2 Discrete-Time Signals and Systems

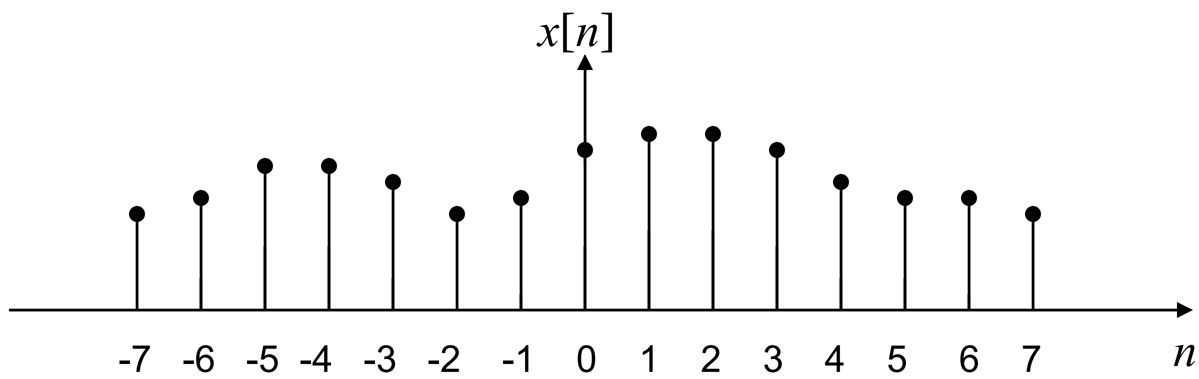
Time-Domain Representations of Discrete-Time Signals and Systems

- Time-domain representation of a discrete-time signal as a *sequence of numbers*
- Basic sequences and operations on sequences
- Discrete-time systems in processing of discrete-time signals

Linear and time-invariant systems

Discrete-Time Signals

- Sequence $\{x[n]\}$ can be considered as a periodically sampled continuous-time signal $x_a(t)$



$$x[n] = x_a(t)\big|_{t=nT} = x_a(nT) \qquad \mathrm{n} = \ldots, -2, -1, 0, 1, 2, \ldots$$

- Sampling interval: T
- Sampling frequency: $F_T = 1/T$

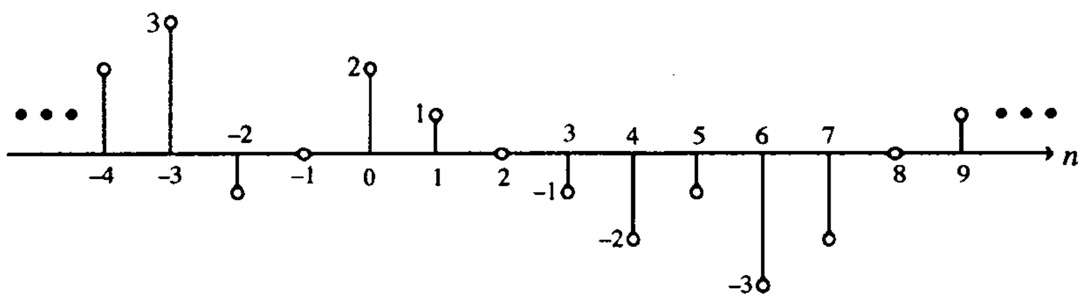
Digital Signals

- ***Digital signal***

  Discrete-time and discrete-valued sequence of numbers

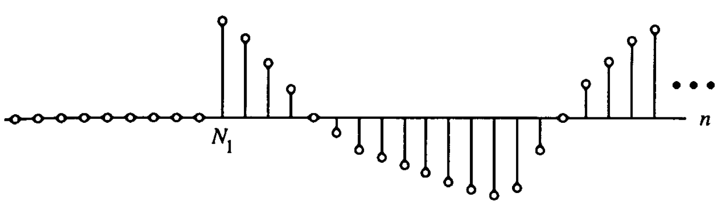
- ***Digital signal processing***

  The sequence is transformed to another sequence by means of arithmetic operations

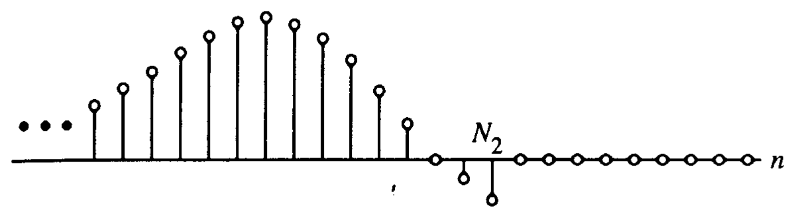


Types of Sequence

- Finite-duration or finite-length sequence:
  Defined in the interval $N_1 \le n \le N_2$, where N_1 and N_2 are finite and $N_2 \ge N_1$
  Length (duration): $N = N_2 - N_1 + 1$
- Infinite-duration or infinite-length sequence:
  a) Right-sided sequence: $x[n]=0,\ n \le N_1$
  b) Left-sided sequence: $x[n]=0,\ n \ge N_2$



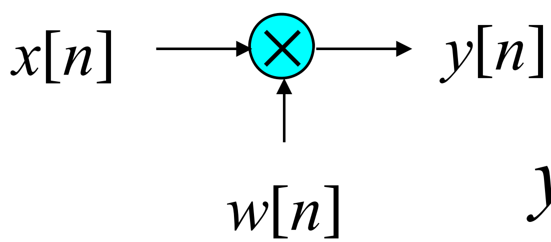
(a)



(b)

Operations on Sequences: Basic Operations

- **Product** (**modulation**) operation:
  - Modulator

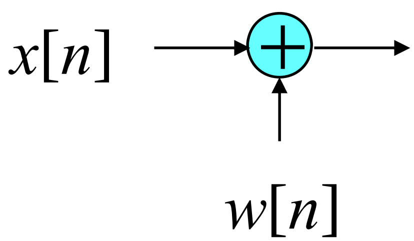


$$y[n] = x[n] \cdot w[n]$$

- An application is in forming a finite-length sequence from an infinite-length sequence by multiplying the latter with a finite-length sequence called an **window sequence**
- Process called **windowing**

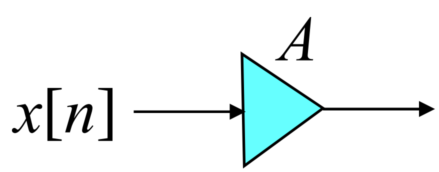
Operations on Sequences: Basic Operations

- **Addition** operation:
  - Adder



$$y[n] = x[n] + w[n]$$

- **Multiplication** operation:
  - Multiplier



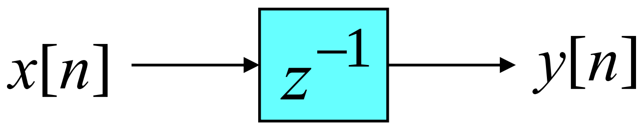
$$y[n] = A \cdot x[n]$$

Operations on Sequences: Basic Operations

- **Time-shifting** operation: $y[n] = x[n-N]$

  where *N* is an integer
- If *N* > 0, it is **delaying** operation

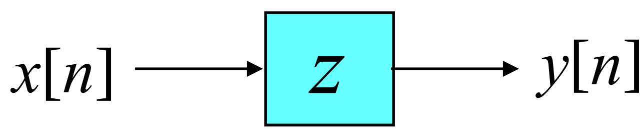
  – Unit delay



$$y[n] = x[n-1]$$

- If *N* < 0, it is an **advance** operation

  – Unit advance



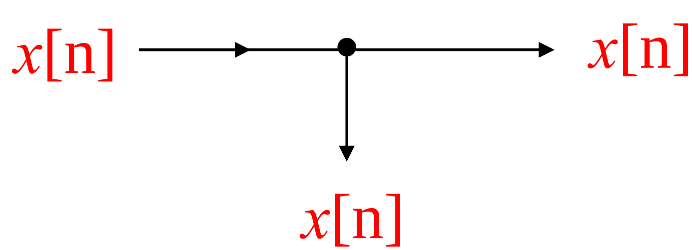
$$y[n] = x[n+1]$$

Operations on Sequences: Basic Operations

- **Time-reversal** (**folding**) operation:

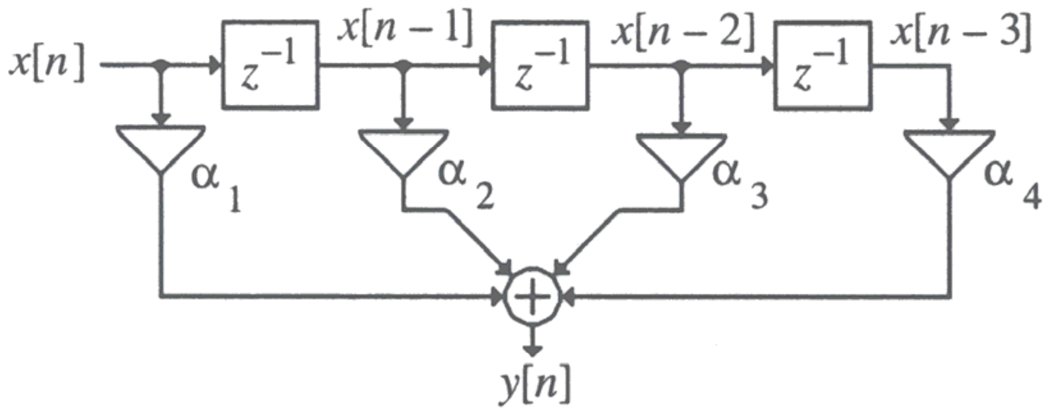
$$y[n] = x[-n]$$

- **Branching** operation: Used to provide multiple copies of a sequence



Combinations of Basic Operations

- Example: Averaging filter



$$y[n] = \alpha_1 x[n] + \alpha_2 x[n-1] + \alpha_3 x[n-2] + \alpha_4 x[n-3]$$

Sampling Rate Alteration: Basic Operations

- Employed to generate a new sequence $y[n]$ with a sampling rate F'_T higher or lower than that of the sampling rate F_T of a given sequence $x[n]$
- **Sampling rate alteration ratio** is: $R = \dfrac{F_T'}{F_T}$
- If *R* > 1, the process called **interpolation**
- If *R* < 1, the process called **decimation**

