3 Discrete-Time Fourier Transform

Introduction

- In time-domain, the input-output relation of a linear and time-invariant (LTI) system is characterized by the convolution
- An alternate description of a sequence in terms of complex exponential sequences of the form $\{e^{-j\omega n}\}$ where ω is the normalized frequency variable
- The frequency domain representation of the discrete-time sequences and discrete-time LTI systems

Continuous-Time Fourier Transform

- Definition:

  The CTFT of a continuous-time signal $x_a(t)$ is given by

$$X_a(j\Omega) = \int_{-\infty}^{+\infty} x_a(t) e^{-j\Omega t} dt$$

- Often referred to as ***Fourier spectrum*** or simply the ***spectrum*** of the continuous-time signal

Continuous-Time Fourier Transform

- Definition:

  The inverse CTFT of a Fourier transform $X_a(j\Omega)$ is given by

$$x_a(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\Omega)e^{j\Omega t}\,d\Omega$$

- Often referred to as ***Fourier integral***

The Continuous-Time Fourier Transform Pair

Analysis equation:

$$\boxed{X_a(j\Omega) = \int_{-\infty}^{+\infty} x_a(t) e^{-j\Omega t} dt}$$

Synthesis equation:

$$\boxed{x_a(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X_a(j\Omega) e^{j\Omega t} d\Omega}$$

A CTFT pair is also denoted as:

$$x_a(t) \quad \overset{\text{CTFT}}{\longleftrightarrow} \quad X_a(j\Omega)$$

Continuous-Time Fourier Transform

- The Fourier transform or Fourier integral $X_a(j\Omega)$ of $x_a(t)$ is also called the ***analysis equation***
- The inverse Fourier transform equation is called the ***synthesis equation***
- For aperiodic signals, the complex exponentials occur at a continuum of frequencies
- The transform $X_a(j\Omega)$ of an aperiodic signal $x_a(t)$ is commonly referred to as the ***spectrum*** of $x_a(t)$

Continuous-Time Fourier Transform

- Variable Ω is real and denotes the continuous-time angular frequency in radians
- In general, the CTFT is a complex function of Ω in the range $-\infty < \Omega < \infty$
- It can be expressed in polar form as

$$X_a(j\Omega) = \left|X_a(j\Omega)\right| e^{j\theta_a(\Omega)}$$

where

$$\theta_a(\Omega) = \arg\{X_a(j\Omega)\}$$

Continuous-Time Fourier Transform

$$X_a(j\Omega) = \left| X_a(j\Omega) \right| e^{j\theta_a(\Omega)}$$

- The quantity $|X_a(j\Omega)|$ is called the ***magnitude spectrum***
- The quantity $\theta_a(\Omega)$ is called the ***phase spectrum***
- Both spectrums are real functions of Ω

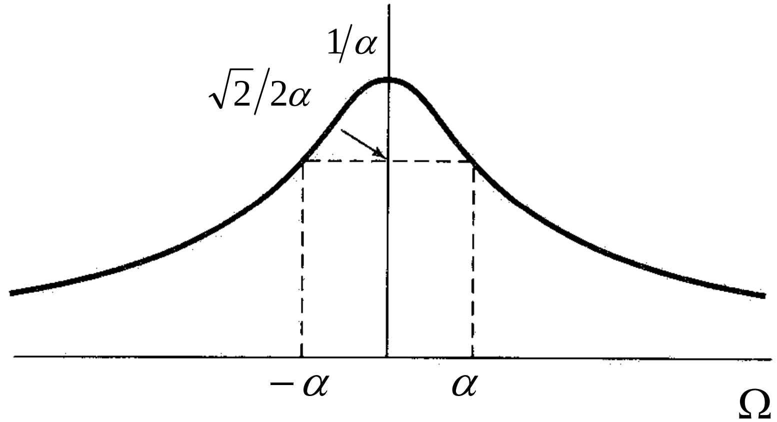
Example 3.1

The Fourier transform of a causal complex exponential

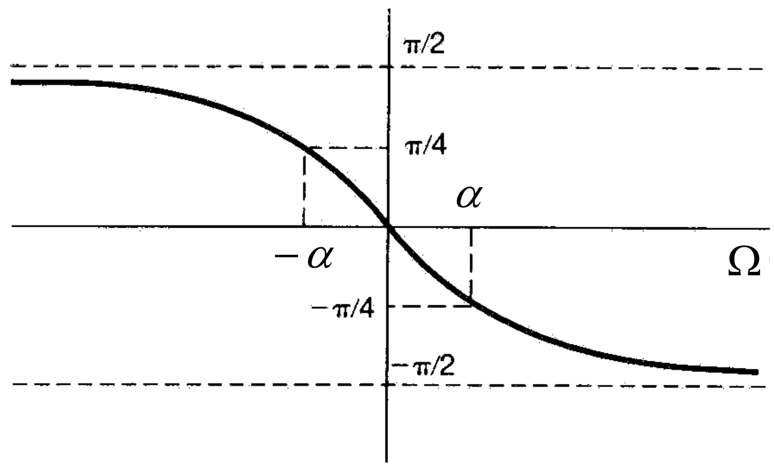
$$x_a(t) = \begin{cases} e^{-\alpha t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$X_a(j\Omega) = \int_0^{\infty} e^{-\alpha t} e^{-j\Omega t} dt$$

$$= \frac{1}{\alpha + j\Omega}, \quad \alpha > 0$$



(a)



