0 f)_-]|_{\partial_- SM} \circ \alpha$$

and $(I_0 f)_-$ denotes the odd continuation of $I_0 f$ to ∂SM.

PROOF. The proof essentially consists in applying the Hilbert transform H twice to the equation $X u_-^f = -f$ and use Proposition 6.11.

Applying H once we derive $(Hf = 0)$:

eq:H1 (9.6)
$$X H u_-^f = -W f$$

since $(u_-^f)_0 = 0$. Applying H again we obtain

$$X H^2 u_-^f + (X_\perp H u_-^f)_0 = 0$$

and using that $H^2 u_-^f = -u_-^f$ we derive

eq:H2 (9.7)
$$-f = X u_-^f = (X_\perp H u_-^f)_0.$$

Using (9.6) we see that

$$H u_-^f = u^{Wf} + w^\sharp$$

where $w := [H u_-^f]|_{\partial_- SM} \circ \alpha \in C^\infty(\partial_+ SM)$. Inserting this expression into (9.7) yields

$$-f - W^2 f = (X_\perp w^\sharp)_0$$

and the proof is completed by observing that

$$u_-^f|_{\partial SM} = \frac{1}{2}(I_0 f)_-.$$

□

EXERCISE 9.11. Using (9.6) show that $I_0 f = 0$ iff $I_0(Wf) = 0$.

9.4. Range

9.5. Tensors

9.6. Tentative: adding a connection plus numerical work

CHAPTER 10

Tensor tomography

This chapter solves the tensor tomography problem for simple surfaces. We shall in fact prove a stronger result in which the absence of conjugate points is replaced by the assumption that I_0^* is surjective.

10.1. Holomorphic integrating factors

In this section we prove an important technical result about the existence of certain solution of the transport equation $Xu = a$ when $a \in \Omega_{-1} \oplus \Omega_1$ (i.e. a represents a 1-form on M). This result will unlock the solution to several geometric inverse problems in 2D.

Proposition 10.1 (Existence of holomorphic integrating factors, Part I). Let (M, g) be a non-trapping surface with strictly convex boundary. Assume that I_0^* is surjective. Given $a_{-1} + a_1 \in \Omega_{-1} \oplus \Omega_1$, there exists $w \in C^\infty(SM)$ such that w is holomorphic and $Xw = a_{-1} + a_1$. Similarly there exists $\tilde{w} \in C^\infty(SM)$ such that
prop:holif $\tilde{w}$ is anti-holomorphic and $X\tilde{w} = a_{-1} + a_1$.

Proof. We do the proof for w holomorphic; the proof for $\tilde{w}$ anti-holomorphic is analogous.

First we note that the equation $\eta_+ f_0 = -a_1$ can always be solved. Indeed this is the case since it is equivalent to solving a ∂-equation on a disc:

$$\eta_+ f_0 = e^{-\lambda} \partial(f_0) e^{i\theta} = -\tilde{a}_1(x_1, x_2) e^{i\theta}$$

and so we just need to solve $\partial(f_0) = -e^{\lambda} \tilde{a}_1$ which is always possible by standard complex analysis.

Since I_0^* is surjective, there exists $q \in C^\infty(SM)$ such that $Xq = 0$ and $q_0 = f_0$. Hence

eq:hol1
$$X(q_2 + q_4 + \cdots) = \eta_- q_2 = -\eta_+ q_0 = a_1. \tag{10.1}$$

Next, we solve $\eta_- g_0 = a_{-1}$ and use surjectivity of I_0^* to find $p \in C^\infty(SM)$ such that $Xp = 0$ and $p_0 = g_0$. Hence

eq:hol2
$$X(p_0 + p_2 + \cdots) = \eta_- p_0 = a_{-1}. \tag{10.2}$$

Combining (10.1) and (10.2) and setting $w = \sum_{k \geq 0} p_{2k} + \sum_{k \geq 1} q_{2k}$ we see that w is holomorphic and $Xw = a_{-1} + a_1$.

□

10.2. Tensor tomography: version I

We begin with a simple observation that holds in any dimensions.

lemma:I0inj

LEMMA 10.2. *Let (M,g) be a non-trapping manifold with strictly convex boundary. If $I_0^*: C_\alpha^\infty(\partial_+ SM) \to C^\infty(M)$ is surjective, then $I_0: C^\infty(M) \to C^\infty(\partial_+ SM)$ is injective.*

PROOF. Suppose there is $f \in C^\infty(M)$ with $I_0 f = 0$. Since I_0^* is sujective, there is $w \in C_\alpha^\infty(\partial_+ SM)$ such that $I_0^* w = f$, hence we can write

$$\|f\|^2 = (f, I_0^* w)_{L^2(M)} = (I_0 f, w)_{L^2_\mu(\partial_+ SM)} = 0$$

and thus $f = 0$. ∎

The next result is the master result from which tensor tomography is derived. It asserts, in terms of the transport equation, that $I|_{\Omega_m}: \Omega_m \to C^\infty(\partial_+ SM)$ is injective whenever I_0^* is surjective.

THEOREM 10.3. *Let (M,g) be a non-trapping surface with strictly convex boundary and I_0^* surjective. Let $u \in C^\infty(SM)$ be such that*

$$Xu = f \in \Omega_m, \qquad u|_{\partial SM} = 0.$$

thm:tt1

Then $u = 0$ and $f = 0$.

PROOF. Let $r := e^{-im\theta}$ and observe that $r^{-1} X r \in \Omega_{-1} \oplus \Omega_1$ since

$$e^{im\theta} \eta_\pm (e^{-im\theta}) \in \Omega_{\pm 1}.$$

By the previous proposition, there are $w, \tilde{w} \in C^\infty(SM)$ holomorphic and anti-holomorphic respectively, such that $Xw = X\tilde{w} = -r^{-1} X r$. Without loss of generality we may assume that both w and $\tilde{w}$ are even. A simple calculation shows that

eq:transport1

$$X(e^w r u) = e^w r f \tag{10.3}$$

with a similar equation for $\tilde{w}$. Since $rf \in \Omega_0$, $e^w r f$ is holomorphic and $e^{\tilde{w}} r f$ is anti-holomorphic.

Assume now that m is even, the proof for m odd being very similar. Then we may assume that u is odd and thus $e^w r u$ and $e^{\tilde{w}} r u$ are odd. Let

$$q := \sum_{-\infty}^{-1} (e^w r u)_k.$$

Using (10.3), the fact that $e^w r f$ is holomorphic and that q is odd we see that

$$Xq = \eta_+ q_{-1} \in \Omega_0, \qquad q|_{\partial SM} = 0.$$

(Note that $u|_{\partial SM} = 0$ iff $u_k | \partial SM = 0$ for all k.) Since we know that I_0 is injective (Lemma 10.2) we deduce that $q = \eta_+ q_{-1} = 0$. Hence $e^w r u$ is holomorphic and thus $ru = e^{-w}(e^w r u)$ is holomorphic. Arguing with $\tilde{w}$ we deduce that ru is also anti-holomorphic and hence $ru \in \Omega_0$. This implies that $u \in \Omega_m$ and using that $Xu \in \Omega_m$ we see that $Xu = 0$ and finally $u = f = 0$ as desired. ∎

G:discuss what happens when $r = he^{-im\theta}$ where h is non-zero

One can explicitly compute $r^{-1} X r$ in the proof above using isothermal coordinates in which the metric is $e^{2\lambda}(dx_1^2 + dx_2^2)$:

Exercise 10.4. Show that

$$r^{-1}Xr = m\eta_+(\lambda) - m\eta_-(\lambda).$$

By inspecting the proof of Proposition 10.1 show that the conclusion of Theorem 10.3 still holds if we assume that I_0 is injective and there is a smooth q such that $Xq = 0$ with $q_0 = \lambda$. Hence surjectivity of I_0^* is only needed for the function λ!

Corollary 10.5. Let (M, g) be a non-trapping surface with strictly convex boundary and I_0^* surjective. Let $u \in C^\infty(SM)$ be such that

$$Xu = f, \qquad u|_{\partial SM} = 0.$$

Suppose $f_k = 0$ for $k \geq m+1$ for some $m \in \mathbb{Z}$. Then $u_k = 0$ for $k \geq m$. Similarly, if $f_k = 0$ for $k \leq m-1$ for some $m \in \mathbb{Z}$. Then $u_k = 0$ for $k \leq m$.

Proof. Suppose $f_k = 0$ for $k \geq m+1$ for some $m \geq 0$. Let $v := \sum_m^\infty u_k$. Using the equation $Xu = f$ and the hypothesis on f we see that

$$Xv = \eta_- u_m + \eta_- u_{m+1} \in \Omega_{m-1} \oplus \Omega_m.$$

Applying Theorem 10.3 to the even and odd components of v we deduce that $v = 0$ and thus $u_k = 0$ for $k \geq m$. Similarly, arguing with $\sum_{-\infty}^m u_k$ we deduce that $u_k = 0$ for $k \leq m$ if $f_k = 0$ for $k \leq m-1$.

□

The next corollary is an obvious consequence of the previous one.

Corollary 10.6 (Tensor tomography, Version I). Let (M, g) be a non-trapping surface with strictly convex boundary and I_0^* surjective. Let $u \in C^\infty(SM)$ be such that

$$Xu = f, \qquad u|_{\partial SM} = 0.$$

Suppose $f_k = 0$ for $|k| \geq m+1$ for some $m \geq 0$. Then $u_k = 0$ for $|k| \geq m$ (when
corollary:tt2 $m = 0$, this means $u = f = 0$).

10.3. Tensor tomography: version II

Let (M, g) be a non-trapping manifold with strictly convex boundary. Using the map

$$\ell_m : C^\infty(S^m(T^*M)) \to C^\infty(SM)$$

we can define the geodesic X-ray transform acting on symmetric tensor of rank m by setting

$$I_m := I \circ \ell_m : C^\infty(S^m(T^*M)) \to C^\infty(\partial_+ SM).$$

When $m \geq 1$, this transform has a kernel. Indeed, if $p \in C^\infty(S^{m-1}(T^*M))$ is such that $p|_{\partial M} = 0$, then

$$I_m(dp) = I(\ell_m dp) = I(X\ell_{m-1}p) = 0,$$

where we used Lemma 6.13.

Tensors of the form dp with $p|_{\partial M} = 0$ are called *potential tensors.*

The *tensor tomography problem* asks: given $h \in C^\infty(S^m(T^*M))$ with $I_m h = 0$, is it true that $h = dp$ where $p \in C^\infty(S^{m-1}(T^*M))$ with $p|_{\partial M} = 0$?

We now give a positive answer to this question in the case of surfaces with I_0^* surjective.

THEOREM 10.7 (Tensor tomography, Version II). *Let (M, g) be a non-trapping surface with strictly convex boundary and I_0^* surjective. Given $h \in C^\infty(S^m(T^*M))$ with $I_m h = 0$, there exists $p \in C^\infty(S^{m-1}(T^*M))$ such that $h = dp$ and $p|_{\partial M} = 0$.*

PROOF. Let $f := \ell_m h$. Since $If = 0$ we know that there exists $u \in C^\infty(SM)$ such that

$$Xu = f, \qquad u|_{\partial SM} = 0.$$

Moreover, by Lemma 6.14 we also know that $f_k = 0$ for $|k| \geq m+1$. From Corollary 10.6 we deduce that $u_k = 0$ for $|k| \geq m$. If m is even (resp. odd) we may take u odd (resp. even). By Proposition 6.15 there is a unique $p \in C^\infty(S^{m-1}(T^*M))$ such that $u = \ell_{m-1}p$. Since $u|_{\partial SM} = 0$, $p|_{\partial M} = 0$. Finally $Xu = f$ can be written using Lemma 6.13 as

$$\ell_m dp = X\ell_{m-1}p = \ell_m h$$

and thus $h = dp$ as desired.

□

CHAPTER 11

Boundary rigidity and Lens rigidity

Let (M, g) be a compact manifold with boundary. The distance function $d_g : M \times M \to \mathbb{R}$ is given by

$$d_g(x, y) = \inf_{\gamma \in \Lambda_{x,y}} \ell_g(\gamma),$$

where $\Lambda_{x,y}$ denotes the set of smooth curves $\gamma : [0, 1] \to M$ such that $\gamma(0) = x$ and $\gamma(1) = y$ and

$$\ell_g(\gamma) := \int_0^1 |\dot{\gamma}(t)|_g \, dt.$$

G: perhaps say that if ∂M is strictly convex, the infimum is realized by a minimizing geodesic whose interior is completely contained in M?

Suppose we know $d_g(x, y)$ for all $(x, y) \in \partial M \times \partial M$. Can we reconstruct g in the interior of M from this information?

If $\psi : M \to M$ is a diffeomorphism such that $\psi|_{\partial M} = \mathrm{Id}$, then $d_{\psi^* g} = d_g$ on $\partial M \times \partial M$ since if $\gamma \in \Lambda_{x,y}$, then $\psi \circ \gamma \in \Lambda_{x,y}$ and $\ell_{\psi^* g}(\gamma) = \ell_g(\psi \circ \gamma)$.

Thus the best we can hope for is to recover g up to an isometry that acts as the identity on the boundary.

Definition 11.1. We say that g is *boundary rigid* if given any other metric h with $d_g|_{\partial M \times \partial M} = d_h|_{\partial M \times \partial M}$, there exists a diffeomorphism $\psi : M \to M$ such that $\psi|_{\partial M} = \mathrm{id}$ and $h = \psi^* g$.

But not all metrics are boundary rigid as the following simple example shows.