Sampling Rate Alteration: Basic Operations

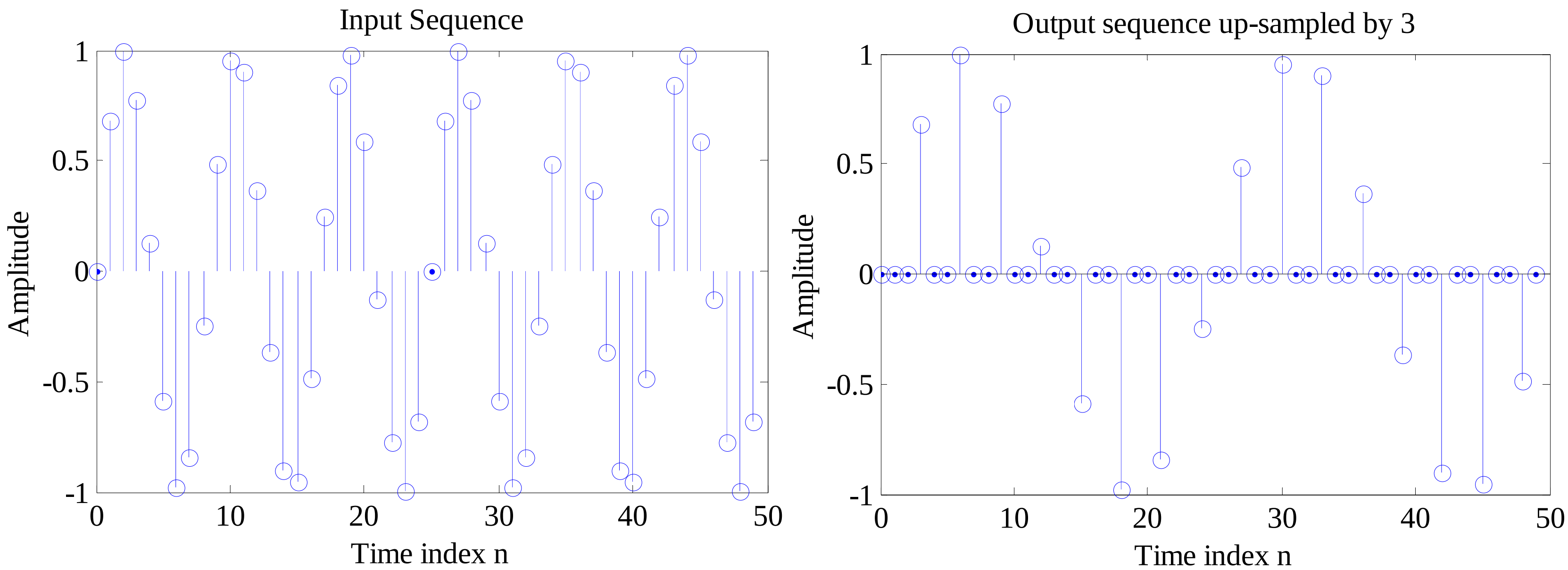
- In **up-sampling** by an integer factor $L > 1$, $L - 1$ equidistant zero-valued samples are inserted by the **up-sampler** between each two consecutive samples of the input sequence $x[n]$:

$$x_u[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \cdots \\ 0, & \text{otherwise} \end{cases}$$

$$x[n] \longrightarrow \boxed{\uparrow L} \longrightarrow x_u[n]$$

Sampling Rate Alteration: Basic Operations

- An example of the up-sampling operation



Sampling Rate Alteration: Basic Operations

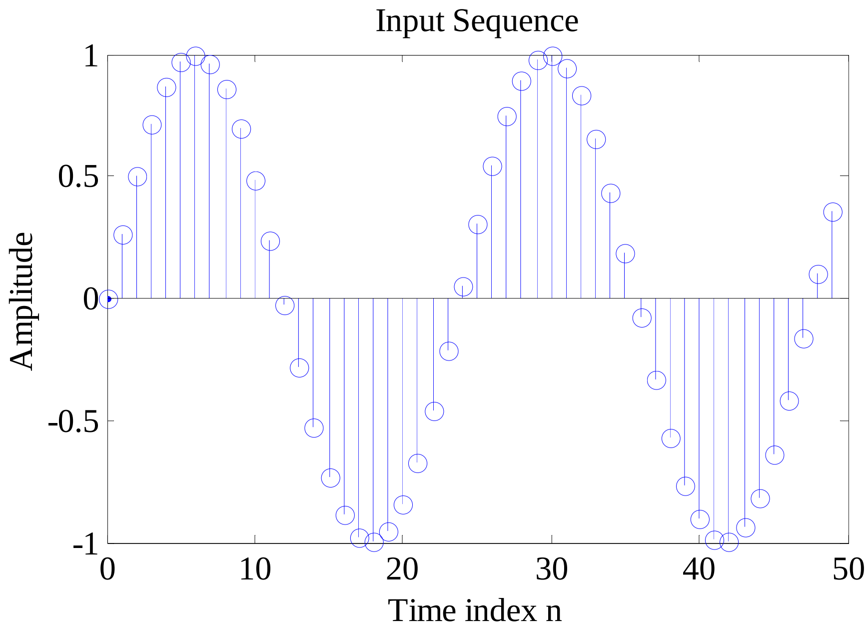
- In **down-sampling** by an integer factor $M > 1$, every M-th samples of the input sequence are kept and M -1 in-between samples are removed:

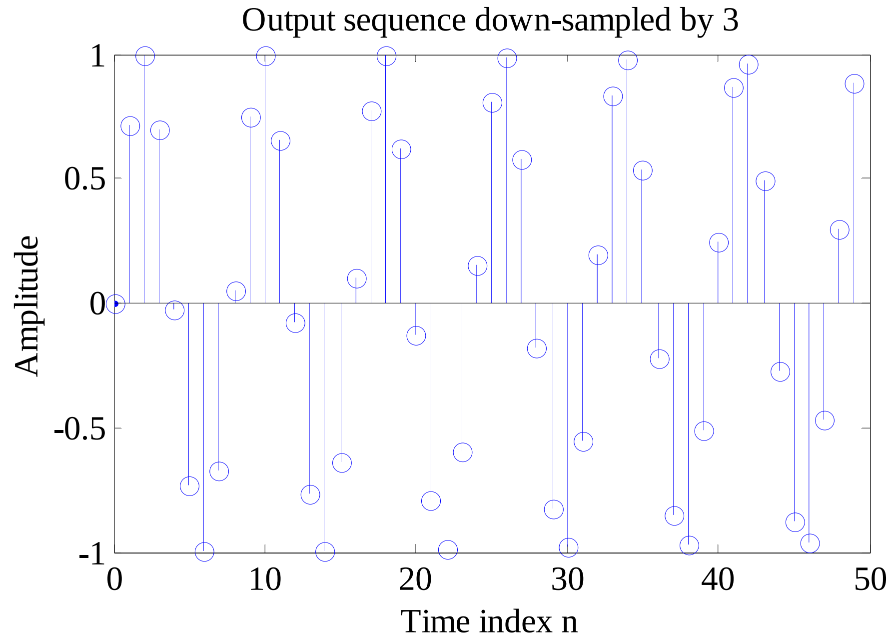
$$y[n] = x[nM]$$

$$x[n] \longrightarrow \boxed{\downarrow M} \longrightarrow y[n]$$

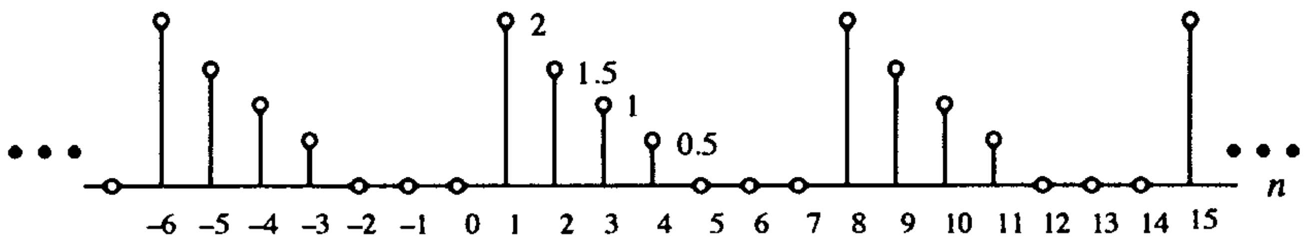
Sampling Rate Alteration: Basic Operations

- An example of the down-sampling operation





Periodic Sequences



- **Periodicity:** $x_p[n]=x_p[n+kN]$, for all n
- The sequence $x_p[n]$ is ***periodic*** with period N where N is a positive integer and k is any integer
- The fundamental period N_f is the smallest N for which the above equation holds
- **Notice!** Sampling of a periodic continuous-time signal does not guarantee the periodicity of the sampled sequence

