Geometric inverse problems (in 2D)

Lectures: Thu and Fri at 12.15–14.00
(starting 19 March, last lecture 8 May (?))

Lectures given using Zoom (interactive, questions welcome)

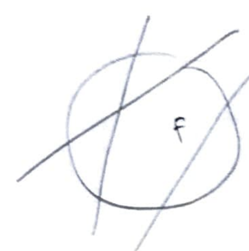
5 credit points, returned exercises (50 % of exercise points)

Prerequisites: differential geometry, functional analysis
(geometry of geodesics recommended)

Material: book draft by Paternain-Salo-Uhlmann

Preface

Inverse problems often arise in imaging methods, when determining interior properties of a medium from external measurements.



X-ray computed tomography (CT): determine an attenuation function f in $\mathbb{R}^2$ from its Radon transform Rf, which encodes the integrals of f over straight lines (X-ray measurements).

- Direct problem: compute Rf if f is known
- Inverse problem: determine f from Rf (i.e. invert the Radon transform)

Different aspects:

1) (Uniqueness) If $Rf_1 = Rf_2$, show that $f_1 = f_2$.
2) (Stability) If $Rf_1 \approx Rf_2$, show that $f_1 \approx f_2$.
3) (Reconstruction) Algorithm for computing f from Rf.
4) (Range characterization) Which functions are of the form Rf?
5) (Partial data) Determine f from partial information on Rf.

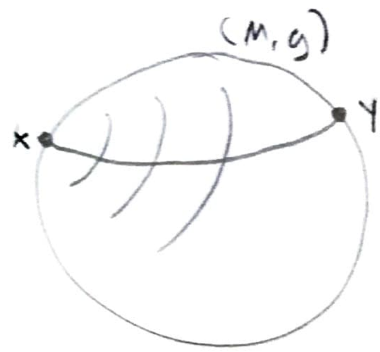
We will study inverse problems in <u>geometric</u> settings. For X-ray type problems, this means that straight lines are replaced by more general curves. We will work on <u>compact Riemannian manifolds</u> (M,g) with smooth boundary, and with corresponding geodesics.

Let us now introduce the main questions in this course. The first one is a geometric version of X-ray problem:

1. Geodesic X-ray transform

Can one determine a function f in (M,g) from its integrals over maximal geodesics?

The second question arises in seismic imaging. Let $M \subset \mathbb{R}^3$ be a ball (Earth), and let g be a Riemannian metric (sound speed). If an earthquake happens at $x \in \partial M$, its first arrival time to a seismometer at $y \in \partial M$ is the geodesic distance $d_g(x,y)$.



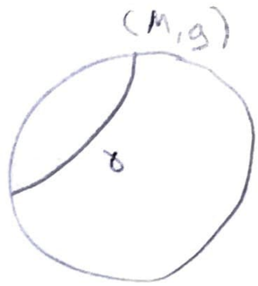
<u>Travel time tomography</u>, where one tries to determine the sound speed in Earth from travel times of earthquakes, reduces to the following question.

2. Boundary rigidity problem

Can one determine the metric in (M,g), up to gauge, from the knowledge of the boundary distance function $d_g|_{\partial M \times \partial M}$?

In fact Question 1 is the <u>linearization</u> of Question 2 for metrics in a fixed conformal class (i.e. $g = c g_0$ for a fixed g_0).

The linearization of Question 2 is a tensor tomography problem. Let (M, g) be a compact Riem. n-manifold with boundary, and let $m \geq 0$. The geodesic X-ray transform is the operator I_m defined by



$$I_m f(\gamma) = \int_\gamma f_{j_1 \cdots j_m}(\gamma(t))\, \dot\gamma^{j_1}(t) \cdots \dot\gamma^{j_m}(t)\, dt$$

where γ is a maximal geodesic and $f = f_{j_1 \cdots j_m}(x)\, dx^{j_1} \otimes \cdots \otimes dx^{j_m}$ is a symmetric m-tensor field. Throughout this course we use the Einstein summation convention (repeated upper and lower indices are summed). If $m = 1$ then $f = f_j(x)\, dx^j$ is a 1-form (i.e. a vector field), and one has the solenoidal, or Helmholtz, decomposition

$$f = v + dh, \qquad \operatorname{div}(v) = 0, \qquad h \in C^\infty(M) \text{ with } h|_{\partial M} = 0.$$

Here v is the solenoidal part and dh is the potential part of f. One always has

$$I_1(dh) = \int_\gamma \partial_{x_j} h(\gamma(t))\, \dot\gamma^j(t)\, dt = \int_\gamma \frac{d}{dt} h(\gamma(t))\, dt = 0$$

since h vanishes at the end points of γ. Similarly, for $m \geq 1$ one can only hope to determine the solenoidal part of f from $I_m f$.

3. Tensor tomography Can one determine the solenoidal part of a symmetric m-tensor field f in (M, g) from the geodesic X-ray transform $I_m f$?

Another variant of Question 1 includes an attenuation factor (this comes up in the medical imaging method SPECT). Here $f \in C^\infty(M)$ is a source function and $a \in C^\infty(M)$ is an attenuation coefficient, and one can measure integrals like

$$I^a f(\gamma) = \int_\gamma e^{\int_0^t a(\gamma(s))\,ds} f(\gamma(t))\,dt, \qquad \gamma \text{ maximal geodesic.}$$

This is the attenuated geodesic X-ray transform of f, and it reduces to standard geodesic X-ray transform when $a=0$. In mathematical physics, one often replaces $a(x)$ by a connection or Higgs field (corresponding to a matrix valued function $a(x)$, or 1-form).

4. Attenuated geodesic X-ray transform

Can one determine a function f in (M,g) from $I^a f$, when a is known?

Finally, we consider the Dirichlet problem for the Laplace equation

$$\begin{cases} \Delta_g u = 0 & \text{in } M \\ u = f & \text{on } \partial M \end{cases}$$

where Δ_g is the Laplace-Beltrami operator given in local coordinates by

$$\Delta_g u = |g|^{-1/2} \partial_{x_j}(|g|^{1/2} g^{jk} \partial_{x_k} u).$$

Here $g = (g_{jk})$, $(g^{jk}) = (g_{jk})^{-1}$, and $|g| = \det(g_{jk})$. There is a unique solution $u \in C^\infty(M)$ for any $f \in C^\infty(\partial M)$. The Dirichlet-to-Neumann map is defined by

$$\Lambda_g : C^\infty(\partial M) \to C^\infty(\partial M), \quad f \mapsto \partial_\nu u|_{\partial M}.$$

This is related to Electrical Impedance Tomography (g is the electrical resistivity of M, and Λ_g encodes voltage and current measurements on ∂M). This leads to:

5. Calderón problem

Can one determine g, up to gauge, from the knowledge of Λ_g?

We will discuss results for Questions 1–5 when (M,g) is <u>two-dimensional</u>, mostly in the context of <u>simple manifolds</u>. (M,g) is simple if

- M is diffeomorphic to a ball,
- ∂M is strictly convex, and
- M has no conjugate points.

Examples: strictly convex smooth domains in Euclidean space, in nonpositively curved spaces (if they are also simply connected), and in the hemisphere.

Questions 1–4 have a positive answer on 2D simple manifolds, and Question 5 has a positive answer on any 2D manifold. We will also give a new proof of the result of Pestov–Uhlmann (2005) solving Question 2 on 2D simple mflds, based on solutions of Question 1 and Question 5.