Example 11.2. Suppose M contains an open set U on which g is very large. Then all length minimizing curves will avoid U, and thus d_g will not carry any information about $g|_U$. Thus we can alter g on U (but keeping it large) and not affect d_g on $\partial M \times \partial M$. Here is a concrete example: take M to be the upper hemisphere of S^2, and let g_0 denote the natural metric on M. Note that $d_{g_0}(x, y)$ for any two boundary points is realized as the length of the shortest arc on ∂M connecting x and y. Now take a non-negative function f supported on U and let $g_1 = (1 + f)g_0$. Then $d_{g_0} = d_{g_1}$ on $\partial M \times \partial M$, but g_0 and g_1 are not isometric since $\mathrm{Vol}(M, g_1) > \mathrm{Vol}(M, g_0)$.

proposition:normalgauge

Proposition 11.3. Let M be a compact manifold with non-empty boundary. Suppose that ∂M is strictly convex with respect to g and g_1 and $d_g = d_{g_1}$ on $\partial M \times \partial M$. Then there exists a diffeomorphism $\psi : M \to M$ such that $\psi|_{\partial M} = \mathrm{id}$ and such that if $g_2 = \psi^* g_1$, then $g_2|_{\partial M} = g|_{\partial M}$, i.e., $g_2(x) = g(x)$ on $T_x M \times T_x M$ for all $x \in \partial M$.

PROOF. Let $(x, v) \in T\partial M$ and take a curve $\tau : (-\varepsilon, \varepsilon) \to \partial M$ such that $\tau(0) = x$ and $\dot{\tau}(0) = v$. Since τ takes values in ∂M for all $s \in (-\varepsilon, \varepsilon)$ we have

$$d_{g_1}(x, \tau(s)) = d_g(x, \tau(s)).$$

Thus (cf. Exercise 11.4 below):

eq:exbelow

$$|v|_{g_1} = \lim_{s \to 0^+} \frac{d_{g_1}(x, \tau(s))}{s} = \lim_{s \to 0^+} \frac{d_g(x, \tau(s))}{s} = |v|_g. \tag{11.1}$$

We now modify g_1 so that we also have agreements of the metrics in the normal direction. Let $\nu(x)$ denote the inward unit normal with respect to g and consider the boundary exponential map

$$\exp_{\partial M} : \partial M \times \mathbb{R}_+ \to M, \quad (x, t) \mapsto \exp_x(t\nu(x))$$

which maps a neighbourhood of $\partial M \times \{0\}$ diffeomorphically onto a neighbourhood of ∂M. Now define

$$\psi := \exp^{g_1}_{\partial M} \circ (\exp^g_{\partial M})^{-1}$$

where superscripts denote which metric they belong to. Then on some collar neighbourhood U of ∂M, ψ is a diffeomorphism. We extend ψ to a diffeomorphism of M.

G: provide reference

We claim that ψ satisfies the requirements of the proposition. Indeed, $\psi|_{\partial M} = \mathrm{Id}$ and given $x \in \partial M$ we have

$$\psi(\gamma^g_{x,\nu(x)}(t)) = \gamma^{g_1}_{x,\nu_1(x)}(t).$$

Differentiating with respect to t we obtain

$$d_x\psi(\nu(x)) = \nu_1(x).$$

Then if $x \in \partial M$ and $v \in T_x\partial M$:

$$\begin{aligned} g_2(v, \nu(x)) &= g_1(d_x\psi(v), d_x\psi(\nu(x))) \\ &= g_1(v, \nu_1(x)) \\ &= 0. \end{aligned}$$

Thus $g_2 = \psi^* g_1$ has unit normal equal to ν and hence

$$g_2|_{\partial M} = g|_{\partial M}.$$

$\square$

ex:lim

EXERCISE 11.4. Prove the first equality in (11.1).

LEMMA 11.5. *Let (M, g) be a simple manifold. Given $x \in M$, let $f : M \to \mathbb{R}$ be $f(y) = d_g(x, y)$. Given $(x, y) \in \partial M \times \partial M$ with $x \neq y$, let $\gamma_{x,y}$ be the unique geodesic connecting x to y and let $\ell_{x,y}$ be its length. Then*

$$\nabla f(y) = \dot{\gamma}_{x,y}(\ell_{x,y}).$$

lemma:grad-dist

EXERCISE 11.6. Prove the lemma.

proposition:ddets

PROPOSITION 11.7. Let g_1 and g_2 be two simple metrics on M such that $d_{g_1} = d_{g_2}$ on $\partial M \times \partial M$ and $g_1|_{\partial M} = g_2|_{\partial M}$. Then $\alpha_{g_1} = \alpha_{g_2}$.

PROOF. Note that we only need to show that $\alpha_{g_1} = \alpha_{g_2}$ on $\partial_+ SM$ since $\alpha_{g_i} : \partial_- SM \to \partial_+ SM$ equals $\left(\alpha_{g_i}|_{\partial_+ SM}\right)^{-1}$. Fix $x, y \in \partial M$ and consider the unique geodesic $\gamma^i_{x,y}$ connecting x to y. By the definition of the scattering relation

$$\alpha_{g_i}(x, \dot\gamma^i_{x,y}(0)) = (y, \dot\gamma^i_{x,y}(\ell^i_{x,y})), \quad i = 1, 2.$$

Let $\ell := \ell^1_{x,y} = \ell^2_{x,y}$. We are required to prove that

$$\dot\gamma^1_{x,y}(0) = \dot\gamma^2_{x,y}(0), \quad \dot\gamma^1_{x,y}(\ell) = \dot\gamma^2_{x,y}(\ell).$$

Let $f_i(y) = d_{g_i}(x, y)$ and $h_i = f_i|_{\partial M}$. Then $\nabla h_i(y)$ is the orthogonal projection of $\nabla f_i(y)$ in the hemisphere $\partial_- S_y M$ onto the "equatorial" unit disk in $T_y \partial M$; in particular $\nabla h_i(y)$ determines $\nabla f_i(y)$. But $h_1 = h_2$, hence by Lemma 11.5

$$\dot\gamma^1_{x,y}(\ell) = \nabla f_1(y) = \nabla f_2(y) = \dot\gamma^2_{x,y}(\ell).$$

To show that $\dot\gamma^1_{x,y}(0) = \dot\gamma^2_{x,y}(0)$ we repeat the argument above only starting at y and running the two geodesics backwards to x.

□

DEFINITION 11.8. Let (M, g) be a non-trapping manifold with strictly convex boundary. The *lens data* of (M, g) consists of $(\tau_g|_{\partial_+ SM}, \alpha_g)$ where τ_g is the exit time function.

The previous proposition shows that when (M, g) is simple, $d_g|_{\partial M \times \partial M}$ determines the lens data.

11.0.1. Volume determination. The next proposition shows that the volume is determined by the exit time function $\tau_g : \partial_+ SM \to \mathbb{R}$.

PROPOSITION 11.9. Let g_1, g_2 be two non-trapping metrics on M such that ∂M is strictly convex with respect to both of them. If $g_1|_{\partial M} = g_2|_{\partial M}$ and $\tau_{g_1}|_{\partial_+ SM} = \tau_{g_2}|_{\partial_+ SM}$, then $\mathrm{Vol}(M, g_1) = \mathrm{Vol}(M, g_2)$.

PROOF. This is an immediate consequence of Santaló's formula that gives

$$\sigma_{n-1} \mathrm{Vol}(M, g_i) = \mathrm{Vol}(SM, g_i) = \int_{\partial_+ SM} \tau_{g_i} \, d\mu, \quad i = 1, 2,$$

where σ_{n-1} is the volume of the standard $(n-1)$-sphere. □

corollary:volume

COROLLARY 11.10. Let g_1, g_2 be two simple metrics on M with the same boundary distance function. Then $\mathrm{Vol}(M, g_1) = \mathrm{Vol}(M, g_2)$.

PROOF. Proposition 11.3 shows that after applying a diffeomorphism that is the identity on the boundary, we may assume $g_1|_{\partial M} = g_2|_{\partial M}$. Since the boundary distance function determines the lens data, the exit time function of both metrics must agree and thus by the previous proposition, the volumes are the same. □

11.1. Boundary determination

THEOREM 11.11 ([**LSU03**]). *Let g_1, g_2 be two metrics on M such that ∂M is strictly convex with respect to both of them. If $d_{g_1} = d_{g_2}$ on $\partial M \times \partial M$, then after modifying g_2 by a diffeomorphism which is the identity on the boundary if necessary, g_1 and g_2 have the same C^∞-jet. This means that given local coordinates $(x^1, \ldots, x^n)$ defined in a neighbourhood of a boundary point, we have $D^\alpha g_1|_{\partial M} = D^\alpha g_2|_{\partial M}$ for any multi-index α.*

PROOF. By Proposition 11.3 we may assume that $g_1|_{\partial M} = g_2|_{\partial M}$. Moreover, the proof of Proposition 11.3, gives that near ∂M, g_1 and g_2 have the same normal geodesics to the boundary. Set $f := g_1 - g_2$. Consider a g_1-geodesic $\gamma : [0,1] \to M$ connecting boundary points x and y in M (not necessarily with speed one). Then we observe that

eq:ineqd

$$\int_0^1 f_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))\, dt \leq 0 \tag{11.2}$$

since

$$\begin{aligned}\int_0^1 f_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))\, dt &= \int_0^1 (g_1)_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))\, dt - \int_0^1 (g_2)_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))\, dt \\ &\leq (d_{g_1}(x,y))^2 - (d_{g_2}(x,y))^2 \\ &= 0.\end{aligned}$$

Fix a point $p \in \partial M$ and consider *boundary normal coordinates* $(u^1, \dots, u^{n-1}, z)$ on a neighbourhood U of p. By definition these are coordinates such that $z \geq 0$ on U and $\partial M \cap U = \{z = 0\}$, and that the length element ds_1^2 of the metric g_1 is given by

$$ds_1^2 = (g_1)_{\alpha\beta} du^\alpha du^\beta + dz^2, \quad \alpha, \beta \in \{1, \dots, n-1\}.$$

The coordinate lines $u =$ constant are geodesics of the metric g_1 orthogonal to the boundary. But we have set up the metrics g_1 and g_2 near the boundary so that $u =$ constant are also geodesics of the metric g_2. It follows that the *same* coordinates are also boundary normal coordinates for g_2; in particular

$$ds_2^2 = (g_2)_{\alpha\beta} du^\alpha du^\beta + dz^2, \quad \alpha, \beta \in \{1, \dots, n-1\}.$$

Since p was arbitrary, to prove the theorem it suffices to show that for all $x \in \partial M \cap U$, $k \in \mathbb{N} \cup \{0\}$ and $1 \leq \alpha, \beta \leq n-1$ we have

eq:zeroderivatives

$$\frac{\partial f_{\alpha\beta}}{\partial z^k}(x) = 0, \tag{11.3}$$

where $f_{\alpha\beta} = (g_1)_{\alpha\beta} - (g_2)_{\alpha\beta}$. The case $k = 0$ is precisely the assertion that $g_1|_{\partial M} = g_2|_{\partial M}$ and so this gives the base step for an inductive proof. Suppose that (11.3) holds for $0 \leq k < l$ but fails for l. This implies the existence of $x_0 \in \partial M \cap U$ and $v_0 \in S_{x_0}\partial M$ such that

$$\frac{\partial^l f_{\alpha\beta}}{\partial z^l}(x_0) v_0^\alpha v_0^\beta \neq 0.$$

Assume

$$\frac{\partial^l f_{\alpha\beta}}{\partial z^l}(x_0) v_0^\alpha v_0^\beta > 0.$$

By continuity, there is a neighbourhood $\mathcal{O} \subset SM$ of (x_0, v_0) such that for all $(x, v) \in \mathcal{O}$,

eq:derf>0

$$\frac{\partial^l f_{\alpha\beta}}{\partial z^l}(x) v^\alpha v^\beta > 0. \tag{11.4}$$

Since the left hand side in (11.4) is a homogeneous polynomial of degree 2, we may assume that if

$$C\mathcal{O} := \left\{ (x, v) \in TM : \ v \neq 0, \left(x, \frac{v}{|v|} \right) \in \mathcal{O} \right\}$$

then (11.4) holds for all $(x, v) \in C\mathcal{O}$. Now we expand $f_{\alpha\beta}$ in a Taylor series; using the inductive hypothese we may write

$$f_{\alpha\beta}(u, z) = \frac{1}{l!}\frac{\partial^l f_{\alpha\beta}}{\partial z^l}(u, 0)z^l + o(|z|^l),$$

and hence shrinking $\mathcal{O}$ if necessary we may assume that for all $(x, v) \in C\mathcal{O}$ we actually have

eq:f>0

$$f_{\alpha\beta}(x)v^\alpha v^\beta > 0. \tag{11.5}$$

Now let $\delta : (-\varepsilon, \varepsilon) \to \partial M$ be a curve such that $\delta(0) = x_0$ and $\dot{\delta}(0) = v_0$, and let $\gamma_\tau : [0, 1] \to M$ be the shortest geodesic of g_1 joining x_0 to $\delta(\tau)$ for $\tau > 0$ and small. Then

$$\left(\gamma_\tau(t), \frac{\dot{\gamma}_\tau(t)}{|\dot{\gamma}_\tau(t)|}\right) \to (x_0, v_0)$$

uniformly in $t \in [0, 1]$ as $\tau \to 0$. Thus for sufficiently small $\tau > 0$, we have $(\gamma_\tau(t), \dot{\gamma}_\tau(t)) \in C\mathcal{O}$ for all $t \in [0, 1]$, and hence

$$\int_0^1 f_{\gamma_\tau(t)}(\dot{\gamma}_\tau(t), \dot{\gamma}_\tau(t))\, dt > 0$$

thus contradicting (11.2). If

$$\frac{\partial^l f_{\alpha\beta}}{\partial z^l}(x_0)v_0^\alpha v_0^\beta < 0$$

a similar contradition is obtained if we integrate f along a g_2-geodesic so that (11.2) changes sign. This completes the proof.

□

11.2. Rigidity in a given conformal class

In thi section we prove boundary rigidity within the conformal class of a simple metric cf. [**Cro91, Muh81, MR78**].

thm:conf

THEOREM 11.12. *Let g_1 and g_2 be simple metrics on M with the same boundary distance function. If g_2 is conformal to g_1, i.e., $g_2 = \omega^2(x)g_1$ for a smooth positive function ω on M, then $\omega \equiv 1$.*

PROOF. In view of (11.4), $\omega = 1$ on the boundary of M. Next, using Proposition 11.7, we see that the scattering relations of g_1 and g_2 coincide. Let us denote by τ their common exit time function.