Example: Sinusoidal Sequences

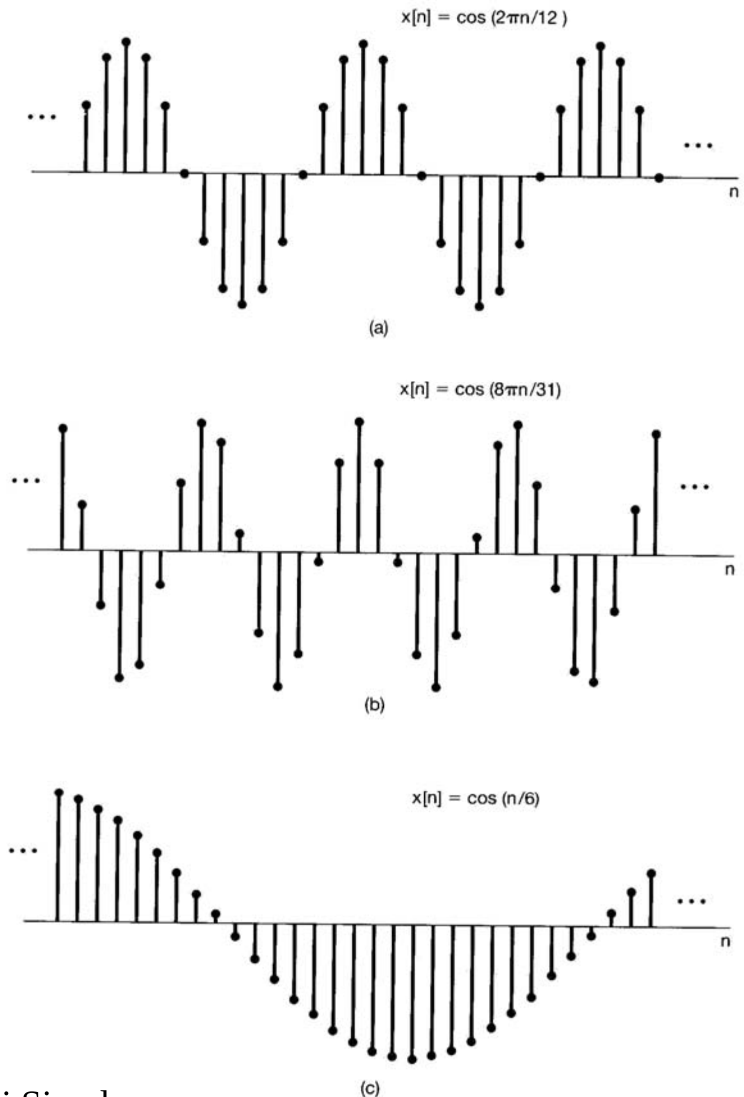


Figure 1.25 Discrete-time sinusoidal signals.

$$x[n] = \cos(2\pi n/12)$$

- Periodic, N=12

$$x[n] = \cos(8\pi n/31)$$

- Periodic, N=31

$$x[n] = \cos(n/6)$$

- Not periodic

Classification of Sequences

- A sequence is **bounded** if $|x[n]| \le B_x < \infty$
- A sequence is **absolutely summable** if

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

- A sequence is **square- summable** if

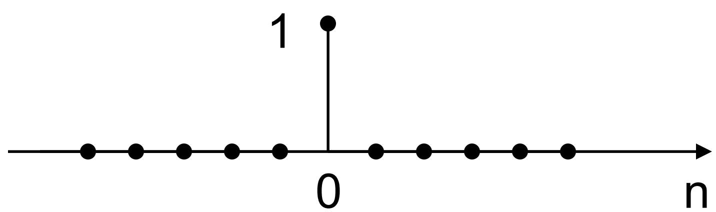
$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

- The **energy** of a sequence is $E = \sum_{n=-\infty}^{\infty} |x[n]|^2$

Some Basic Sequences

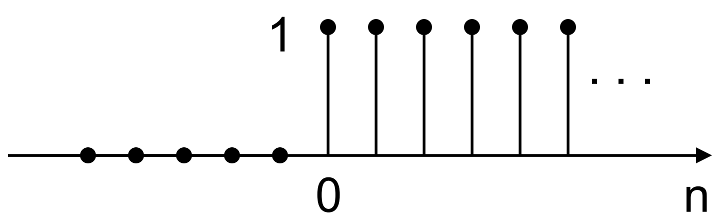
- ***Unit sample sequence***

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$



- ***Unit step sequence***

$$\mu[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



Relations between Basic Sequences

- Unit sample and unit step sequences are related as follows:

$$\mu[n] = \sum_{k=-\infty}^{n} \delta[k]$$

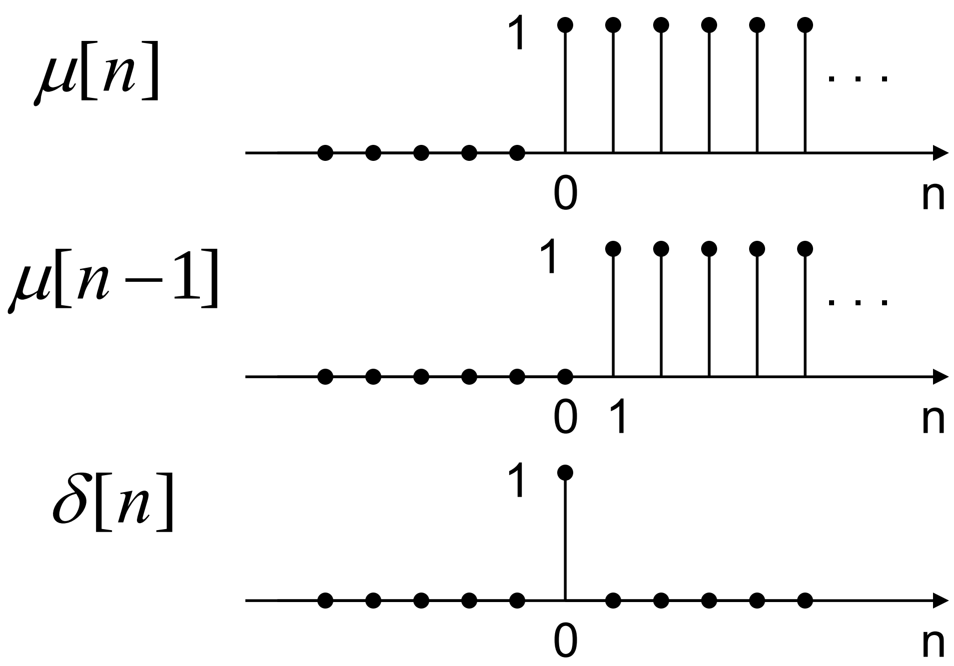
$$\delta[n] = \mu[n] - \mu[n-1]$$

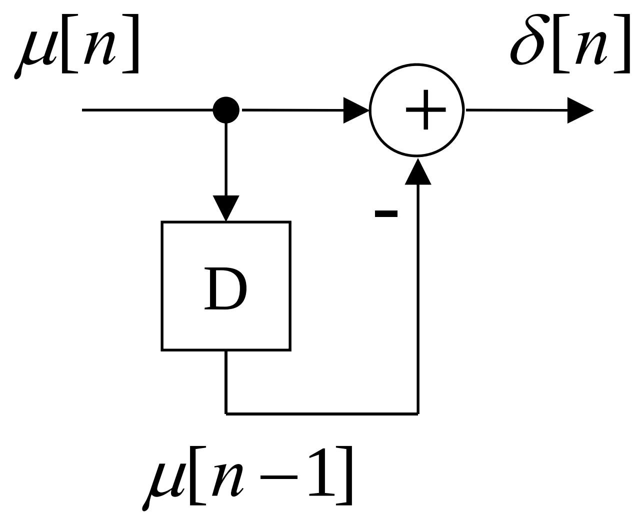
- The above relations can be implemented with simple computational structures consisting of basic arithmetic operations

Relations between Basic Sequences

- The unit sample is the first difference of the unit step:

$$\delta[n] = \mu[n] - \mu[n-1]$$



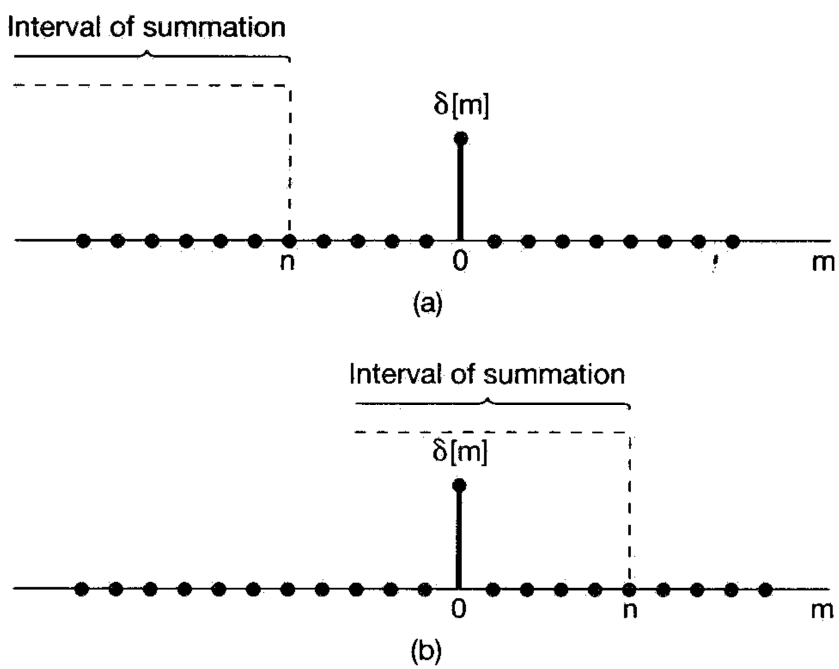


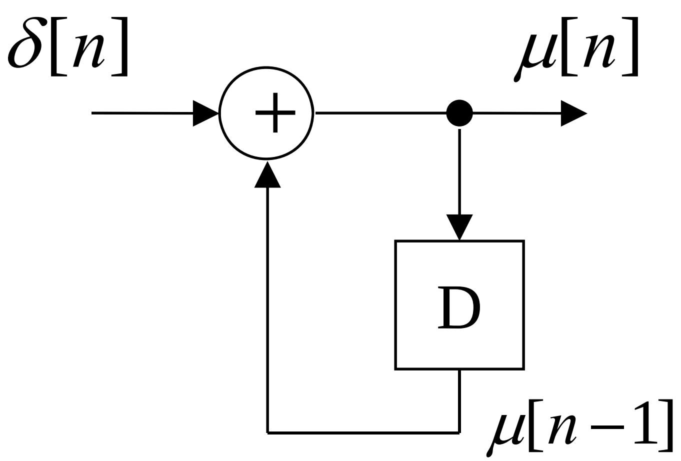
Realization

Relations between Basic Sequences

- Unit step is the running sum of the unit sample:

$$\mu[n] = \sum_{m=-\infty}^{n} \delta[m] = \sum_{m=-\infty}^{n-1} \delta[m] + \delta[n] = \mu[n-1] + \delta[n]$$