Let us show that $\omega = 1$ on the whole of M. Since geodesics on a simple manifold minimize the energy

$$E_g(\gamma) = \int_0^T |\dot{\gamma}(t)|_g^2\, dt$$

among all curves $\gamma : [0, T] \to M$ with the same endpoints we deduce for $(x, v) \in \partial_+ SM$:

eq:a'

$$\tau(x, v) = E_{g_2}(\gamma_{x,v}^2) \le E_{g_2}(\gamma_{x,v}^1) = \int_0^{\tau(x,v)} \omega^2(\gamma_{x,v}^1(t))\, dt. \tag{11.6}$$

Using Santaló's formula we obtain

$$\begin{aligned}\operatorname{Vol}(M,g_2) &= \frac{1}{\sigma_{n-1}}\int_{\partial_+SM}\tau\,d\mu\\ &\leq \frac{1}{\sigma_{n-1}}\int_{\partial_+SM}\left\{\int_0^{\tau(x,v)}\omega^2(\gamma^1_{x,v}(t))\,dt\right\}d\mu\\ &= \int_M \omega^2\,dV^n_{g_1}.\end{aligned}$$

On the other hand, by Hölder's inequality

hoelder (11.7)

$$\int_M \omega^2\,dV^n_{g_1} \leq \left\{\int_M \omega^n\,dV^n_{g_1}\right\}^{\frac{2}{n}}\left\{\int_M dV^n_{g_1}\right\}^{\frac{n-2}{n}} = \operatorname{Vol}(M,g_2)^{\frac{2}{n}}\operatorname{Vol}(M,g_1)^{\frac{n-2}{n}},$$

with equality if and only if $\omega\equiv 1$.

It follows that

v-v
$$\operatorname{Vol}(M,g_2)\leq \operatorname{Vol}(M,g_2)^{\frac{2}{n}}\operatorname{Vol}(M,g_1)^{\frac{n-2}{n}}. \tag{11.8}$$

However, by Corollary 11.10, $\operatorname{Vol}(M,g_1)=\operatorname{Vol}(M,g_2)$, which implies that (11.8) holds with the equality sign. This means that (11.7) holds with the equality sign. Thus, $\omega\equiv 1$.

□

11.3. Scattering rigidity

In what follows we shall assume that (M,g_1) and (M,g_2) are non-trapping surfaces with strictly convex boundary such that $g_1|_{\partial M}=g_2|_{\partial M}$. Given a function $\varphi\in C^\infty(\partial_+SM)$ we denote by $\varphi^{\sharp g_i}$ the function uniquely determined by $X_{g_i}\varphi^{\sharp g_i}=0$ and $\varphi^{\sharp g_i}|_{\partial_+SM}=\varphi$.

Recall that the scattering relation is a smooth map $\alpha_g:\partial_+SM\to\partial_-SM$ that extends to a diffeomorphism $\alpha_g:\partial SM\to\partial SM$ such that $\alpha_g^2=\mathrm{id}$. Note that if $\alpha_{g_1}=\alpha_{g_2}$ then $\varphi^{\sharp g_1}|_{\partial SM}=\varphi^{\sharp g_2}|_{\partial SM}$ since $\varphi^{\sharp g_i}|_{\partial_-SM}=\varphi\circ\alpha_{g_i}$. Thus the scattering relation determines the boundary values of invariant functions.

Observe that if $\alpha_{g_1}=\alpha_{g_2}$, then $C^\infty_\alpha(\partial_+SM)$ is the same space for both metrics since it only depends on α.

Recall that the Dirichlet-to-Neumann map $\Lambda_g:C^\infty(\partial M)\to C^\infty(\partial M)$ is defined as follows. Given $f\in C^\infty(\partial M)$, consider the unique harmonic extension u of f to (M,g) and set

$$\Lambda_g f := du(\nu)|_{\partial M}.$$

We are now ready to show:

thm:DN

Theorem 11.13. *Let (M,g_1) and (M,g_2) be non-trapping surfaces with strictly convex boundary, I_0^* surjective and $g_1|_{\partial M}=g_2|_{\partial M}$. If $\alpha_{g_1}=\alpha_{g_2}$, then $\Lambda_{g_1}=\Lambda_{g_2}$.*

Proof. First we show that α_g controls the Fourier series of invariant functions (here holomorphic means fibre-wise holomorphic), in other words, if $\alpha_{g_1}=\alpha_{g_2}$, then

$$\varphi^{\sharp g_1}\text{ holomorphic in }(M,g_1)\implies \varphi^{\sharp g_2}\text{ holomorphic in }(M,g_2).$$

(Note that $\varphi^{\sharp g_1}$ is smooth iff $\varphi^{\sharp g_2}$ is.) Indeed, let

$$w:=\sum_{-\infty}^{-1}(\varphi^{\sharp g_2})_k.$$

Then $w|_{\partial SM} = 0$ since $\varphi^{\sharp g_1}$ is holomorphic and the boundary values of invariant functions are the same. Since

$$X_{g_2} w = \eta_{+,g_2} w_{-1} + \eta_{+,g_2} w_{-2}$$

splitting into even and odd components and applying Theorem 10.3 we deduce that $w = 0$.

Next show that α_g determines the boundary values of holomorphic functions in M (here holomorphic means with respect to $x \in M$), in other words, if $\alpha_{g_1} = \alpha_{g_2}$, then

$$h \text{ holomorphic in } (M, g_1) \implies \exists\ \tilde{h} \text{ holomorphic in } (M, g_2) \text{ with } \tilde{h}|_{\partial M} = h|_{\partial M}.$$

Given h is holomorphic in (M, g_1) use surjectivity of I_0^* to choose w with $X_{g_1} w = 0$ and $w_0 = h$. Replacing w by its holomorphic projection if necessary, we have that $w = \varphi^{\sharp g_1}$ is fibre-wise holomorphic and $(\varphi^{\sharp g_1})_{0,g_1} = h$. Then also $\varphi^{\sharp g_2}$ is fibre-wise holomorphic and $X_{g_2} \varphi^{\sharp g_2} = 0$, so $\eta_{-,g_2}(\varphi^{\sharp g_2})_{0,g_2} = 0$. This means that $\tilde{h} = (\varphi^{\sharp g_2})_{0,g_2}$ is holomorphic in (M, g_2) and it has the same boundary values as h.

The fact that boundary values of holomorphic functions in (M, g_1) and (M, g_2) coincide is just another way of saying that $\Lambda_{g_1} = \Lambda_{g_2}$. If $f \in C^\infty(\partial M)$, let u be the harmonic extension of f in (M, g_1), let v be a harmonic conjugate of u in (M, g_1) and let $h = u + iv$. Then there is $\tilde{h} = \tilde{u} + i\tilde{v}$ holomorphic in (M, g_2) with $\tilde{h}|_{\partial M} = h|_{\partial M}$. Then if $\nu_\perp$ denotes the rotation of ν by $\pi/2$ according to the orientation, holomorphicity of h and $\tilde{h}$ implies

$$du(\nu) = dv(\nu_\perp),$$
$$d\tilde{u}(\nu) = d\tilde{v}(\nu_\perp).$$

Thus

$$\begin{aligned} \Lambda_{g_1} f &= du(\nu) = dv(\nu_\perp) \\ &= d\tilde{v}(\nu_\perp) = d\tilde{u}(\nu) \\ &= \Lambda_{g_2} f. \end{aligned}$$

The equality $dv(\nu_\perp) = d\tilde{v}(\nu_\perp)$ holds because $v|_{\partial M} = \tilde{v}|_{\partial M}$.

□

11.4. Royden's proof [**Roy56**]

We now wish to study the consequences of Theorem 11.13. We note that a metric g on M makes M into a compact Riemann surface with boundary. As we saw in the proof of the theorem, α_g determines the boundary values of holomorphic functions on M. Let A_i denote the ring of holomorphic functions in (M, g_i) (smooth all the way up to the boundary). The proof of Theorem 11.13 gives a map $F : A_1 \to A_2$ by setting $F(h) = \tilde{h}$. It is immediate to check that this map is a ring isomorphism mapping constant functions to constant functions.

We now invoke the following fact about the ring A; this should hold given the comments in [**Roy56**, Page 272].

prop:ideal

PROPOSITION 11.14. An ideal I is an ideal consisting of all functions which vanish at some point $x \in M$ if and only if it is the kernel of a homomorphism $\pi : A \to \mathbb{C}$ such that $\pi(c) = c$ for all complex constants c.

G: check carefully that the boundary does not cause any problems. Keep in mind that (M, g) can be embedded into an open Riemann surface S without boundary, so one can -if needed- consider holomorphic functions on S and restrict to M. But also note that there could be elements in $A(M)$ having ∂M as its natural boundary.

We now follow the proof of [**Roy56**, Theorem 1]. Let $x \in M$ and denote by I_x the ideal consisting of all functions in A_2 that vanish at x. By Proposition 11.14 there is a homomorphism $\pi : A_2 \to \mathbb{C}$ which preserves constants and whose kernel is I_x. Now $\pi \circ F : A_1 \to \mathbb{C}$ is a homomorphism preserving constants and by Proposition 11.14, its kernel is an ideal I_y in A_1. This defines a map $\psi : M \to M$, by setting $\psi(x) = y$. (For this map to be well-defined we need to note that given x there is a holomorphic function that vanishes only at x.)

Let $h \in A_1$ and suppose $F(h)(x) = c$. Then $F(h) - c \in I_x$ and $h - c \in I_{\psi(x)}$. Thus the value of h at $\psi(x)$ is also c, and thus $F(h) = h \circ \psi$. Note also that since $F(h)|_{\partial M} = h|_{\partial M}$, the map ψ must be the identity on ∂M. It is also straightforward to check that ψ is unique: if ϕ is another map such that $F(h) = h \circ \psi = h \circ \phi$ for all h and $\psi(x) \neq \phi(x)$ for some x we could construct a function h with different values at these points and arrive at a contradiction.

Let us prove that ψ is continuous. Consider a sequence $x_n \to x$ and suppose by contradiction that $\psi(x_n)$ does not converge to $\psi(x)$. We may consider a subsequence still denoted x_n such that $\psi(x_n) \to z \neq \psi(x)$. Let $h \in A_1$ such that $h(\psi(x)) = 0$ and $h(z) \neq 0$. Then $F(h)(x_n) \to F(h)(x) = 0$, while $h(\psi(x_n)) \to h(z) \neq 0$, a contradiction since $F(h) = h \circ \psi$. Thus ψ must be continuous.

Let x be a point in the interior of M and let $h \in A_1$ be such that it has a simple zero at $\psi(x)$. Set $g := F(h)$. Then there is a neighbourhood U of $\psi(x)$ in which h is $1-1$. Take a neighbourhood V of x such that $V \subset \psi^{-1}(U)$ and such that $g(U) \subset h(U)$. Then in V we can represent ψ by $h^{-1} \circ g$ and hence ψ is holomorphic.

Finally to conclude that $\psi : (M, g_2) \to (M, g_1)$ is a conformal equivalence, observe that F is an isomorphism, hence its inverse induces a holomorphic map ψ' such that $\psi \circ \psi'$ induces the identity map in A_1 and by uniqueness $\psi \circ \psi'$ is the identity.

Hence combining the argument above with Theorem 11.13 we have proved:

`thm:scarigid`

THEOREM 11.15. *Let (M, g_1) and (M, g_2) be non-trapping surfaces with strictly convex boundary, I_0^* surjective and $g_1|_{\partial M} = g_2|_{\partial M}$. If $\alpha_{g_1} = \alpha_{g_2}$, then there exists a conformal equivalence $\psi : (M, g_1) \to (M, g_2)$ such that $\psi|_{\partial M} = id$.*

G: check that in the proof above ψ is smooth up to the boundary. But this seems perfectly fine: just take h holomorphic in a slightly larger open Riemann surface containing (M, g_1) and run the same proof as above around a boundary point. One still gets $\psi = h^{-1} \circ g$ as above where h^{-1} and g are smooth up to the boundary, (even though in principle g might not extend holomorphically outside M).

11.4.1. Using Royden's result directly. We might wish to avoid checking Proposition 11.14 and use [**Roy56**, Theorem 1] directly for open surfaces as follows. The following argument requires to know that g_1 and g_2 have the same C^∞-jet at the boundary, something that we might eventually prove elsewhere. Consider M inside an open surface S. Extend the metrics g_1 and g_2 smoothly and equally

We only need the jets of j_1 and j_2 to agree at the boundary and this might follow from equality of the DN maps, even using less regular complex structures might be fine outside M. Hence we have two open Riemann surfaces (S, j_1) and (S, j_2), where j_i is the complex structure associated with g_i. Clearly $j_1 = j_2$ outside M. Let $A(S, j_i)$ denote the ring of holomorphic functions of (S, j_i). We will define an isomorphism $F : A(S, j_1) \to A(S, j_2)$ using the fact that j_1 and j_2 have the same boundary values for holomorphic functions in M. Consider $h \in A(S, j_1)$ and restrict it to M. We know there exists a holomorphic function $\tilde{h}$ in (M, j_2) such that $h|_{\partial M} = \tilde{h}|_{\partial M}$. Define a function

eq:3

$$\tilde{h}_S(x) := \begin{cases} \tilde{h}(x), & x \in M \\ h(x), & x \in S \setminus M. \end{cases} \tag{11.9}$$

The function $\tilde{h}_S$ is clearly continuous since h and $\tilde{h}$ agree on ∂M. Moreover, $\tilde{h}_S$ is j_2-holomorphic in $S \setminus \partial M$. Since ∂M is removable, we deduce that $\tilde{h}_S \in A(S, j_2)$. Define $F(h) := \tilde{h}_S$. Now it is straightforward to check that F is an isomorphism since the roles j_1 and j_2 can be swapped. Moreover, F maps constants to constants. Hence by [**Roy56**, Theorem 1], there is a unique conformal equivalence $\psi : (S, j_2) \to (S, j_1)$ such that $F(h) = h \circ \psi$. Since h and $\tilde{h}_S$ agree on the complement of the interior of M, ψ must be the identity on that set.