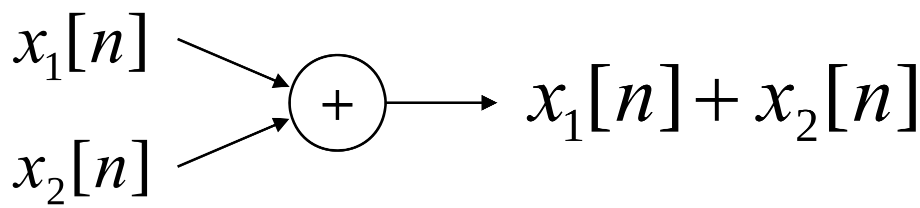




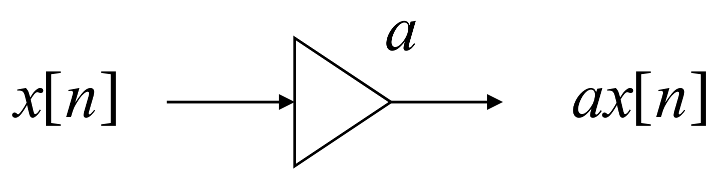
Realization

Basic Operations on Sequences

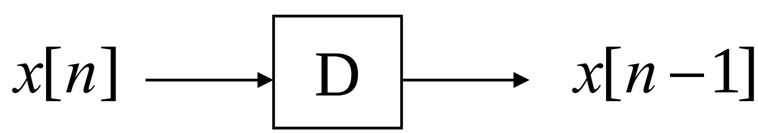
- Addition:



- Multiplication:



- Unit delay:



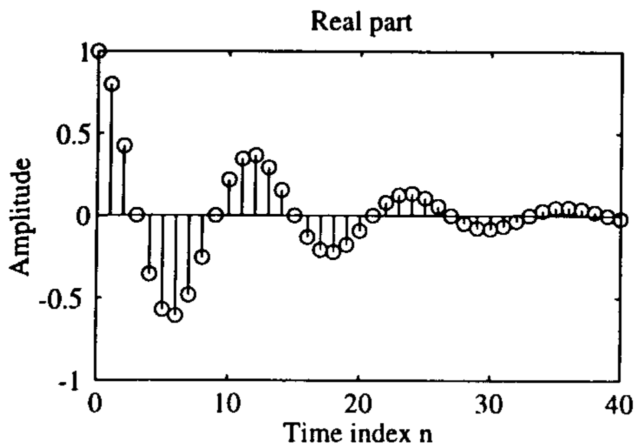
Exponential and Sinusoidal Sequences

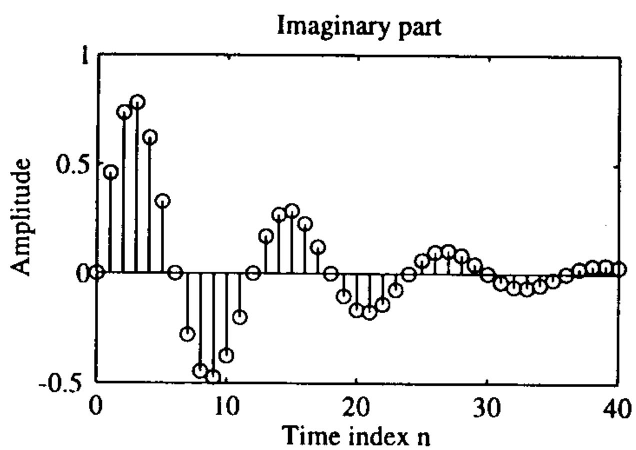
- ***Complex exponential sequence*** $x[n] = A\alpha^n$

  where A and α are complex

$$x[n] = |A|e^{j\phi}e^{(\sigma_0 + j\omega_0)n} = |A|e^{\sigma_0 n}e^{j(\omega_0 n+\phi)}$$

$$= |A|e^{\sigma_0 n}\left[\cos(\omega_0 n + \phi) + j\sin(\omega_0 n + \phi)\right]$$





Real Exponential Sequences

- With both A and α real, the sequence reduces to a ***real exponential sequence***

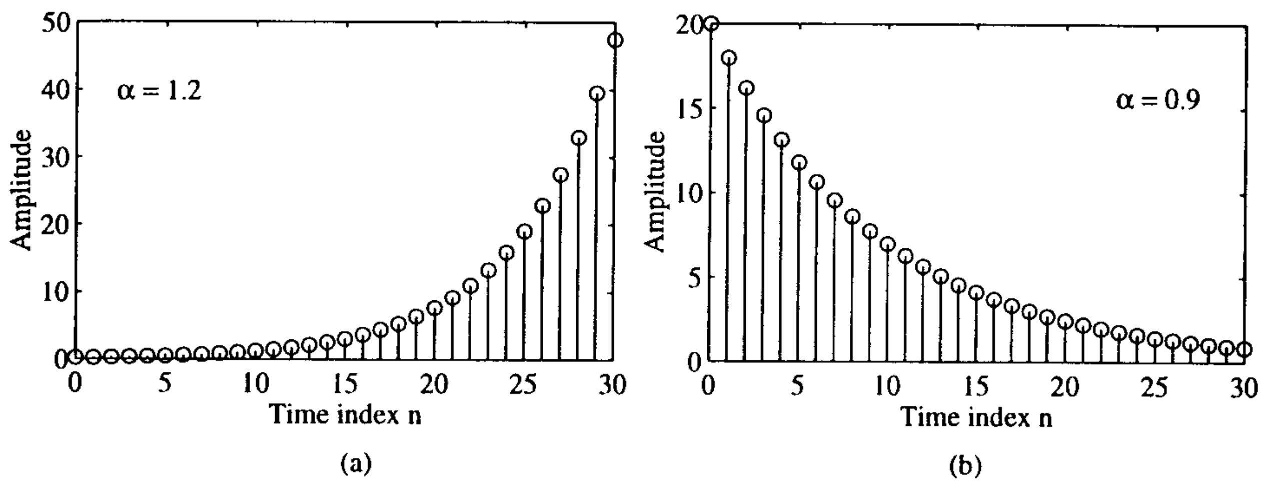
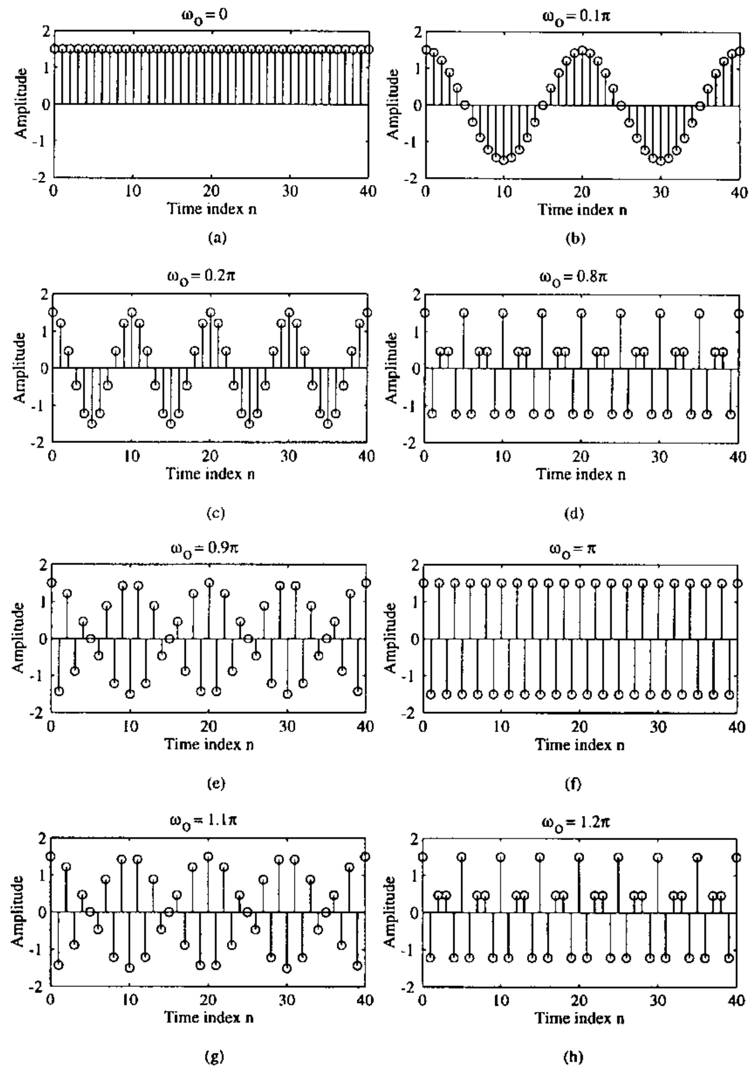


Figure 2.10 Examples of real exponential sequences: (a) $x[n] = 0.2(1.2)^n$, (b) $x[n] = 20(0.9)^n$.