11.5. Boundary rigidity for simple surfaces

We are now ready to prove the main result of this chapter.

THEOREM 11.16 (Pestov-Uhlmann [**PU05**]). *Let g_1 and g_2 be two simple metrics on a surface M with the same boundary distance function. Then there exists a diffeomorphism $\psi : M \to M$ such that $\psi|_{\partial M} = id$ and $g_2 = \psi^* g_1$.*

PROOF. After applying a diffeomorphism if necessary we may assume by Proposition 11.3 that $g_2|_{\partial M} = g_1|_{\partial M}$. We also know that on a simple manifold I_0^* is surjective by Theorem 8.6. Since the boundary distance function determines the lens data, we may apply Theorem 11.15 to deduce that there exists a diffeomorphism $\psi : M \to M$ such that $\psi|_{\partial M} = \mathrm{Id}$ and $g_2 = \omega^2 \psi^* g_1$, where ω is a smooth positive function. Finally Theorem 11.12 gives $\omega \equiv 1$ and the proof is completed.

□

11.6. Relation to Calderón problem in 2D.

Linearized Calderón? We know how to solve this (it is in an old file)

CHAPTER 12

Attenuated geodesic X-ray transform

12.1. Novikov's formula? Finch's survey

12.2. Salo-Uhlmann result, attenuation of the form $a(x) + \theta_x(v) \in GL(1, \mathbb{C})$

In this section we consider the case when the attenuation $\mathcal{A}$ is scalar and has the special form $\mathcal{A}(x, v) = a(x) + \theta_x(v)$ where $a \in C^\infty(M, \mathbb{C})$ is a function and θ is a smooth complex-valued 1-form. Since we are working in two dimensions, we may equivalently say that we shall consider attenuations for which

$$\mathcal{A} = a_{-1} + a_0 + a_1 \in \Omega_{-1} \oplus \Omega_0 \oplus \Omega_1.$$

We will consider first the case $a_0 = 0$ (i.e. purely a 1-form). In this setting we can prove a fairly general result:

`thm:injectiveatt`

THEOREM 12.1. *Let (M, g) be a non-trapping surface with strictly convex boundary and I_0^* surjective. Let θ be any smooth complex-valued 1-form. Then $I_{0,\theta}$ is injective.*

PROOF. Suppose there is a smooth $f \in \Omega_0$ such that $I_{0,\theta} f = 0$. By Theorem 5.10 there is a smooth function u such that $Xu + \theta u = -f$ and $u|_{\partial SM} = 0$. Since $X + \theta$ maps even (odd) functions to odd (even) and $f \in \Omega_0$ we may assume without loss of generality that u is odd.

Using Proposition 10.1 we know that there exists w holomorphic and even with $Xw = \theta$. Thus we may write

`eq:noatt`

$$X(e^w u) = e^w((Xw)u + Xu) = -fe^w. \tag{12.1}$$

Note that $e^w u$ is odd and consider

$$q := \sum_{-\infty}^{-1} (e^w u)_k.$$

Since fe^w is holomorphic, (12.1) gives

$$Xq = \eta_+ q_{-1} \in \Omega_0.$$

But $q|_{\partial SM} = 0$, hence injectivity of I_0 gives $q = 0$. This means that $e^w u$ is holomorphic and thus u is holomorphic. Using Proposition 10.1 again but with $\tilde{w}$ anti-holomorphic we deduce that u is also anti-holomorphic. Since we assumed u odd we must have $u = 0$ and thus $f = 0$ as claimed.

$\square$

This result has the following important corollary.

COROLLARY 12.2. Let (M,g) be a simple surface and let θ be a smooth complex-valued 1-form. Then, given $f\in C^\infty(M,\mathbb{C})$ there exists $u\in C^\infty(SM,\mathbb{C})$ such that

$$\begin{cases} Xu+\theta u=0, \\ u_0=f \end{cases}$$

cor:thetatransport

PROOF. Consider any smooth function $R: SM\to\mathbb{C}\backslash\{0\}$ such that $XR+\theta R=0$ and set $W=\bar{R}$. Then by Remark 5.15 injectivity of $I_{0,\theta}$ is equivalent to injectivity of $I_{0,W}$. Combining Theorem 12.1 with Corollary 8.10 we deduce right away the existence of u as claimed.

□

The next theorem may be seen as the dual statement at the level of the transport equation to the injectivity of the geodesic X-ray transform on the spaces Ω_k.

THEOREM 12.3. *Let (M,g) be a simple surface. Given $f\in\Omega_k$ there exists $u\in C^\infty(SM)$ such that*

$$\begin{cases} Xu=0, \\ u_k=f \end{cases}$$

thm:fulltransportsol

PROOF. Let $r:=e^{ik\theta}\in\Omega_k$. Then $\theta:=r^{-1}X(r)\in\Omega_{-1}\oplus\Omega_1$ is a 1-form. By Corollary 12.2, there exists a smooth u such that $Xu+\theta u=0$ and $u_0=r^{-1}f\in\Omega_0$. Now observe

$$X(ru)=r(Xu+\theta u)=0$$

and since $(ru)_k=ru_0=f\in\Omega_k$ the theorem is proved.

□

Armed with this theorem we can now prove the existence of holomorphic integrating factors for $a\in C^\infty(M,\mathbb{C})$:

PROPOSITION 12.4 (Existence of holomorphic integrating factors, Part II). Let (M,g) be a simple surface. Given $a\in\Omega_0$, there exists $w\in C^\infty(SM)$ such that w is holomorphic and $Xw=a$. Similarly there exists $\tilde{w}\in C^\infty(SM)$ such that $\tilde{w}$ is anti-holomorphic and $X\tilde{w}=a$.

prop:holif2

PROOF. We do the proof for w holomorphic; the proof for $\tilde{w}$ anti-holomorphic is analogous.

First we note -as in the proof of Proposition 10.1- that the equation $\eta_- f_1=a$ can always be solved. Indeed this is the case since it is equivalent to solving a $\bar\partial$-equation on a disc:

$$\eta_- f_1=e^{-2\lambda}\bar\partial(fe^\lambda)=a$$

where $f_1=fe^{i\theta}$. Hence we just need to solve $\bar\partial(fe^\lambda)=e^{2\lambda}a$ which is always possible by standard complex analysis.

Next, using Theorem 12.3 there is a smooth function u such that $Xu=0$ and $u_1=f_1$. Now take $w=u_1+u_3+u_5+\dots$. Then $Xw=\eta_- u_1=a$ and w is the desired holomorphic integrating factor.

□

We now state the final version on the existence of holomorphic integrating factors.

prop:holiffinal

PROPOSITION 12.5 (Existence of holomorphic integrating factors, Final version). Let (M,g) be a simple surface. Given $a = a_{-1} + a_0 + a_{-1} \in \Omega_{-1} \oplus \Omega_0 \oplus \Omega_1$, there exists $w \in C^\infty(SM)$ such that w is holomorphic and $Xw = a$. Similarly there exists $\tilde{w} \in C^\infty(SM)$ such that $\tilde{w}$ is anti-holomorphic and $X\tilde{w} = a$.

PROOF. This is a direct consequence of Propositions 10.1 and 12.4. □

We can now prove the main result of this section.

THEOREM 12.6. *Let (M,g) be a simple surface. Let $a = a_{-1} + a_0 + a_{-1} \in \Omega_{-1} \oplus \Omega_0 \oplus \Omega_1$. Then $I_{0,a}$ is injective.*

PROOF. This proof is very similar in spirit that of Theorems 12.1 and 10.3. Suppose there is a smooth $f \in \Omega_0$ such that $I_{0,a}f = 0$. By Theorem 5.10 there is a smooth function u such that $Xu + au = -f$ and $u|_{\partial SM} = 0$.

Using Proposition 12.5 we know that there exists w holomorphic with $Xw = a$. Thus we may write

eq:noatt-2

$$X(e^w u) = e^w((Xw)u + Xu) = -fe^w. \tag{12.2}$$

Consider

$$q := \sum_{-\infty}^{-1} (e^w u)_k.$$

Since fe^w is holomorphic, (12.2) gives

$$Xq = \eta_+ q_{-2} + \eta_+ q_{-1} \in \Omega_{-1} \oplus \Omega_0.$$

But $q|_{\partial SM} = 0$, hence splitting into even and odd degrees, Theorem 10.3 gives that $q = 0$. This means that $e^w u$ is holomorphic and thus u is holomorphic. Using Proposition 12.5 again but with $\tilde{w}$ anti-holomorphic we deduce that u is also anti-holomorphic. Hence $u = u_0$. To complete the proof we need to show that u_0 also vanishes (and hence $f = 0$ as well).

Going back to the transport equation $Xu + au = -f$ we see that if we focus on degree -1 we have $\eta_- u_0 + a_{-1}u_0 = 0$ with $u_0|_{\partial M} = 0$. Solve for $b \in \Omega_0$, $\eta_- b = a_{-1}$. Then

$$\eta_-(e^b u_0) = 0$$

and $e^b u_0$ is a holomorphic function on M that vanishes on the boundary, so it must be zero everywhere.

□

G: After this result there is not much else to do except discuss tensor tomography with attenuation a as above. We can also set that up as exercise.

12.3. Relation to Boman's results, analiticity, counterexamples

12.4. Open problem with polynomial attenuation

CHAPTER 13

Non-abelian X-ray transforms

Let (M, g) be a non-trapping manifold of dimension d with strictly convex boundary ∂M. Consider a matrix attenuation $\mathcal{A}$ as in Section 5.3, namely, let $\mathcal{A} : SM \to \mathbb{C}^{n\times n}$ be a smooth function. Consider (M, g) isometrically embedded in a closed manifold (N, g) and we extend $\mathcal{A}$ smoothly to N. Under these assumptions $\mathcal{A}$ on N defines a *smooth* cocycle over the geodesic flow φ_t of (N, g). Recall that the cocycle takes values in the group $GL(n, \mathbb{C})$ and is determined by the following matrix ODE along the orbits of the geodesic flow

$$\frac{d}{dt}C(x, v, t) + \mathcal{A}(\varphi_t(x, v))C(x, v, t) = 0, \qquad C(x, v, 0) = \mathrm{Id}.$$

In Lemma 5.6 we have seen that the function

$$U_+(x, v) := [C(x, v, \tau(x, v)]^{-1}$$

solves

eq:weightplus

$$\begin{cases} XU_+ + \mathcal{A}U_+ = 0, \\ U_+|_{\partial_- SM} = \mathrm{Id}. \end{cases} \tag{13.1}$$

Definition 13.1. The *scattering data* of $\mathcal{A}$ is the map $C_{\mathcal{A},+} : \partial_+ SM \to GL(n, \mathbb{C})$ given by

$$C_{\mathcal{A},+} := U_+|_{\partial_+ SM}.$$

We shall also call $C_{\mathcal{A},+}$ the non-abelian X-ray transform of $\mathcal{A}$.

Note that $C_{\mathcal{A},+} \in C^\infty(\partial_+ SM, \mathbb{C}^{n\times n})$. We can also consider the unique solution of

eq:weightminus

$$\begin{cases} XU_- + \mathcal{A}U_- = 0, \\ U_-|_{\partial_+ SM} = \mathrm{Id} \end{cases} \tag{13.2}$$

and define scattering data $C_{\mathcal{A},-} : \partial_- SM \to GL(n, \mathbb{C})$ by setting

$$C_{\mathcal{A},-} := U_-|_{\partial_- SM}.$$

Both quantities are related by

eq:relplusminus

$$C_{\mathcal{A},-} = [C_{\mathcal{A},+}]^{-1} \circ \alpha. \tag{13.3}$$

Exercise 13.2. Prove (13.3).

From now on we shall only work with $C_{\mathcal{A},+}$ and we shall drop the subscript $+$ from the notation.

13.1. Pseudo-linearization identity

Given two $\mathcal{A}, \mathcal{B} \in C^\infty(SM, \mathbb{C}^{n\times n})$ we would like to have a formula relating $C_\mathcal{A}$ and $C_\mathcal{B}$ with certain attenuated X-ray transform. We first introduce the map $E(\mathcal{A}, \mathcal{B}) : SM \to \mathrm{End}(\mathbb{C}^{n\times n})$ given by

$$E(\mathcal{A}, \mathcal{B})U := \mathcal{A}U - U\mathcal{B}.$$

Here, $\mathrm{End}(\mathbb{C}^{n\times n})$ denotes the linear endomorphisms of $\mathbb{C}^{n\times n}$.

Proposition 13.3. Let (M, g) be a non-trapping manifold with strictly convex boundary. Given $\mathcal{A}, \mathcal{B} \in C^\infty(SM, \mathbb{C}^{n\times n})$, we have

eq:pseudo-linearization

$$C_\mathcal{A} C_\mathcal{B}^{-1} = \mathrm{Id} + I_{E(\mathcal{A},\mathcal{B})}(\mathcal{A} - \mathcal{B}), \tag{13.4}$$

where $I_{E(\mathcal{A},\mathcal{B})}$ denotes the attenuated X-ray transform with attenuation $E(\mathcal{A}, \mathcal{B})$ as defined in Definition 5.7.

Proof. Consider the fundamental solutions for both $\mathcal{A}$ and $\mathcal{B}$, namely

$$\begin{cases} XU_\mathcal{A} + \mathcal{A}U_\mathcal{A} = 0, \\ U_\mathcal{A}|_{\partial_- SM} = \mathrm{Id}, \end{cases}$$

and

$$\begin{cases} XU_\mathcal{B} + \mathcal{A}U_\mathcal{B} = 0, \\ U_\mathcal{B}|_{\partial_- SM} = \mathrm{Id}. \end{cases}$$

Let $W := U_\mathcal{A} U_\mathcal{B}^{-1} - \mathrm{Id}$. A direct computation shows that

$$\begin{cases} XW + \mathcal{A}W - W\mathcal{B} = -(\mathcal{A} - \mathcal{B}), \\ W|_{\partial_- SM} = 0. \end{cases}$$

By definition of $I_{E(\mathcal{A},\mathcal{B})}$ we have

$$I_{E(\mathcal{A},\mathcal{B})}(\mathcal{A} - \mathcal{B}) = W|_{\partial_+ SM}$$

and since by construction $W|_{\partial_+ SM} = C_\mathcal{A} C_\mathcal{B}^{-1} - \mathrm{Id}$, the proposition follows.

□

Remark 13.4. Note that the function $U := U_\mathcal{A} U_\mathcal{B}^{-1}$ satisfies

$$\begin{cases} \mathcal{B} = U^{-1}XU + U^{-1}\mathcal{A}U, \\ U|_{\partial_- SM} = \mathrm{Id}. \end{cases}$$

remark:gaugeU

Using this identity we can establish when two attenuations $\mathcal{A}, \mathcal{B} \in C^\infty(SM, \mathbb{C}^{n\times n})$ have the same non-abelian X-ray data:

Proposition 13.5. Let (M, g) be a non-trapping manifold with strictly convex boundary. Given $\mathcal{A}, \mathcal{B} \in C^\infty(SM, \mathbb{C}^{n\times n})$, we have $C_\mathcal{A} = C_\mathcal{B}$ if and only if there exists a smooth $U : SM \to GL(n, \mathbb{C})$ with $U|_{\partial SM} = \mathrm{Id}$ and such that

$$\mathcal{B} = U^{-1}XU + U^{-1}\mathcal{A}U.$$

proposition:gaugeU

Proof. If such a smooth function U exists it is straightforward from the definitions that $C_\mathcal{A} = C_\mathcal{B}$. Indeed the function $V = UU_\mathcal{B}$ satisfies $XV + \mathcal{A}V = 0$ and $V|_{\partial_- SM} = \mathrm{Id}$ and thus $V = U_\mathcal{A}$ and consequently $C_\mathcal{A} = C_\mathcal{B}$. Conversely, if the non-abelian X-ray transforms agree, the function W in the proof of Proposition 13.4 has zero boundary value and by Theorem 5.10 is must be smooth. Hence $U = W + \mathrm{Id}$ is smooth and by Remark 13.4 it satisfies the required equation.

$\square$

EXERCISE 13.6. Consider the Hermitian inner product on the set of $n \times n$ matrices $\mathbb{C}^{n\times n}$ given by $(U,V) = \operatorname{trace}(UV^*)$ where V^* denotes the conjugate transpose of V. Show that the adjoint of $E(\mathcal{A},\mathcal{B})$ with respect to this inner product is

$$[E(\mathcal{A},\mathcal{B})]^* U = E(\mathcal{A}^*,\mathcal{B}^*)U.$$

Conclude that if both $\mathcal{A}$ and $\mathcal{B}$ are skew-hermitian, i.e. $\mathcal{A}^* = -\mathcal{A}$ and $\mathcal{B}^* = -\mathcal{B}$, then $E^* = -E$ as well. `exe:unitary`

13.2. Elementary background on connections

Consider the trivial bundle $M \times \mathbb{C}^n$. For us a connection A will be a complex $n \times n$ matrix whose entries are smooth 1-forms on M. Another way to think of A is to regard it as a smooth map $A : TM \to \mathbb{C}^{n\times n}$ which is linear in $v \in T_xM$ for each $x \in M$.

Very often in physics and geometry one considers *unitary* or *Hermitian* connections. This means that the range of A is restricted to skew-Hermitian matrices. In other words, if we denote by $\mathfrak{u}(n)$ the Lie algebra of the unitary group $U(n)$, we have a smooth map $A : TM \to \mathfrak{u}(n)$ which is linear in the velocities. There is yet another equivalent way to phrase this. The connection A induces a covariant derivative d_A on sections $s \in C^\infty(M,\mathbb{C}^n)$ by setting $d_A s = ds + As$. Then A being Hermitian or unitary is equivalent to requiring compatibility with the standard Hermitian inner product of $\mathbb{C}^n$ in the sense that

$$d\langle s_1, s_2\rangle = \langle d_A s_1, s_2\rangle + \langle s_1, d_A s_2\rangle$$

for any pair of functions s_1, s_2.

Given two unitary connections A and B we shall say that A and B are gauge equivalent if there exists a smooth map $u : M \to U(n)$ such that

`eq:1`
$$B = u^{-1}du + u^{-1}Au. \tag{13.5}$$

The *curvature* of the connection is the 2-form F_A with values in $\mathfrak{u}(n)$ given by

$$F_A := dA + A \wedge A.$$

If A and B are related by (13.5) then:

$$F_B = u^{-1}\, F_A\, u.$$

Given a smooth curve $\gamma : [a,b] \to M$, the *parallel transport* along γ is obtained by solving the linear differential equation in $\mathbb{C}^n$:

`eq:2`
$$\begin{cases} \dot{s} + A(\gamma(t),\dot{\gamma}(t))s = 0, \\ s(a) = w \in \mathbb{C}^n. \end{cases} \tag{13.6}$$

The isometry $P_A(\gamma) : \mathbb{C}^n \to \mathbb{C}^n$ is defined as $P_A(\gamma)(w) := s(b)$. We may also consider the fundamental unitary matrix solution $U : [a,b] \to U(n)$ of (13.6). It solves

`eq:3`
$$\begin{cases} \dot{U} + A(\gamma(t),\dot{\gamma}(t))U = 0, \\ U(a) = \mathrm{Id}. \end{cases} \tag{13.7}$$

Clearly $P_A(\gamma)(w) = U(b)w$.

A connection A naturally gives rise to a matrix attenuation of a special type simply setting $\mathcal{A}(x,v) := A(x,v)$. Note that since A is a matrix of 1-forms, it is completely determined by its values on SM. The scattering data $C_A : \partial_+ M \to GL(n,\mathbb{C})$ encapsulates the parallel transport of A along geodesics running between boundary points.

13.3. Structure equations including a connection

section:auxback

In this section we consider an oriented Riemannian surface (M,g) and a connection A on the trivial bundle $M\times\mathbb{C}^n$. Recall that the metric g induces a Hodge star operator $\star$ acting on forms. If we regard connections as functions $A : SM \to \mathbb{C}^{n\times n}$ with $A\in\Omega_{-1}\oplus\Omega_1$, then $\star A = -VA$. If iv denotes rotation of $v\in T_xM$ by $\pi/2$ according to the orientation of the surface (or multiplication by i using the complex structure induced by g), then $VA(x,v) = A(x,iv) = -\star A(x,v)$.

The main purpose of this section is to establish the following lemma that describes backet relations when X is replaced by $X+A$ and $X_\perp$ by $X_\perp+\star A$. Here we understand that A and $\star A$ act on functions by multiplication.

LEMMA 13.7. *The following equations hold:*

$$[V, X+A] = -(X_\perp + \star A)$$
$$[V, X_\perp + \star A] = X + A$$
$$[X+A, X_\perp+\star A] = -KV - \star F_A.$$

lemma:bracketsA

PROOF. Let us recall the standard brackets:

$$[V,X] = -X_\perp$$
$$[V,X_\perp] = X$$
$$[X,X_\perp] = -KV.$$

Hence the first two bracket relations in the lemma follow from $[V,A] = V(A) = -\star A$ and $[V,\star A] = -V^2(A) = A$. To check the third and last bracket it suffices to prove that

eq:FA

$$\star F_A = X_\perp(A) - X(\star A) + [\star A, A]. \tag{13.8}$$

GIven a unit norm vector $v\in T_xM$, $\{v, iv\}$ is a positively oriented orthonormal basis. Thus

$$\star F_A(x) = F_A(v,iv) = dA(v,iv) + (A\wedge A)(v,iv) = dA(v,iv) + [A(v), A(iv)].$$

But $[\star A, A](x,v) = [-A(iv), A(v)]$, and thus to complete the proof of (13.8) we just have to show that

$$X_\perp(A)(x,v) - X(\star A)(x,v) = dA(v,iv).$$

Let $\pi : SM \to M$ be the canonical projection. Recall that $d\pi(X(x,v)) = v$ and $d\pi(X_\perp(x,v)) = -iv$. Consider π^*A and note (using the standard formula for d applied to π^*A):

$$d(\pi^*A)(X,X_\perp) = X\pi^*A(X_\perp) - X_\perp(\pi^*A(X)) - \pi^*A([X,X_\perp]).$$

By the structure equations, the term $[X,X_\perp]$ is purely vertical, hence it is killed by π^*A. Next note that $\pi^*A(X_\perp)(x,v) = A(-iv) = (\star A)(v)$ and $\pi^*A(X) = A(v)$. Finally since

$$d(\pi^*A)(X,X_\perp) = \pi^*dA(X,X_\perp) = dA(d\pi(X), d\pi(X_\perp)) = -dA(v,iv)$$

we are done.

□

Given a connection $A \in \Omega_{-1} \oplus \Omega_1$ we write it as $A = A_{-1} + A_1$ with $A_{\pm 1} \in \Omega_{\pm 1}$. We can consider modified operators

$$\mu_\pm := \eta_\pm + A_{\pm 1}.$$

Clearly $X + A = \mu_+ + \mu_-$. These operators also satisfy nice brackets relations:

LEMMA 13.8. *The following bracket relations hold*

$$[\mu_\pm, iV] = \pm\mu_\pm, \quad [\mu_+, \mu_-] = \frac{i}{2}(KV + \star F_A).$$

lemma:bracketsmu *Moreover*

$$\mu_+ : \Omega_k \to \Omega_{k+1}, \quad \mu_- : \Omega_k \to \Omega_{k-1}$$

and if A is unitary $(\mu_\pm)^* = -\mu_\mp$.

PROOF. We only prove the relation $[\mu_+, \mu_-] = \frac{i}{2}(KV + \star F_A)$, the rest is left as exercise. First we note that

$$\mu_\pm = \frac{(X + A) \pm i(X_\perp + \star A)}{2}.$$

Hence

$$[\mu_+, \mu_-] = \frac{i}{2}[X_\perp + \star A, X + A]$$

and the desired relation follows from Lemma 13.7.

□

EXERCISE 13.9. Show that $X + A$ maps even functions to odd functions and odd functions to even functions.

EXERCISE 13.10. If H denotes the Hilbert transform, show that for any smooth function $u \in C^\infty(SM, \mathbb{C}^n)$:

$$[H, X + A]u = (X_\perp + \star A)(u_0) + ((X_\perp + \star A)(u))_0.$$

13.4. Scattering rigidity for connections

In this section we would like to consider the following geometric inverse problem: is a connection A determined by C_A?

Right away, we see that the problem has a gauge: if $u : M \to GL(n, \mathbb{C})$ is a smooth map with $u|_{\partial M} = \mathrm{Id}$, then

$$C_{u^{-1}du + u^{-1}Au} = C_A.$$

Our goal would be to show:

THEOREM 13.11. *Let (M, g) be a simple surface and let A and B be two unitary connections with $C_A = C_B$. Then there exists a smooth $u : M \to U(n)$ with* thm:scatteringA *$u|_{\partial M} = \mathrm{Id}$ such that $B = u^{-1}du + u^{-1}Au$.*

From Proposition 13.5 we know that $C_A = C_B$ means that there exists a smooth $U : SM \to U(n)$ such that $U|_{\partial SM} = \mathrm{Id}$ and

$$B = U^{-1}XU + U^{-1}AU.$$

Notice the similarity of this equation with our goal, which is to show that

$$B = u^{-1}du + u^{-1}Au.$$

In fact if U only had dependence on x and not on v, then $U = u$, $XU(x, v) = d_x u(v)$ and we would be done. We will accomplish this for a simple surface. We start by rephrasing our problem in terms of an attenuated ray transform. Showing that U depends only on x is equivalent to showing that $W = U - \mathrm{Id}$ depends only on x. But as we have seen, W is associated precisely with the attenuated X-ray transform $I_{E(A,B)}(A-B)$ and if $C_A = C_B$, then this transform vanishes. Note that $A - B \in \Omega_{-1} \oplus \Omega_1$.

Hence, making the choice to ignore the specific form $E(A, B)$ but noting that it is unitary by Exercise 13.6, it suffices to show:

THEOREM 13.12. *Let (M, g) be a simple surface and let A be a unitary connection. Suppose there is a smooth function $u : SM \to \mathbb{C}^n$ such that*

$$\begin{cases} Xu + Au = f \in \Omega_{-1} \oplus \Omega_1, \\ u|_{\partial SM} = 0. \end{cases}$$

THM:KEYA *Then $u = u_0$ and $f = d_A u_0 = du_0 + Au_0$ with $u_0|_{\partial M} = 0$.*

We will prove this theorem in the next section.

13.5. Proof of Theorem 13.12

The first key ingredient is an energy identity which generalizes the standard Pestov identity from Proposition 4.12 to the case when a connection is present. Recall that the curvature F_A of the connection A is defined as $F_A = dA + A \wedge A$ and $\star F_A$ is a function $\star F_A : M \to \mathfrak{u}(n)$.