- A ***real sinusoidal sequence***: $x[n] = A\cos(\omega_0 n + \phi)$

A Family of Sinusoidal Sequences



The Sampling Process

- A ***discrete-time sequence*** is developed by uniformly sampling the continuous-time signal $x_a(t)$

$$x[n] = x_a(t)\big|_{t=nT} = x_a(nT)$$

- The time variable -time t is related to the discrete time variable n only at discrete-time instants t_n

$$t_n = nT = \frac{n}{F_T} = \frac{2\pi n}{\Omega_T}$$

with $\quad F_T = 1/T \quad$ (sampling frequency)

and $\quad \Omega_T = 2\pi F_T \quad$ (sampling angular frequency)

The Sampling Process

- Consider $x_a(t) = A\cos(\Omega_0 t + \phi)$
- Now $x[n] = A\cos(\Omega_0 nT + \phi)$

$$= A\cos\left(\frac{2\pi\Omega_0}{\Omega_T} n + \phi\right) = A\cos(\omega_0 n + \phi)$$

where $$\omega_0 = \frac{2\pi\Omega_0}{\Omega_T} = \Omega_0 T$$

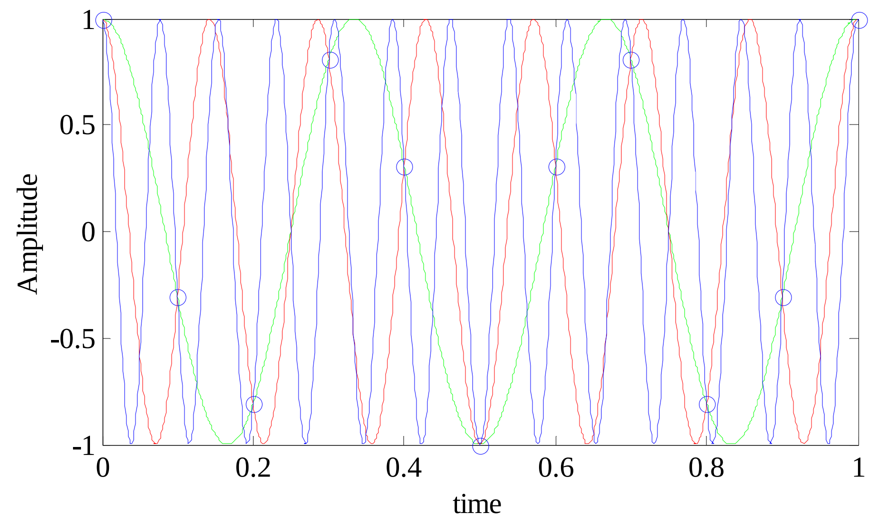
- ω_0 is the normalized angular frequency

Example: Three Sinusoidal Sequences

$$g_1[n] = \cos(0.6\pi n)$$

$$g_2[n] = \cos(1.4\pi n)$$

$$g_3[n] = \cos(2.6\pi n)$$



$$g_2[n] = \cos((2\pi - 0.6\pi)n) = \cos(0.6\pi n) = g_1[n]$$

$$g_3[n] = \cos((2\pi + 0.6\pi)n) = \cos(0.6\pi n) = g_1[n]$$

The Aliasing Phenomenon

- In general, the family of continuous-time sinusoids

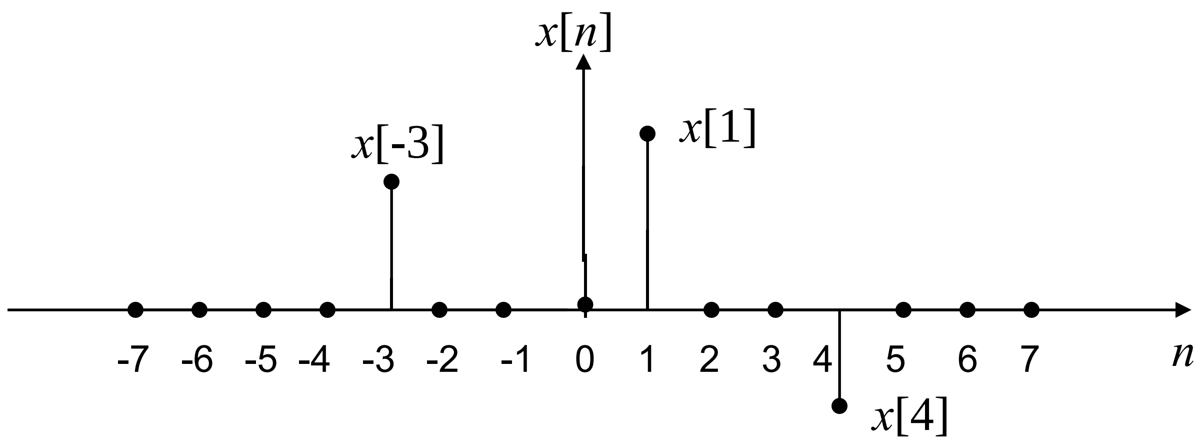
$$x_{a,k}(t) = A\cos((\Omega_0 + k\Omega_T)t + \phi) \quad k = 0, \pm 1, \pm 2, \ldots$$

lead to identical sampled signals

$$\begin{aligned} x_{a,k}(nT) &= A\cos((\Omega_0 + k\Omega_T)nT + \phi) \\ &= A\cos\left(\frac{2\pi(\Omega_0 + k\Omega_T)}{\Omega_T}n + \phi\right) = A\cos\left(\frac{2\pi\Omega_0}{\Omega_T}n + \phi\right) \\ &= A\cos(\omega_0 n + \phi) = x[n] \end{aligned}$$

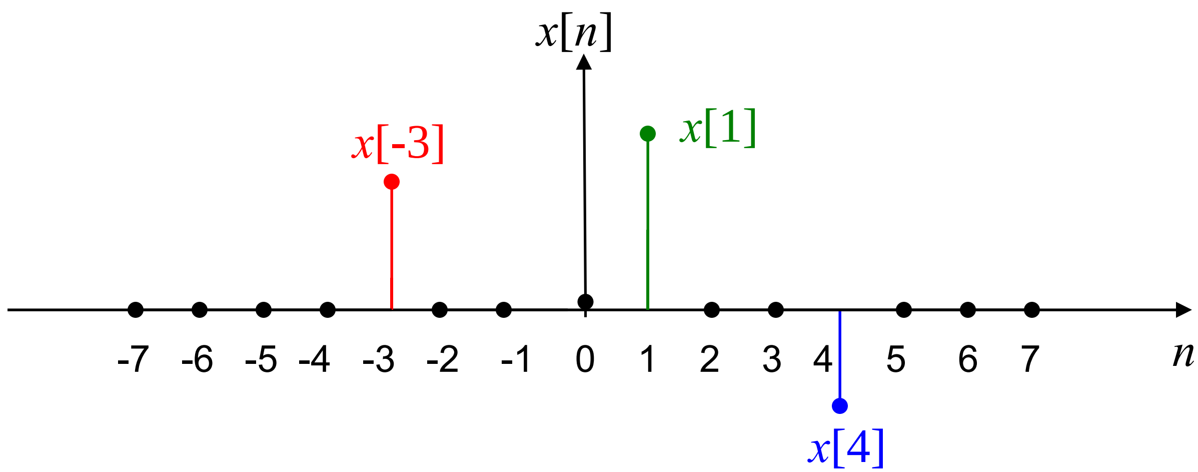
- The phenomenon is called ***aliasing***

Arbitrary Sequence



- An arbitrary sequence $x[n]$ can be expressed as a superposition of scaled versions of shifted unit impulses, $\delta[n\text{-}k]$

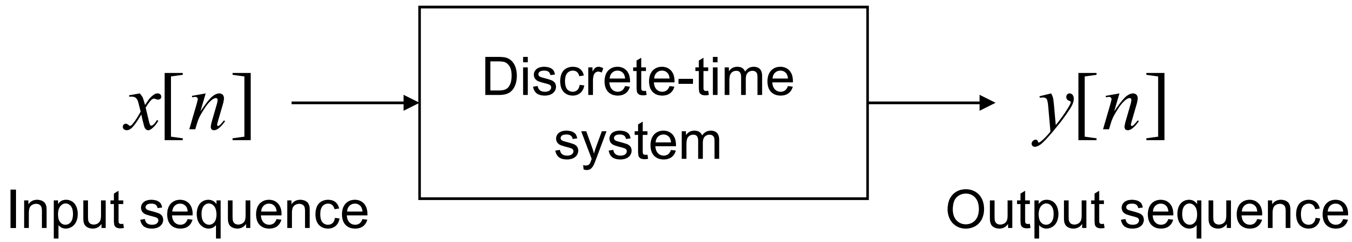
Arbitrary Sequence



$$x[n] = \boxed{x[-3]\delta[n+3]} + \boxed{x[1]\delta[n-1]} - \boxed{x[4]\delta[n-4]}$$

- In general: $$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

Discrete-Time Systems



Single-input single-output system

- Output sequence is generated sequentially, beginning with a certain time index value n

A certain class of discrete-time systems, linear and time invariant (LTI) systems will be discussed

Linearity

- A ***linear system*** is a system that possesses the important property of superposition

Additivity:

*T*he response to $x_1[n]+x_2[n]$ is $y_1[n]+y_2[n]$

Scaling or homogeneity:

*T*he response to $ax_1[n]$ is $ay_1[n]$

where *a* is any complex constant

Linearity

- Combining the two properties of superposition into a single statement

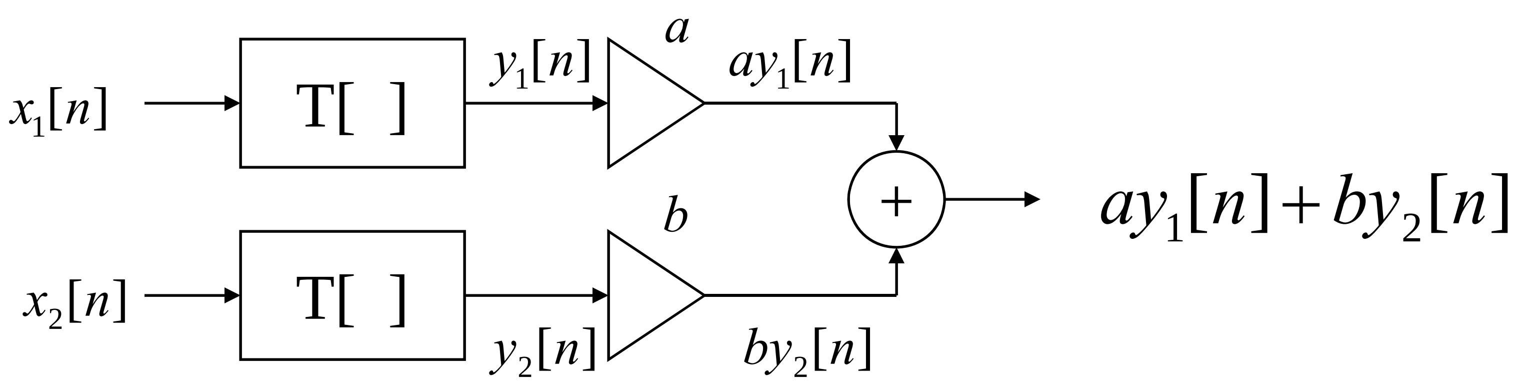
Discrete-time:

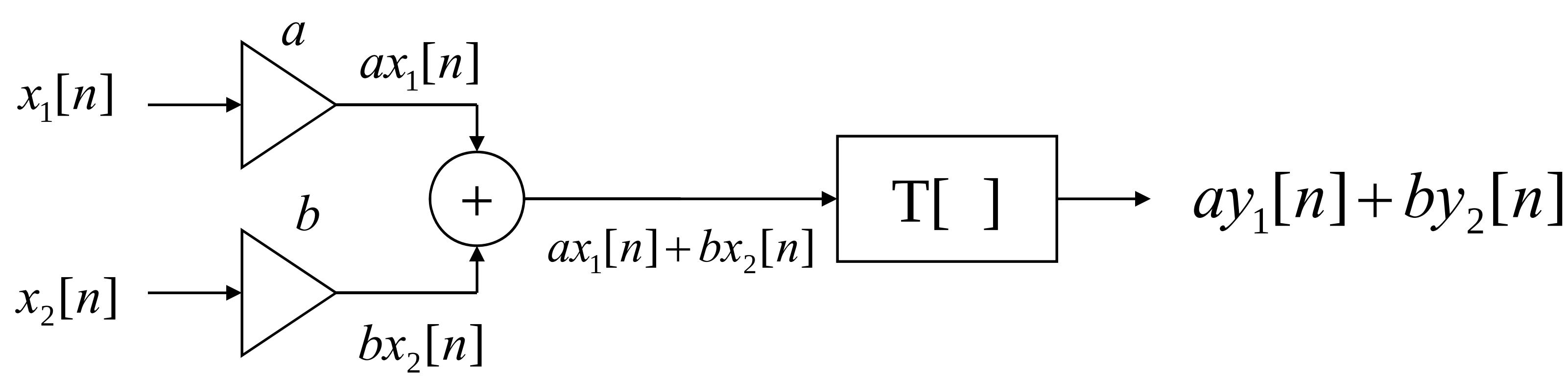
$$a x_1[n] + b x_2[n] \rightarrow a y_1[n] + b y_2[n]$$

where *a* and *b* are any complex constants

The superposition property holds for linear systems

Linearity


$x_1[n]$
$x_2[n]$
T[ ]
T[ ]
$y_1[n]$
$y_2[n]$
a
b
$ay_1[n]$
$by_2[n]$
+
$ay_1[n] + by_2[n]$


$x_1[n]$
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a
b
$ax_1[n]$
$bx_2[n]$
+
$ax_1[n] + bx_2[n]$
T[ ]
$ay_1[n] + by_2[n]$

Time Invariance

- A system is time-invariant (or shift-invariant) if a time shift in the input signal results in an identical time shift in the output signal

$$y[n] = T\left(x[n]\right)$$

$$y[n-n_0] = T\left(x[n-n_0]\right)$$

- For time-invariant systems the system properties do not change with time

Time Invariance

- A ***time invariant*** discrete-time system

$$y[n] = \sin\left[x[n]\right]$$

- A ***time variant*** discrete-time system

$$y[n] = nx[n]$$

Coefficient *n* is changing with time

Causality

- In a ***causal*** discrete-time system the output sample $y[n_0]$ at time instant n_0 depends only on the input samples $x[n]$ for $n \leq n_0$ and does not depend on input samples for $n > n_0$
- If $y_1[n]$ and $y_2[n]$ are the responses of a causal system to two inputs $u_1[n]$ and $u_2[n]$, respectively, then

$$u_1[n] = u_2[n], \quad \text{for} \quad n < N$$

implies that

$$y_1[n] = y_2[n], \quad \text{for} \quad n < N$$

Stability

- A discrete-time system is ***stable*** if and only if, for every bounded input, the output is also bounded
- If the response to $x[n]$ is the sequence $y[n]$, and if

$$|x[n]| \leq B_x$$

for all values of n, then

$$|y[n]| \leq B_y$$

for all values of n, where B_x and B_y are finite constants

Bounded-input bounded-output (BIBO) stability

Impulse and Step Response

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

- ***Unit sample response*** or ***(unit) impulse response*** is the response of the system to a unit impulse

$$x[n] = \delta[n]; \quad y[n] = h[n]$$

- ***Unit step response*** or ***step response*** is the output sequence when the input sequence is the unit step

$$x[n] = \mu[n]; \quad y[n] = s[n]$$

Convolution

- ***Linearity:*** The response of a linear system to $x[n]$ will be the superposition of the scaled responses of the system to each of these shifted impulses
- ***Time invariance:*** The responses of a time-invariant system to time-shifted unit impulses are the time-shifted versions of one another

Convolution

- The unit impulse response of a system is $h[n]$

$$\delta[n] \longrightarrow \boxed{\mathrm{T}(\ \ )} \longrightarrow h[n]$$

- The unit impulse response $h[n]$ is the response of the system to a unit impulse

Convolution

$$y[n] = T\left(x[n]\right) = T\left(\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right)$$

Additivity: $\quad y[n] = \sum_{k=-\infty}^{\infty} T\left(x[k]\delta[n-k]\right)$

Homogeneity: $\quad y[n] = \sum_{k=-\infty}^{\infty} x[k]T\left(\delta[n-k]\right)$

Shift – invariance: $\quad y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

Basic Properties of LTI Systems

- The Commutative Property
- The Distributive Property
- The Associative Property

The Commutative Property

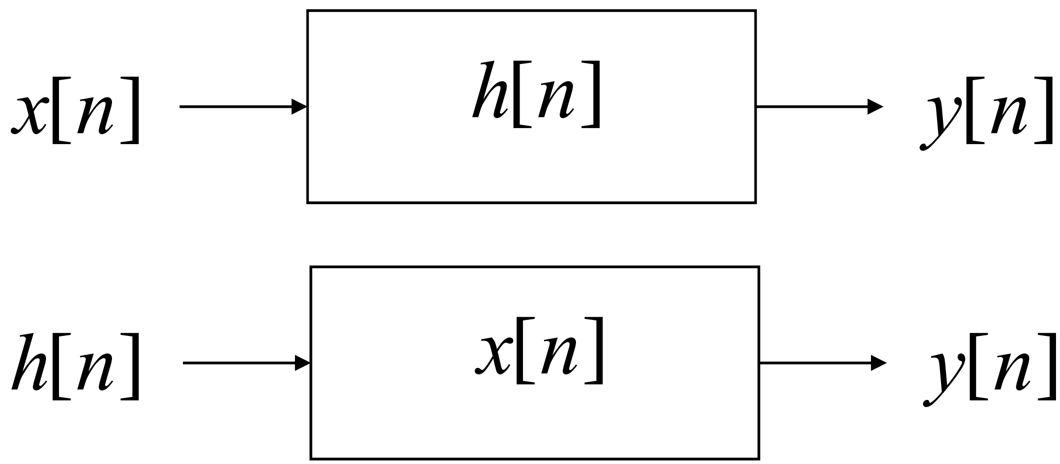
$$x[n] * h[n] = h[n] * x[n]$$

- Let $r=n-k$ or $k=n-r$; substituting to convolution sum:

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] =$$

$$\sum_{r=-\infty}^{\infty} x[n-r]h[r] = h[n] * x[n]$$

The Commutative Property



- The output of an LTI system with input $x[n]$ and unit impulse response $h[n]$ is identical to the output of an LTI system with input $h[n]$ and unit impulse response $x[n]$

The Distributive Property

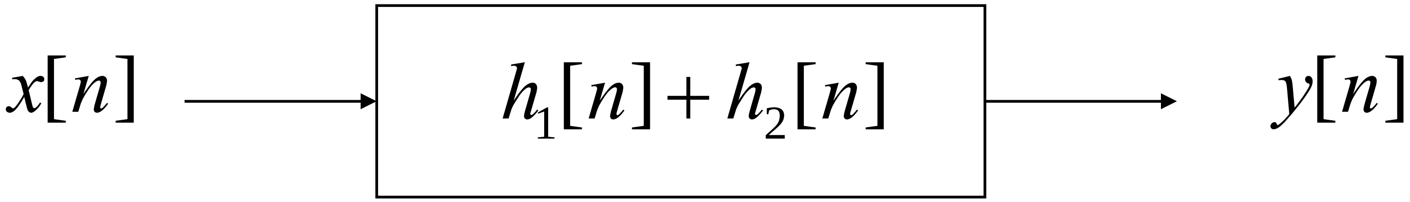
$$x[n] * \left(h_1[n] + h_2[n]\right) =$$