LEMMA 13.13 (Energy identity). *If $u : SM \to \mathbb{C}^n$ is a smooth function such that $u|_{\partial SM} = 0$, then*

$$\|(X + A)Vu\|^2 - (K\,Vu, Vu) - (\star F_A u, Vu) = \|V(X + A)(u)\|^2 - \|(X + A)u\|^2.$$

lemma:pestov

PROOF. We adopt the same approach as in the proof of Proposition 4.12 we define $P = V(X + A)$. Since A is a unitary connection, $A^* = -A$ and hence $P^* = (X + A)V$. Let us compute using the structure equations from Lemma 13.7:

$$\begin{aligned}
[P^*, P] &= (X + A)VV(X + A) - V(X + A)(X + A)V \\
&= V(X + A)V(X + A) + (X_\perp + \star A)V(X + A) \\
&\quad - V(X + A)V(X + A) - V(X + A)(X_\perp + \star A) \\
&= V[X_\perp + \star A, X + A] - (X + A)^2 = -(X + A)^2 + VKV + \star F_A V.
\end{aligned}$$

The identity in the lemma now follows from this bracket calculation and

$$\|Pu\|^2 = \|P^*u\|^2 + ([P^*, P]u, u)$$

for a smooth u with $u|_{\partial SM} = 0$. $\square$

REMARK 13.14. The same Energy identity holds true for *closed* surfaces.

To use the Energy identity we need to control the signs of various terms. The first easy observation is the following:

LEMMA 13.15. *Assume* $(X+A)u = f = f_{-1}+f_0+f_1 \in \Omega_{-1}\oplus\Omega_0\oplus\Omega_1$. *Then*

$$\|V(X+A)u\|^2 - \|(X+A)u\|^2 = -\|f_0\|^2 \leq 0.$$

lemma:easy

PROOF. It suffices to note the identities:

$$\|V(X+A)u\|^2 = \|V(f_{-1}+f_1)\|^2 = \|-if_{-1}+if_1\|^2 = \|f_{-1}\|^2+\|f_1\|^2,$$
$$\|f\|^2 = \|f_{-1}+f_1\|^2+\|f_0\|^2.$$

□

Next we have the following lemma due to the absence of conjugate points on simple surfaces (compare with Proposition 4.13).

LEMMA 13.16. *Let M be a compact simple surface. If $u : SM \to \mathbb{C}^n$ is a smooth function such that $u|_{\partial SM} = 0$, then*

$$\|(X+A)Vu\|^2 - (K\,Vu, Vu) \geq 0.$$

lemma:nonconj

G: this is the original proof and a bit different from the proof of Proposition 4.13, should we leave both?

PROOF. Consider a smooth function $a : SM \to \mathbb{R}$ which solves the Riccati equation $X(a)+a^2+K=0$. These exist by the absence of conjugate points. Set for simplicity $\psi = V(u)$. Clearly $\psi|_{\partial SM} = 0$. G: prove it.

Let us compute using that A is skew-Hermitian:

$$\begin{aligned}|(X+A)(\psi)-a\psi|^2_{\mathbb{C}^n} &= |(X+A)(\psi)|^2_{\mathbb{C}^n} - 2\Re\langle (X+A)(\psi), a\psi\rangle_{\mathbb{C}^n} + a^2|\psi|^2_{\mathbb{C}^n}\\ &= |(X+A)(\psi)|^2_{\mathbb{C}^n} - 2a\Re\langle X(\psi),\psi\rangle_{\mathbb{C}^n} + a^2|\psi|^2_{\mathbb{C}^n}.\end{aligned}$$

Using the Riccati equation we have

$$X(a|\psi|^2) = (-a^2-K)|\psi|^2 + 2a\Re\langle X(\psi),\psi\rangle_{\mathbb{C}^n}$$

thus

$$|(X+A)(\psi)-a\psi|^2_{\mathbb{C}^n} = |(X+A)(\psi)|^2_{\mathbb{C}^n} - K|\psi|^2_{\mathbb{C}^n} - X(a|\psi|^2_{\mathbb{C}^n}).$$

Integrating this equality with respect to $d\Sigma^3$ and using that ψ vanishes on $\partial(SM)$ we obtain

$$\|(X+A)(\psi)\|^2 - (K\,\psi,\psi) = \|(X+A)(\psi)-a\psi\|^2 \geq 0.$$

□

We now show:

THEOREM 13.17. *Let $f : SM \to \mathbb{C}^n$ be a smooth function. Suppose $u : SM \to \mathbb{C}^n$ satisfies*

$$\begin{cases} Xu + Au = f,\\ u|_{\partial SM} = 0.\end{cases}$$

Then if $f_k = 0$ for all $k \leq -2$ and $i\star F_A(x)$ is a negative definite Hermitian matrix for all $x \in M$, the function u must be holomorphic. Moreover, if $f_k = 0$ for all $k \geq 2$ and $i\star F_A(x)$ is a positive definite Hermitian matrix for all $x \in M$, the function u must be antiholomorphic.

thm:pi

PROOF. Let us assume that $f_k = 0$ for $k \leq -2$ and $i \star F_A$ is a negative definite Hermitian matrix; the proof of the other claim is similar.

Let $q := \sum_{-\infty}^{-1} u_k$. We need to show that $q = 0$. Since $A = A_{-1} + A_1$ and $f_k = 0$ for $k \leq -2$, we see that $(X + A)q \in \Omega_{-1} \oplus \Omega_0$. Now we are in good shape to use the Energy identity from Lemma 13.13. We will apply it to q, note that $q|_{\partial SM} = 0$. We know from Lemma 13.15 that its right hand side is ≤ 0 and using Lemma 13.16 we deduce

$$(\star F_A v, V v) \geq 0.$$

But on the other hand

$$(\star F_A v, V v) = -4 \sum_{k=-\infty}^{-1} k(i \star F_A u_k, u_k)$$

and since $i \star F_A$ is negative definite this forces $u_k = 0$ for all $k < 0$.

□

We are now ready to complete the proof of Theorem 13.12.

PROOF. Consider the area form ω_g of the metric g. Since M is a disk there exists a smooth 1-form φ such that $\omega_g = d\varphi$. Given $s \in \mathbb{R}$, consider the Hermitian connection

$$A_s := A - is\varphi\,\mathrm{Id}.$$

Clearly its curvature is given by

$$F_{A_s} = F_A - is\omega_g \mathrm{Id}$$

therefore

$$i \star F_{A_s} = i \star F_A + s\mathrm{Id},$$

from which we see that there exists $s_0 > 0$ such that for $s > s_0$, $i \star F_{A_s}$ is positive definite and for $s < -s_0$, $i \star F_{A_s}$ is negative definite.

Let e^{sw} be an integrating factor of $-is\varphi$. In other words $w : SM \to \mathbb{C}$ satisfies $X(w) = i\varphi$. By Proposition 10.1 we know we can choose w to be holomorphic or antiholomorphic. Observe now that $u_s := e^{sw}u$ satisfies $u_s|_{\partial SM} = 0$ and solves

$$(X + A_s)(u_s) = e^{sw} f.$$

Choose w to be holomorphic. Since $f \in \Omega_{-1} \oplus \Omega_1$, the function $e^{sw} f$ has the property that its Fourier coefficients $(e^{sw} f)_k$ vanish for $k \leq -2$. Choose s such that $s < -s_0$ so that $i \star F_{A_s}$ is negative definite. Then Theorem 13.17 implies that u_s is holomorphic and thus $u = e^{-sw}u_s$ is also holomorphic.

Choosing w antiholomorphic and $s > s_0$ we show similarly that u is antiholomorphic. This implies that $u = u_0$ which together with $(X + A)u = f$, gives $du_0 + Au_0 = f$. □

13.6. An alternative proof of tensor tomography

In this section we shall use the ideas from the previous section to give an alternative proof of Corollary 10.6 for the case of (M, g) a simple surface.

Corollary 10.6 is an immediate consequence of the next two results.

osition_onesidedfourier1

PROPOSITION 13.18. Let (M, g) be a simple surface, and assume that $u \in C^\infty(SM)$ satisfies $Xu = -f$ in SM with $u|_{\partial SM} = 0$. If $m \geq 0$ and if $f \in C^\infty(SM)$ is such that $f_k = 0$ for $k \leq -m - 1$, then $u_k = 0$ for $k \leq -m$.

osition_onesidedfourier2 PROPOSITION 13.19. Let (M,g) be a simple surface, and assume that $u \in C^\infty(SM)$ satisfies $Xu = -f$ in SM with $u|_{\partial SM} = 0$. If $m \geq 0$ and if $f \in C^\infty(SM)$ is such that $f_k = 0$ for $k \geq m+1$, then $u_k = 0$ for $k \geq m$.

Below we will use the operators $\mu_\pm$ introduced in Section 13.3. Recall that when A is unitary

mu_adjoint
$$(\mu_\pm u, v) = -(u, \mu_\mp v), \quad u, v \in C^\infty(SM) \text{ with } u|_{\partial SM} = 0 \text{ or } v|_{\partial SM} = 0. \tag{13.9}$$

We also have commutator formula from Lemma 13.8:

mu_commutator
$$[\mu_+, \mu_-]u = \frac{i}{2}(KVu + (\star F_A)u). \tag{13.10}$$

We will only prove Proposition 13.18, the proof of the other result being completely analogous.

PROOF OF PROPOSITION 13.18. Assume that f is even, m is even, and u is odd. Let ω_g be the area form of (M,g) and choose a real valued 1-form φ with $d\varphi = \omega_g$. Consider the unitary connection

$$A(x,v) := is\varphi_x(v)$$

where $s > 0$ is a fixed number to be chosen later. Then $i\star F_A = -s$. By Proposition 10.1, there exists a holomorphic $w \in C^\infty(SM)$ satisfying $Xw = -i\varphi$. We may assume that w is even. The functions $\tilde{u} := e^{sw}u$ and $\tilde{f} := e^{sw}f$ then satisfy

$$(X + A)\tilde{u} = -\tilde{f} \text{ in } SM, \quad \tilde{u}|_{\partial SM} = 0.$$

Using that e^{sw} is holomorphic, we have $\tilde{f}_k = 0$ for $k \leq -m-1$. Also, since e^{sw} is even, $\tilde{f}$ is even and $\tilde{u}$ is odd. We now define

$$v := \sum_{k=-\infty}^{-m-1} \tilde{u}_k.$$

Then $v \in C^\infty(SM)$, $v|_{\partial SM} = 0$, and v is odd. Also, $((X+A)v)_k = \mu_+ v_{k-1} + \mu_- v_{k+1}$. If $k \leq -m-2$ one has $((X+A)v)_k = ((X+A)\tilde{u})_k = 0$, and if $k \geq -m+1$ then $((X+A)v)_k = 0$ since $v_j = 0$ for $j \geq -m$. Also $((X+A)v)_{-m-1} = 0$ because v is odd. Therefore the only nonzero Fourier coefficient is $((X+A)v)_{-m}$, and

$$(X + A)v = \mu_+ v_{-m-1} \text{ in } SM, \quad v|_{\partial SM} = 0.$$

We apply the Energy identity in Lemma 13.13 with attenuation A to v, so that

$$\|(X+A)Vv\|^2 - (KVv, Vv) + (\star F_A Vv, v) + \|(X+A)v\|^2 - \|V(X+A)v\|^2 = 0.$$

We know from Lemma 13.16 that if (M,g) is simple and $v|_{\partial SM} = 0$, then

pestov_estimate1
$$\|(X+A)Vv\|^2 - (KVv, Vv) \geq 0. \tag{13.11}$$

We also have

pestov_estimate2
$$(\star F_A Vv, v) = -\sum_{k=-\infty}^{-m-1} i|k|(\star F_A v_k, v_k) = s\sum_{k=-\infty}^{-m-1} |k|\|v_k\|^2. \tag{13.12}$$

For the remaining two terms, we compute

$$\|(X+A)v\|^2 - \|V(X+A)v\|^2 = \|\mu_+ v_{-m-1}\|^2 - m^2\|\mu_+ v_{-m-1}\|^2.$$

If $m = 0$, then this expression is nonnegative and we obtain from the energy identity that $v = 0$. Assume from now on that $m \geq 2$. Using (13.9), (13.10), and the fact that $v_k|_{\partial SM} = 0$ for all k, we have

$$\begin{aligned}\|\mu_+ v_k\|^2 &= \|\mu_- v_k\|^2 + \frac{i}{2}(KVv_k + (\star F_A)v_k, v_k) \\ &= \|\mu_- v_k\|^2 - \frac{s}{2}\|v_k\|^2 - \frac{k}{2}(Kv_k, v_k).\end{aligned}$$

If $k \leq -m-1$ we also have

$$\mu_+ v_{k-1} + \mu_- v_{k+1} = ((X+A)v)_k = 0.$$

We thus obtain

$$\begin{aligned}&\|(X+A)v\|^2 - \|V(X+A)v\|^2 = -(m^2-1)\|\mu_+ v_{-m-1}\|^2 \\ &= -(m^2-1)\left[\|\mu_- v_{-m-1}\|^2 - \frac{s}{2}\|v_{-m-1}\|^2 + \frac{m+1}{2}(Kv_{-m-1}, v_{-m-1})\right] \\ &= -(m^2-1)\left[\|\mu_+ v_{-m-3}\|^2 - \frac{s}{2}\|v_{-m-1}\|^2 + \frac{m+1}{2}(Kv_{-m-1}, v_{-m-1})\right] \\ &= -(m^2-1)\Bigg[\|\mu_- v_{-m-3}\|^2 - \frac{s}{2}(\|v_{-m-1}\|^2 + \|v_{-m-3}\|^2) \\ &\qquad + \frac{m+1}{2}(Kv_{-m-1}, v_{-m-1}) + \frac{m+3}{2}(Kv_{-m-3}, v_{-m-3})\Bigg].\end{aligned}$$

Continuing this process, and noting that $\mu_- v_k \to 0$ in $L^2(SM)$ as $k \to -\infty$ (which follows since $\mu_- v \in L^2(SM)$), we obtain

pestov_estimate3 (13.13)

$$\|(X+A)v\|^2 - \|V(X+A)v\|^2 = \frac{m^2-1}{2} s \sum \|v_k\|^2 - \frac{m^2-1}{2}\sum |k|(Kv_k, v_k).$$

Collecting (13.11)–(13.13) and using them in the energy identity implies that

$$0 \geq \frac{m^2-1}{2} s \sum \|v_k\|^2 + \left(s - \frac{m^2-1}{2}\sup_M K\right)\sum |k| \|v_k\|^2.$$

If we choose $s > \frac{m^2-1}{2}\sup_M K$, then both terms above are nonnegative and therefore have to be zero. It follows that $v = 0$, so $\tilde{u}_k = 0$ for $k \leq -m-1$ and also $u_k = 0$ for $k \leq -m-1$ since $u = e^{-sw}\tilde{u}$ where e^{-sw} is holomorphic.

□

13.7. General skew-Hermitian attenuations

Remarkably, many aspects of the arguments done in the previous sections work for general attenuations $\mathcal{A} : SM \to \mathbb{C}^{n\times n}$ as long as $\mathcal{A}^* = -\mathcal{A}$. We begin with the Pestov identity. Define

eq:F1 (13.14) $$F_{\mathcal{A}} := XV(\mathcal{A}) + X_\perp(\mathcal{A}) + [\mathcal{A}, V(\mathcal{A})],$$

eq:F2 (13.15) $$\varphi(\mathcal{A}) := -V^2(\mathcal{A}) - \mathcal{A}.$$

Lemma 13.20 (Energy identity). *Let (M, g) be a compact oriented Riemannian surface with boundary. Assume $\mathcal{A} \in C^\infty(SM, \mathbb{C}^{n\times n})$ is skew-Hermitian, i.e. $\mathcal{A} =$*

$-\mathcal{A}$. *If* $u : SM \to \mathbb{C}^n$ *is a smooth function such that* $u|_{\partial SM} = 0$, *then*

$$\|(X+\mathcal{A})Vu\|^2 - (KVu, Vu) - (F_{\mathcal{A}}u, Vu) + ((X+\mathcal{A})u, \varphi(\mathcal{A})u) = \|V(X+\mathcal{A})(u)\|^2 - \|(X+\mathcal{A})u\|^2.$$

lemma:pestovA

PROOF. If we let $G := X + \mathcal{A}$, then routine calculations show

$$[V, G] = -(X_\perp - V(\mathcal{A})) := -G_\perp$$
$$[V, G_\perp] = G + \varphi(\mathcal{A})$$
$$[G, G_\perp] = -KV - F_{\mathcal{A}}.$$

We adopt the standard approach (as in the proof of Proposition 4.12) and define $P = VG$. Since $\mathcal{A}^* = -\mathcal{A}$ we have $P^* = GV$. Using the bracket relations above we compute:

$$\begin{aligned}[P^*, P] &= GVVG - VGGV \\ &= VGVG + G_\perp VG - VGVG - VGG_\perp \\ &= V[G_\perp, G] - G^2 - \varphi(\mathcal{A})G = -G^2 - \varphi(\mathcal{A})G + VKV + VF_{\mathcal{A}}.\end{aligned}$$

The identity in the lemma now follows from this bracket calculation and

$$\|Pu\|^2 = \|P^*u\|^2 + ([P^*, P]u, u)$$

for a smooth u with $u|_{\partial SM} = 0$. □

LEMMA 13.21. *Let* M *be a compact simple surface and* $\mathcal{A} : SM \to \mathbb{C}^{n\times n}$ *such that* $\mathcal{A}^* = -\mathcal{A}$. *If* $u : SM \to \mathbb{C}^n$ *is a smooth function such that* $u|_{\partial SM} = 0$, *then*

$$\|(X+\mathcal{A})Vu\|^2 - (KVu, Vu) \geq 0.$$

lemma:nonconjv2

The proof of this lemma is exactly the same as the proof of Lemma 13.16. Finally, in Lemma 13.15 we may replace A by $\mathcal{A}$ without trouble.

We can now interpret the quantities (13.14) and (13.15) as naturally appearing as curvature terms of a suitable connection in SM.

Consider the co-frame of 1-forms $\{\omega_1, \omega_2, \psi\}$ dual to the frame of vector fields $\{X, X_\perp, V\}$. The structure equations (3.5), (3.6) and (3.7) imply

eq:structure1 (13.16) $$d\omega_1 = -\psi \wedge \omega_2$$

eq:structure2 (13.17) $$d\omega_2 = \psi \wedge \omega_1$$

eq:structure3 (13.18) $$d\psi = K\omega_1 \wedge \omega_2$$

Given $\mathcal{A} \in C^\infty(SM, \mathbb{C}^{n\times n})$ with $\mathcal{A}^* = -\mathcal{A}$ we define a unitary connection $\mathbb{A}$ on SM by setting

$$\mathbb{A} := \mathcal{A}\,\omega_1 - V(\mathcal{A})\,\omega_2.$$

EXERCISE 13.22. If A is a connection in M, show that

$$\pi^* A = A\omega_1 - V(A)\omega_2.$$

LEMMA 13.23. *With* $\mathbb{A}$ *defined as above we have*

$$F_{\mathbb{A}} = -F_{\mathcal{A}}\,\omega_1 \wedge \omega_2 + \varphi(\mathcal{A})\psi \wedge \omega_2.$$

PROOF. Recall that $F_{\mathbb{A}} = d\mathbb{A} + \mathbb{A}\wedge\mathbb{A}$. We compute

$$\mathbb{A}\wedge\mathbb{A} = (\mathcal{A}\omega_1 - V(\mathcal{A})\omega_2)\wedge(\mathcal{A}\omega_1 - V(\mathcal{A})\omega_2) = -[\mathcal{A}, V(\mathcal{A})]\,\omega_1\wedge\omega_2.$$

Next note

$$\begin{aligned} d\mathbb{A} &= X_\perp(\mathcal{A})\omega_2\wedge\omega_1 + V(\mathcal{A})\psi\wedge\omega_1 + \mathcal{A}d\omega_1 \\ &\quad - XV(\mathcal{A})\omega_1\wedge\omega_2 - V^2(\mathcal{A})\psi\wedge\omega_2 - V(\mathcal{A})d\omega_2. \end{aligned}$$

Using the structure equations (13.16) and (13.17) we see that

$$d\mathbb{A} = -(XV(\mathcal{A}) + X_\perp(\mathcal{A})\omega_1\wedge\omega_2 - (V^2(\mathcal{A}) + \mathcal{A})\psi\wedge\omega_2$$

and the lemma follows.

□

13.8. Injectivity for connections and Higgs fields

We now wish to extend the key Theorem 13.12 to include a *Higgs field*. For us this means an element $\Phi \in C^\infty(M, \mathbb{C}^{n\times n})$. We will assume that Φ is skew-Hermitian, i.e. $\Phi^* = -\Phi$.

THEOREM 13.24. *Let (M, g) be a simple surface, A a unitary connection and Φ a skew-Hermitian Higgs field. Suppose there is a smooth function $u : SM \to \mathbb{C}^n$ such that*

$$\begin{cases} Xu + (A+\Phi)u = f \in \Omega_{-1}\oplus\Omega_0\oplus\Omega_1, \\ u|_{\partial SM} = 0. \end{cases}$$

THM:KEYB *Then $u = u_0$ and $f = d_A u_0 + \Phi u_0 = du_0 + Au_0 + \Phi u_0$ with $u_0|_{\partial M} = 0$.*

PROOF. We will prove that u is both holomorphic and antiholomorphic. If this is the case then $u = u_0$ only depends on x and $u_0|_{\partial M} = 0$, and we have

$$du_0 + Au_0 = f_{-1} + f_1, \quad \Phi u_0 = f_0.$$

The first step, as in the proof of Theorem 13.12, is to replace A by a connection whose curvature has a definite sign. We choose a real valued 1-form φ such that $d\varphi = \omega_g$, and let

$$A_s := A + is\varphi\mathrm{Id}.$$

Here $s > 0$ so that A_s is unitary and $i\star F_{A_s} = i\star F_A - s\mathrm{Id}$. We use Proposition 10.1 to find a holomorphic scalar function $w \in C^\infty(SM)$ satisfying $Xw = -i\varphi$. Then $u_s = e^{sw}u$ satisfies

$$(X + A_s + \Phi)u_s = -e^{sw}f.$$

Let $v := \sum_{-\infty}^{-1}(u_s)_k$. Since $(e^{sw}f)_k = 0$ for $k \leq -2$

$$(X + A_s + \Phi)v \in \Omega_{-1}\oplus\Omega_0.$$

Let $h := [(X + A_s + \Phi)v]_0$.

We apply the Energy identity in Lemma 13.20 with attenuation $\mathcal{A} := A_s + \Phi$ to the function v, which also satisfies $v|_{\partial SM} = 0$ to obtain

pestov_higgs_identity (13.19)

$$\begin{aligned} &\|(X+A_s+\Phi)(Vv)\|^2 - (K\,V(v), V(v)) + \|(X+A_s+\Phi)v\|^2 - \|V[(X+A_s+\Phi)v]\|^2 \\ &\qquad - (\star F_{A_s}v, V(v)) - \Re((\star d_{A_s}\Phi)v, V(v)) - \Re(\Phi v, (X+A_s+\Phi)v) = 0. \end{aligned}$$

Note that $\varphi(\mathcal{A}) = -\Phi$ and $F_{\mathcal{A}} = \star F_{A_s} + \star d_{A_s}\Phi$, where $d_{A_s}\Phi = d\Phi + [A_s, \Phi]$. It was proved in Lemmas 13.15 and 13.21 that

stov_higgs_intermediate1
$$\|(X + A_s + \Phi)(Vv)\|^2 - (K\,V(v), V(v)) \geq 0, \tag{13.20}$$

stov_higgs_intermediate2
$$\|(X + A_s + \Phi)v\|^2 - \|V[(X + A_s + \Phi)v]\|^2 = \|h\|^2 \geq 0. \tag{13.21}$$

The term involving the curvature of A_s satisfies

stov_higgs_intermediate3
$$-(\star F_{A_s}v, V(v)) = \sum_{k=-\infty}^{-1} |k|(-i \star F_{A_s}v_k, v_k) \geq (s - \|F_A\|_{L^\infty(M)}) \sum_{k=-\infty}^{-1} |k|\|v_k\|^2. \tag{13.22}$$

Here we can choose $s > 0$ large to obtain a positive term. For the next term in (13.19), we consider the Fourier expansion of $d_{A_s}\Phi = d_A\Phi = a_1 + a_{-1}$ where $a_{\pm 1} \in \Omega_{\pm 1}$. Note that $\star d_A\Phi = -V(d_A\Phi) = -ia_1 + ia_{-1}$. Then, since $v_k = 0$ for $k \geq 0$,

$$\begin{aligned}
((\star d_A\Phi)v, V(v)) &= \sum_{k=-\infty}^{-1} (-ia_1v_{k-1} + ia_{-1}v_{k+1}), ikv_k) \\
&= \sum_{k=-\infty}^{-1} |k| \left[(a_1v_{k-1}, v_k) - (a_{-1}v_{k+1}, v_k)\right].
\end{aligned}$$

Consequently

stov_higgs_intermediate4
$$\Re((\star d_A\Phi)v, V(v)) \leq C_{A,\Phi} \sum_{k=-\infty}^{-1} |k|\|v_k\|^2. \tag{13.23}$$

The last term in (13.19) requires the most work. We note that $v_k = 0$ for $k \geq 0$ and that $(X + A_s + \Phi)v \in \Omega_{-1} \oplus \Omega_0$. Therefore

$$(\Phi v, (X + A_s + \Phi)v) = (\Phi v_{-1}, ((X + A_s + \Phi)v)_{-1}).$$

Recall that we may write $X = \eta_+ + \eta_-$ where $\eta_+ = (X + iX_\perp)/2 : \Omega_k \to \Omega_{k+1}$ and $\eta_- = (X - iX_\perp)/2 : \Omega_k \to \Omega_{k-1}$. Expand $A = A_1 + A_{-1}$ and $\varphi = \varphi_1 + \varphi_{-1}$ so that $A_s = (A_1 + is\varphi_1\mathrm{Id}) + (A_{-1} + si\varphi_{-1}\mathrm{Id}) := a_1 + a_{-1}$ where $a_j \in \Omega_j$. Since A_s is unitary we have $a_{\pm 1}^* = -a_{\mp 1}$.

The fact that $(X + A_s + \Phi)v \in \Omega_{-1} \oplus \Omega_0$ implies that

$$\begin{gathered}
\eta_+v_{-2} + a_1v_{-2} + \Phi v_{-1} = ((X + A_s + \Phi)v)_{-1}, \\
\eta_+v_{-k-1} + a_1v_{-k-1} + \eta_-v_{-k+1} + a_{-1}v_{-k+1} + \Phi v_{-k} = 0, \qquad k \geq 2.
\end{gathered}$$

Note that $(\eta_\pm a, b) = -(a, \eta_\mp b)$ when $a|_{\partial(SM)} = 0$. Using this and that Φ is skew-Hermitian, we have

$$\begin{aligned}
&\Re(\Phi v_{-1}, ((X + A_s + \Phi)v)_{-1}) = \Re(\Phi v_{-1}, \eta_+v_{-2} + a_1v_{-2} + \Phi v_{-1}) \\
&= \Re\left[(\eta_-v_{-1}, \Phi v_{-2}) - ((\eta_-\Phi)v_{-1}, v_{-2}) + (\Phi v_{-1}, a_1v_{-2}) + \|\Phi v_{-1}\|^2\right].
\end{aligned}$$

We claim that for any $N \geq 1$ one has

$$\Re(\Phi v_{-1}, ((X + A_s + \Phi)v)_{-1}) = p_N + q_N$$

where

$$p_N := (-1)^{N-1}\Re(\eta_- v_{-N}, \Phi v_{-N-1}),$$

$$q_N := \Re\sum_{j=1}^{N}\left[(-1)^j((\eta_-\Phi)v_{-j}, v_{-j-1}) + (-1)^{j-1}(\Phi v_{-j}, a_1 v_{-j-1}) + (-1)^{j-1}\|\Phi v_{-j}\|^2\right]$$
$$+\Re\sum_{j=1}^{N-1}(-1)^j(a_{-1}v_{-j}, \Phi v_{-j-1}).$$

We have proved the claim when $N = 1$. If $N \geq 1$ we compute

$$\begin{aligned} p_N &= (-1)^N\Re((\eta_+ + a_1)v_{-N-2} + a_{-1}v_{-N} + \Phi v_{-N-1}, \Phi v_{-N-1}) \\ &= (-1)^N\Re\big[(\Phi v_{-N-2}, \eta_- v_{-N-1}) - (v_{-N-2}, (\eta_-\Phi)v_{-N-1}) \\ &\qquad\qquad + (a_1 v_{-N-2} + a_{-1}v_{-N} + \Phi v_{-N-1}, \Phi v_{-N-1})\big] \\ &= p_{N+1} + q_{N+1} - q_N. \end{aligned}$$

This proves the claim for any N.