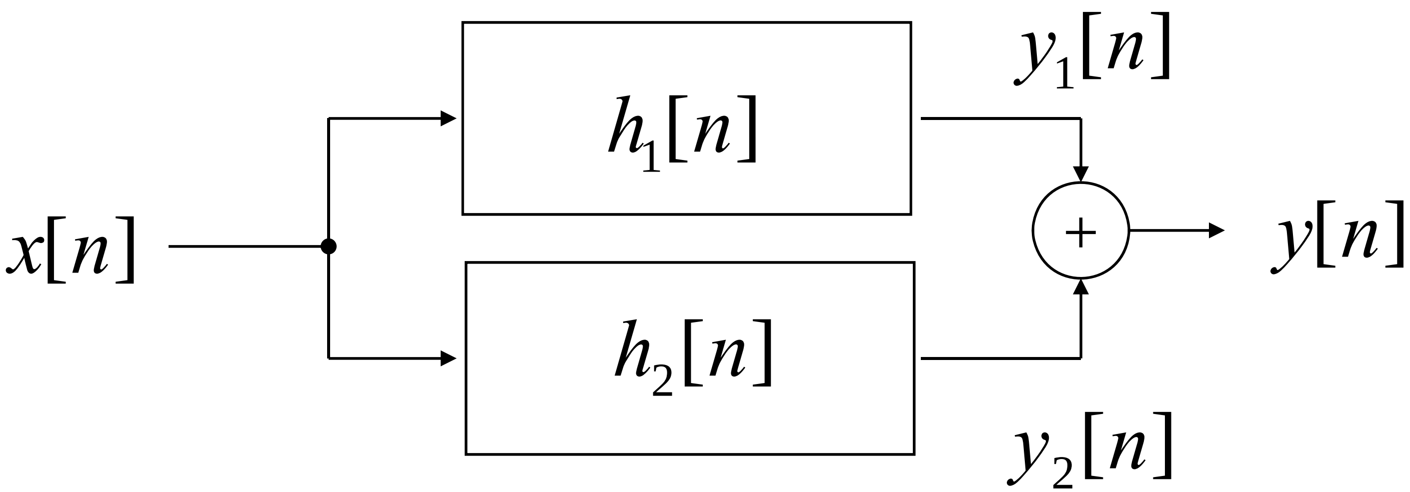
$$= x[n] * h_1[n] + x[n] * h_2[n]$$

- The distributive property has a useful interpretation in terms of system interconnections

=> *PARALLEL INTERCONNECTION*

The Distributive Property





The Associative Property

$$x[n]*\left(h_1[n]*h_2[n]\right)=$$

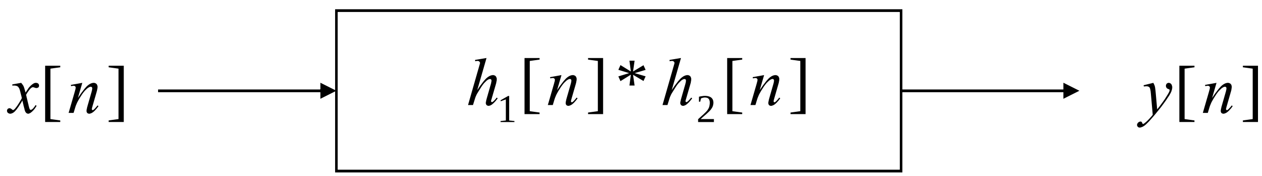
$$=\left(x[n]*h_1[n]\right)*h_2[n]$$

- As a consequence of associative property the following expression is unambiguous

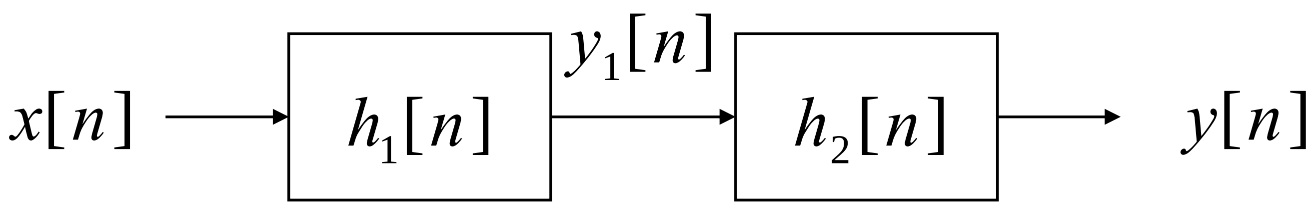
$$y[n]=x[n]*h_1[n]*h_2[n]$$

The Associative Property

$$y[n] = x[n] * \left(h_1[n] * h_2[n]\right)$$



$$y[n] = \left(x[n] * h_1[n]\right) * h_2[n] = y_1[n] * h_2[n]$$



The Associative Property

- The associative property can be interpreted as

  => ***SERIES (OR CASCADE) INTERCONNECTION OF SYSTEMS***

The Associative and Commutative Property

$$y[n] = x[n] * \left(h_1[n] * h_2[n]\right) = x[n] * \left(h_2[n] * h_1[n]\right)$$

$$x[n] \longrightarrow \boxed{h_2[n] * h_1[n]} \longrightarrow y[n]$$

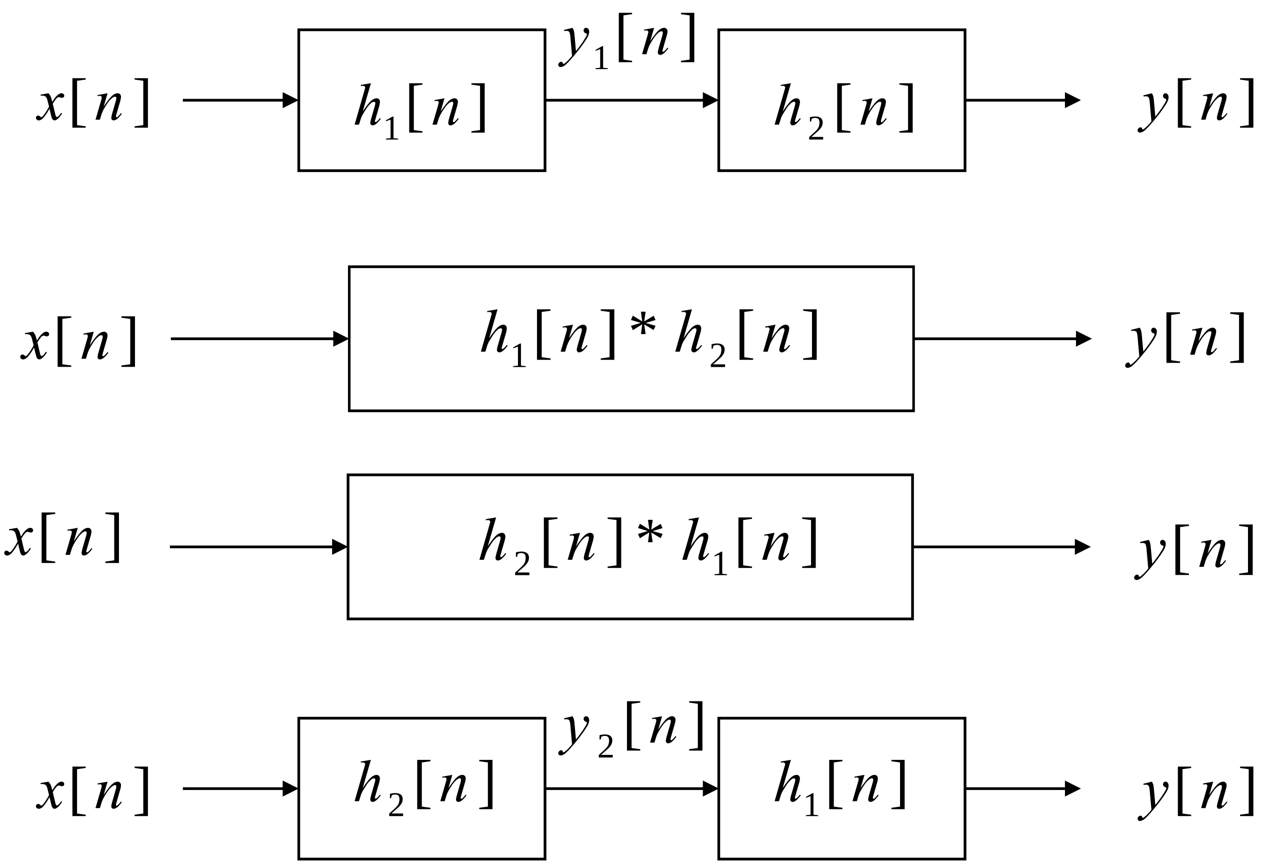
$$y[n] = \left(x[n] * h_2[n]\right) * h_1[n] = y_2[n] * h_1[n]$$

$$x[n] \longrightarrow \boxed{h_2[n]} \xrightarrow{y_2[n]} \boxed{h_1[n]} \longrightarrow y[n]$$

The Properties of Cascade Connection of Systems

- The order of the systems in cascade can be interchanged
- The intermediate signal values, $w_i[n]$, between the systems are different
- Different structures have different properties when implemented using finite precision arithmetic

The Cascade Connection of Systems



The Cascade Connection of Systems

- The properties of the cascade system depend on the sequential order of cascaded blocks
- The behavior of discrete-time systems with finite wordlength is sensitive to signal values, $w_i[n]$, between the blocks
- ***What is the optimal sequential order of cascaded blocks ?***

Stability for LTI Systems

- Consider an input $x[n]$ that is bounded in magnitude

$$|x[n]| \leq B \text{ for all } n$$

- The output is given by the convolution sum

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right|$$

$$|y[n]| \leq \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]|$$

Stability for LTI Systems

- For bounded input $|x[n\text{-}k]| \leq B$

$$|\, y[n] \,| \leq B \sum_{k=-\infty}^{\infty} |\, h[k] \,| \qquad \textit{for} \quad \textit{all} \quad n$$

- The output $y[n]$ is bounded if the the impulse response is absolutely summable

$$\sum_{k=-\infty}^{\infty} \left| h[k] \right| < \infty$$

A SUFFICIENT CONDITION FOR STABILITY !