Note that since $\|\eta_- v\|^2 = \sum\|\eta_- v_k\|^2$, we have $\eta_- v_k \to 0$ and similarly $v_k \to 0$ in $L^2(SM)$ as $k \to -\infty$. Therefore $p_N \to 0$ as $N \to \infty$. We also have

$$\|q_N\| \leq C_\Phi \sum\|v_k\|^2 + \left|\sum_{j=1}^{N}(-1)^j([a_{-1},\Phi]v_{-j}, v_{-j-1})\right| \leq C_{A,\Phi}\sum\|v_k\|^2.$$

Here it was important that the term in a_{-1} involving s is a scalar, so it goes away when taking the commutator $[a_{-1}, \Phi]$. After taking a subsequence, (q_N) converges to some q having a similar bound. We finally obtain

stov_higgs_intermediate5

$$\Re(\Phi v, (X + A_s + \Phi)v) = \lim_{N\to\infty}(p_N + q_N) \leq C_{A,\Phi}\sum\|v_k\|^2. \tag{13.24}$$

Collecting the estimates (13.20)–(13.24) and using them in (13.19) shows that

$$0 \geq \|h\|^2 + (s - C_{A,\Phi})\sum_{k=-\infty}^{-1}|k|\|v_k\|^2.$$

Choosing s large enough implies $v_k = 0$ for all k. This proves that u_s is holomorphic, and therefore $u = e^{-sw}u_s$ is holomorphic as required. □

G: I still find this proof a bit baffling; but I am unable to write anything better.

We now rephrase Theorem 13.24 as an injectivity result for a matrix attenuated X-ray transform. We let $\mathcal{A}(x, v) := A_x(v) + \Phi(x)$ and we let $I_{A,\Phi} = I_{\mathcal{A}}$ be the associated attenuated X-ray transform.

thm:injective_higgs

THEOREM 13.25. *Let M be a compact simple surface. Assume that $f : SM \to \mathbb{C}^n$ is a smooth function of the form $F(x) + \alpha_x(v)$, where $F : M \to \mathbb{C}^n$ is a smooth function and α is a $\mathbb{C}^n$-valued 1-form. Let also A be a unitary connection and Φ a skew-Hermitian matrix function. If $I_{A,\Phi}(f) = 0$, then $F = \Phi p$ and $\alpha = d_A p$, where $p : M \to \mathbb{C}^n$ is a smooth function with $p|_{\partial M} = 0$.*

PROOF. If $I_{A,\Phi}(f) = 0$, we know by Theorem 5.10 that u^f is C^∞ and satisfies

$$(X + A + \Phi)u^f = -f \in \Omega_{-1} \oplus \Omega_0 \oplus \Omega_1$$

with $u^f|_{\partial SM} = 0$. Thus by Theorem 13.24, u^f only depends on x and upon setting $p = -u_0$, the result follows.

□

13.9. Scattering rigidity for connections and Higgs fields

In this section we extend the scattering rigidity result for unitary connections in Theorem 13.11 to pairs (A, Φ), where A is a unitary connection and Φ is a skew-Hermitian matrix valued function. We let $C_{A,\Phi} = C_{\mathcal{A}}$ be the scattering data associated with the attenuation $\mathcal{A}(x, v) = A_x(v) + \Phi(x)$.

THEOREM 13.26. *Assume M is a compact simple surface, let A and B be two unitary connections, and let Φ and Ψ be two skew-Hermitian Higgs fields. Then $C_{A,\Phi} = C_{B,\Psi}$ implies that there exists a smooth $u : M \to U(n)$ such that $u|_{\partial M} = \mathrm{Id}$*
thm:inverse *and $B = u^{-1}du + u^{-1}Au$, $\Psi = u^{-1}\Phi u$.*

PROOF. From Proposition 13.5 we know that $C_{A,\Phi} = C_{B,\Psi}$ means that there exists a smooth $U : SM \to U(n)$ such that $U|_{\partial SM} = \mathrm{Id}$ and

eq:geq
$$\mathcal{B} = U^{-1}XU + U^{-1}\mathcal{A}U, \tag{13.25}$$

where $\mathcal{B}(x, v) = B_x(v) + \Psi(x)$. We rephrase this information in terms of an attenuated ray transform. If we let $W = U - \mathrm{Id}$, then $W|_{\partial SM} = 0$ and

$$XW + \mathcal{A}W - W\mathcal{B} = -(\mathcal{A} - \mathcal{B}).$$

Hence W is associated with the attenuated X-ray transform $I_{E(\mathcal{A},\mathcal{B})}(\mathcal{A} - \mathcal{B})$ and if $C_{A,\Phi} = C_{B,\Psi}$, then this transform vanishes. Note that $\mathcal{A} - \mathcal{B} \in \Omega_{-1} \oplus \Omega_0 \oplus \Omega_1$.

Hence, making the choice to ignore the specific form $E(\mathcal{A}, \mathcal{B})$ but noting that it is unitary by Exercise 13.6, we can apply Theorem 13.24 to deduce that W only depends on x. Hence U only depends on x and if we set $u(x) = U_0$, then (13.25) easily translates into $B = u^{-1}du + u^{-1}Au$ and $\Psi = u^{-1}\Phi u$ just by looking at the components of degree 0 and ± 1.

□

REMARK 13.27. Note that the theorem implies in particular that scattering ridigity just for Higgs fields does not have a gauge. Indeed, if $C_\Phi = C_\Psi$, where Φ and Ψ are two skew-Hermitian matrix fields, Theorem 13.26 applied with $A = B = 0$ implies that $u = \mathrm{Id}$ and thus $\Phi = \Psi$.

13.10. Matrix holomorphic integrating factors

Unfortunately, it is not possible to extend the proof of Theorem 13.24 to the case of non skew-Hermitian attenuations. The main issue is that the Pestov identity given by Lemma 13.20 has a particularly nice form when $\mathcal{A}$ is skew-Hermitian. While it is possible to derive a general Pestov identity, new terms appear and there is a priori no clear way as to how to control them.

An alternative approach would be to try to prove the existence of certain *matrix holomorphic integrating factors*. Note that the proof of Theorem 13.24 uses the existence of *scalar* holomorphic integrating factors. In this section we try to explain the main difficulties with this approach and state some open problems.

We start with a general definition.

DEFINITION 13.28. Let (M,g) be a compact oriented Riemannian surface and let $\mathcal{A} \in C^\infty(SM, \mathbb{C}^{n\times n})$ be given. We say that $R : SM \to GL(n,\mathbb{C})$ is a *matrix holomorphic integrating factor for* $\mathcal{A}$ if

(1) R solves $(X + \mathcal{A})R = 0$;
(2) Both R and R^{-1} are fibre-wise holomorphic.

There is an analogous definition for *anti-holomorphic* integrating factors. The existence of these integrating factors imposes conditions on $\mathcal{A}$:

LEMMA 13.29. *If $\mathcal{A}$ admits a holomorphic integrating factor then $\mathcal{A} \in \oplus_{k\geq -1}\Omega_k$. If $\mathcal{A}$ admits both, holomorphic and anti-holomorphic integrating factors, then $\mathcal{A} \in \Omega_{-1} \oplus \Omega_0 \oplus \Omega_1$.*

PROOF. This follows right away from writing $\mathcal{A} = -X(R)R^{-1}$, since R^{-1} is holomorphic and $X(R) \in \oplus_{k\geq -1}\Omega_k$. The second statement in the lemma follows immediately.

□

Thus if we wish to use holomorphic and anti-holomorphic integrating factors the attenutation $\mathcal{A}$ must be of the form $\mathcal{A}(x,v) = A_x(v) + \Phi(x)$ where A is a connection and Φ a matrix-valued field. The relevance of these type of integrating factors can be seen in the following proposition.

PROPOSITION 13.30. Let (M,g) be a non-trapping surface with strictly convex boundary such that I_0 is injective and I_1 is solenoidal injective. Let (A,Φ) be a pair given by a connection A and a matrix valued field Φ. If (A,Φ) admits holomorphic and anti-holomorphic integrating factors, then $I_{A,\Phi}$ has the same kernel as in Theorem 13.25.

PROOF. Assume there is a smooth $u \in C^\infty(SM,\mathbb{C}^n)$ such that $u|_{\partial SM} = 0$ and $(X + A + \Phi)u = -f \in \Omega_{-1}\oplus\Omega_0\oplus\Omega_1$. We wish to show that $u = u_0$. For this it is enough to show that u is both holomorphic and anti-holomorphic.

Let R be a matrix holomorphic integrating factor for $A + \Phi$. Since R^{-1} solves $XR^{-1} - R^{-1}(A+\Phi) = 0$ a computation shows that

$$X(R^{-1}u) = -R^{-1}f.$$

Since R^{-1} is holomorphic, $(R^{-1}f)_k = 0$ for $k \leq -2$. Thus if we set $v = \sum_{-\infty}^{-1}(R^{-1}u)_k$, then $v|_{\partial SM} = 0$ and

$$Xv \in \Omega_{-1}\oplus\Omega_0.$$

Using the hypotheses on I_0 and I_1, we deduce that $v = 0$ and thus $R^{-1}u$ is holomorphic. It follows that $u = RR^{-1}u$ is also holomorphic since R is holomorphic.

An analogous argument using anti-holomorphic integrating factors shows that u is anti-holomorphic and hence $u = u_0$ as desired.

□

We can now state the following open problem.

Open Problem. Let (M,g) be a simple surface and let (A,Φ) be a pair, where A is a connection and Φ is a matrix field. Do holomorphic (anti-holomorphic) integrating factors exist for (A,Φ)?

Note that Proposition 12.5 gives a positive answer to this question when $n = 1$. It suffices to take $R := e^{-w}$ where w is given by the proposition. In the non-abelian case $n \geq 2$ we can no longer argue using an exponential. While we can certainly find a holomorphic matrix W such that $XW = A + \Phi$, the exponential of W might not solve the relevant transport problem since XW and W do not necessarily commute.

EXERCISE 13.31. Show that for any $W \in C^\infty(SM, \mathbb{C}^{n\times n})$ we have

$$e^W X(e^{-W}) = \int_0^1 e^{-sW}(XW)e^{sW}\, ds.$$

13.10.1. On the group of invertible first integrals. In this subsection we study the group of all smooth $R : SM \to GL(n, \mathbb{C})$ such that $XR = 0$ for (M, g) a simple surface.

We start with an auxiliary lemma.

LEMMA 13.32. *Let $F : M \to GL(n, \mathbb{C})$ be such that $\eta_- F = 0$. Then we can write F as*

$$F = F_1 \cdots F_r$$

where each $F_j : M \to GL(n, \mathbb{C})$ has the property that $\eta_- F_j = 0$ and $|\mathrm{Id} - F_j(x)| < 1$ for all $x \in M$ and $1 \leq j \leq r$.

lemma:GR

PROOF. (Sketch) The set G of all $F : M \to GL(n, \mathbb{C})$ with $\eta_- F = 0$ clearly forms a group. In fact it is a connected topological group with the supremum norm. Such groups are generated by any open neighbourhood of the identity. Considering a neighbourhood of the form

$$U = \{F \in G : \ \|F - \mathrm{Id}\|_{L^\infty} < 1\}$$

the result follows. □

We now prove:

THEOREM 13.33. *Let (M, g) be a simple surface and let $F : M \to GL(n, \mathbb{C})$ with $\eta_- F = 0$ be given. Then there exists a smooth $R : SM \to GL(n, \mathbb{C})$ such that*

(1) *$XR = 0$ and $R_0 = F$.*
(2) *Both R and R^{-1} are fibre-wise holomorphic.*

PROOF. By Lemma 13.32 we may write $F = F_1 \cdots F_r$ where each $F_j : M \to GL(n, \mathbb{C})$ is such that $\eta_- F_j = 0$ and $|\mathrm{Id} - F_j(x)| < 1$ for all x. Hence we can write $F_j = e^{P_j}$, where $P_j : M \to \mathbb{C}^{n\times n}$ is such that $\eta_- P_j = 0$. By the surjectivity of I_0^*, there is a smooth W_j such that $XW_j = 0$, W_j is fibre-wise holomorphic and $(W_j)_0 = P_j$. Now set

$$R := e^{W_1} \cdots e^{W_r}.$$

We claim that R has all the desired properties. Since each e^{W_j} is a first integral, so is R. By construction, each e^{W_j} is holomorphic, hence so is their product. Since

$$R^{-1} = e^{-W_r} \cdots e^{-W_1},$$

it follows that R^{-1} is also fibre-wise holomorphic. It remains to prove that $R_0 = F$. But since R is holomorphic we must have

$$R_0 = (e^{W_1})_0 \cdots (e^{W_r})_0.$$

But for each j, $(e^{W_j})_0 = e^{(W_j)_0} = e^{P_j} = F_j$ and the theorem is proved. □

13.11. Carleman estimates in 2D? Stability estimate?

CHAPTER 14

Miscellaneous Topics

14.1. Other curves rather than geodesics

14.2. Obstacles

14.3. Non-convex boundary, hyperbolic trapping, Croke's results on cylinders

14.4. Open Problems

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