Causality Condition

- Let $x_1[n]$ and $x_2[n]$ be two input sequences with

$$x_1[n] = x_2[n] \qquad \text{for} \quad n \leq n_0$$

then the corresponding output sequence of a causal system

$$y_1[n] = y_2[n] \qquad \text{for} \quad n \leq n_0$$

- The system is causal if and only if

$$h[n] = 0 \qquad \text{for} \quad n < 0$$

Finite-Dimensional LTI Discrete-Time Systems

- An important subclass of LTI discrete-time is characterized by a linear constant coefficient difference equation

$$\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k]$$

where $x[n]$ and $y[n]$ are, respectively, the input and output of the system and $\{d_k\}$ and $\{p_k\}$ are constants

- The ***order*** of the system is given by $\max\{N,M\}$

Finite-Dimensional LTI Discrete-Time Systems

- The output can be computed recursively by solving $y[n]$

$$y[n] = -\sum_{k=1}^{N} \frac{d_k}{d_0} y[n-k] + \sum_{k=0}^{M} \frac{p_k}{d_0} x[n-k]$$

provided that $d_0 \neq 0$.

- The output $y[n]$ can be computed for all $n \geq n_0$, knowing the input $x[n]$ and the initial conditions $y[n_0\text{-}1]$, $y[n_0\text{-}2]$, ..., $y[n_0\text{-}N]$

Classification of LTI Discrete-Time Systems

- LTI discrete-time are usually classified either according to the length of the their impulse responses or according to the method of calculation employed to determine the output samples
- Impulse response classification:
  – ***Finite impulse response*** (FIR) systems
  – ***Infinite impulse response*** (IIR) systems

Classification Based on Impulse Response

- If $h[n]$ is of finite length, i.e.,

  $$h[n] = 0, \quad \text{for } n < N_1 \text{ and } n > N_2\,, \quad \text{with } \; N_1 < N_2$$

  then it is known as a ***finite impulse response*** (FIR) discrete-time system
- The convolution sum reduces to

$$y[n] = \sum_{k=N_1}^{N_2} h[k]x[n-k]$$

- $y[n]$ can be calculated directly from the finite sum

Classification Based on Impulse Response

- If $h[n]$ is of infinite length then the system is known as an ***infinite impulse response*** (IIR) discrete-time system
- For a causal IIR discrete-time system with causal input $x[n]$, the convolution sum can be expressed as

$$y[n] = \sum_{k=0}^{n} h[k]x[n-k]$$

$y[n]$ can now be calculated sample by sample

Classification Based on Output Calculation Process

- If the output sample can be calculated sequentially, knowing only the present and past input samples, the filter is said to be ***nonrecursive*** discrete-time system
- If, on the other hand, the computation of the output involves past output samples in addition to the present and past input samples, the filter is known as ***recursive*** discrete-time system

$$y[n] = -\sum_{k=1}^{N} \frac{d_k}{d_0} y[n-k] + \sum_{k=0}^{M} \frac{p_k}{d_0} x[n-k]$$

Classification Based on Output Calculation Process

- A different terminology is used to classify causal finite-dimensional LTI systems in different applications, such as model-based spectral analysis
- The classes assigned here are based on the form of the linear constant coefficient difference equation modeling the system

Moving Average (MA) Model

- The simplest model is described by the input-output relation

$$y[n] = \sum_{k=0}^{M} p_k x[n-k]$$

- A ***moving average*** (MA) model is an FIR discrete-time system
- It can be considered as a generalization of the M-point moving average filter with different weights assigned to input samples

Autoregressive Models

- The simplest IIR, called an ***autoregresive*** (AR) model is characterized by the input-output relation

$$y[n] = x[n] - \sum_{k=0}^{N} d_k y[n-k]$$

- The second type of IIR system, called an ***autoregresive moving average*** (ARMA) model is described by the input-output relation

$$y[n] = \sum_{k=0}^{M} p_k x[n-k] - \sum_{k=0}^{N} d_k y[n-k]$$

Correlation of Signals
and
Matched Filters

Correlation of Signals

- There are applications where it is necessary to compare one reference signal with one or more signals to determine the similarity between the pair and to determine additional information based on the similarity

Example: Communications

- In digital communications, a set of data symbols are represented by a set of unique discrete-time sequences
- If one of these sequences has been transmitted, the receiver has to determine which particular sequence has been received
- The received signal is compared with every member of possible sequences from the set

→ ***Correlation***

Example: Radar Applications

- Similarly, in radar and sonar applications, the received signal reflected from the target is a delayed version of the transmitted signal
- By measuring the delay, one can determine the location of the target
- The detection problem gets more complicated in practice, as often the received signal is corrupted by additive random noise

Correlation of Signals

Definitions

- A measure of similarity between a pair of energy signals, $x[n]$ and $y[n]$, is given by the ***cross-correlation sequence*** $r_{xy}[l]$ defined by

$$r_{xy}[l] = \sum_{n=-\infty}^{\infty} x[n]y[n-l], \qquad l = 0, \pm 1, \pm 2, \ldots$$

- The parameter l called ***lag***, indicates the time-shift between the pair of signals

Correlation of Signals

- Sequence $y[n]$ is said to be shifted by l samples to the right with respect to the reference sequence $x[n]$ for positive values of l, and shifted by l samples to the left for negative values of l
- The ordering of the subscripts xy in the definition of $r_{xy}[l]$ specifies that $x[n]$ is the reference sequence which remains fixed in time while $y[n]$ is being shifted with respect to $x[n]$

Correlation of Signals

- If $y[n]$ is made the reference signal and $x[n]$ is shifted with respect to $y[n]$, the corresponding cross-correlation sequence is given by

$$r_{yx}[l] = \sum_{n=-\infty}^{\infty} y[n]x[n-l]$$

$$= \sum_{m=-\infty}^{\infty} y[m+l]x[m] = r_{xy}[-l]$$

- Thus, $r_{yx}[l]$ is obtained by time-reversing $r_{xy}[l]$

Correlation of Signals

- The ***autocorrelation sequence*** of $x[n]$ is given by

$$r_{xx}[l] = \sum_{n=-\infty}^{\infty} x[n]x[n-l]$$

obtained by setting $y[n] = x[n]$ in the definition of the cross-correlation sequence $r_{xy}[l]$

- <u>Note:</u> The energy of the signal $x[n]$ is

$$r_{xx}[0] = \sum_{n=-\infty}^{\infty} x^2[n] = \mathrm{E}_x$$

Correlation and Convolution

- From the relation $r_{yx}[l] = r_{xy}[-l]$ it follows that $r_{xx}[l] = r_{xx}[-l]$ implying that $r_{xx}[l]$ is an even function for real $x[n]$
- An examination of

$$r_{xy}[l] = \sum_{n=-\infty}^{\infty} x[n]y[n-l]$$

reveals that the expression for the cross-correlation looks quite similar to that of the linear convolution

Convolution Revisited

- The convolution of $x[m]$ and $h[m]$ was defined as

$$y[m] = \sum_{k=-\infty}^{\infty} x[k]h[m-k]$$

- Compare to correlation

$$r_{xy}[l] = \sum_{n=-\infty}^{\infty} x[n]y[n-l]$$

- Replacing now m by l and k by n, we obtain

$$r_{xy}[l] = \sum_{n=-\infty}^{\infty} x[n]y[-(l-n)]$$

Correlation and Convolution

- The expression for the cross-correlation is now similar to the convolution, i.e.,

$$r_{xy}[l] = \sum_{n=-\infty}^{\infty} x[n]y[-(l-n)] = x[l] * y[-l]$$

- The equations of correlation and convolution are the same, except the minus sign inside the summation
- In step-by-step calculation of the convolution, the other sequence is time-reversed; in correlation, it is not

Matched Filter

- The cross-correlation of $x[n]$ with the reference signal $y[n]$ can be computed by processing $x[n]$ with an LTI discrete-time system of impulse response $y[-n]$

$$x[n] \longrightarrow \boxed{y[-n]} \longrightarrow r_{xy}[n]$$

- The impulse response, $h[n]$, of the ***matched filter*** is the time-reversed version of the of reference signal $y[n]$, i.e., $h[n] = y[-n]$

Applications of Matched Filters

- In matched filters, the impulse response of the filter is "matched" to the signal, or signal pattern of interest
- Applications:
  - Radar, the impulse response of the filter is the time-reversed version of the signal to be detected
  - Pattern recognition
  - Template matching in image analysis, i.e., sub-areas of the image are correlated with the desired template

Autocorrelation

- Likewise, the autocorrelation of $x[n]$ can be computed by processing $x[n]$ with an LTI discrete-time system of impulse response $x[-n]$

$$x[n] \longrightarrow \boxed{x[-n]} \longrightarrow r_{xx}[n]$$