(b)

The Frequency Response of an LTI Continuous-Time System

- The output response of $y_a(t)$ of an initially relaxed linear, time-invariant continuous-time system characterized by an impulse response $h_a(t)$ for an input signal $x_a(t)$ is given by the convolution integral

$$y_a(t) = \int_{-\infty}^{+\infty} h_a(t-\tau) x_a(\tau) d\tau$$

- Applying CTFT to both sides

$$Y_a(j\Omega) = H_a(j\Omega) X_a(j\Omega)$$

- $H_a(j\Omega)$ is the ***frequency response*** of the system

The Discrete-Time Fourier Transform

- The ***discrete-time Fourier transform*** (DTFT) of a discrete-time sequence $x[n]$ is a representation of the sequence in terms of the complex exponential sequence $\{e^{-j\omega n}\}$ where ω is the real frequency variable
- The DTFT representation of a sequence, if it exists, is unique and the original sequence can be computed from its DTFT by an inverse transform operation

The Discrete-Time Fourier Transform

- The ***discrete-time Fourier transform*** (DTFT) $X(e^{j\omega})$ of a sequence $x[n]$ is defined by:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

- The Fourier transforms of most practical discrete-time sequences can be expressed in terms of a sum of a convergent geometric series
- They can be summed in a simple closed form

Example:

Consider a causal sequence: $x[n] = \alpha^n \mu[n], \quad |\alpha| < 1$

The Fourier transform $X(e^{j\omega})$ is obtained as:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n]\ e^{-j\omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(\alpha e^{-j\omega}\right)^n = \frac{1}{1-\alpha e^{-j\omega}}$$

Discrete-Time Fourier Transform (DTFT)

- As can be seen from definition, DTFT $X(e^{j\omega})$ of a sequence $x[n]$ is a continuous function of ω
- Unlike the continuous-time Fourier transform, DTFT is a periodic function in ω with a period 2π

$$X(e^{j(\omega+2\pi k)}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j(\omega+2\pi k)n} = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}e^{-j2\pi kn}$$

$$= \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n} = X(e^{j\omega}), \quad \text{for all values of } k$$

where $e^{-j2\pi kn} = 1$

Inverse Discrete-Time Fourier Transform

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

- The inverse discrete-time Fourier transform can be interpreted as a linear combination of infinitesimally small complex exponential signals of the form $\frac{1}{2\pi} e^{j\omega n} d\omega$, weighted by the complex constant $X(e^{j\omega})$ over the angular frequency range from $-\pi$ to π

The Discrete-Time Fourier Transform (DTFT) Pair

Analysis equation, denoted by operator $\mathcal{F}\{x[n]\}$:

$$\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}}$$

Synthesis equation, denoted by operator $\mathcal{F}^{-1}\{x[n]\}$:

$$\boxed{x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega}$$

Basic Properties of the DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

- $X(e^{j\omega})$ is a complex function the real variable ω :

$$X(e^{j\omega}) = X_{re}(e^{j\omega}) + jX_{im}(e^{j\omega})$$

$$X(e^{j\omega}) = \left|X(e^{j\omega})\right| e^{j\theta(\omega)}, \text{where} \quad \theta(\omega) = \arg\{X(e^{j\omega})\}$$

- $|X(e^{j\omega})|$ is the magnitude function
- $\theta(e^{j\omega})$ is called the phase function

Basic Properties of the DTFT

$$X(e^{j\omega}) = \left|X(e^{j\omega})\right| e^{j\theta(\omega)}$$

- In many applications, the Fourier transform $X(e^{j\omega})$ is called the ***Fourier spectrum***
- $|X(e^{j\omega})|$ is called the ***magnitude spectrum*** and
- $\theta(\omega)$ is the ***phase spectrum***
- It is usually assumed that the phase function $\theta(\omega)$ is restricted to the *principal value*

$$-\pi \leq \theta(\omega) < \pi$$

Commonly Used DTFT Pairs

Sequence		DTFT
$\delta[n]$	$\leftrightarrow$	1
1	$\leftrightarrow$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$\mu[n]$	$\leftrightarrow$	$\dfrac{1}{1 - e^{-j\omega}}$
$e^{j\omega_0 n}$	$\leftrightarrow$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$
$\alpha^n \mu[n], \quad (\lvert\alpha\rvert < 1)$	$\leftrightarrow$	$\dfrac{1}{1 - \alpha e^{-j\omega}}$

DTFT Properties

- There are a number of important properties of the DTFT that are useful in signal processing applications
- These are listed here without proof
- Their proofs are straightforward
- The applications of some of the properties are illustrated

Table 3.1: DTFT Properties: Symmetry Relations

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega})$
$x[-n]$	$X(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$\mathrm{Re}\{x[n]\}$	$X_{\mathrm{cs}}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) + X^*(e^{-j\omega})\}$
$j\mathrm{Im}\{x[n]\}$	$X_{\mathrm{ca}}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) - X^*(e^{-j\omega})\}$
$x_{\mathrm{cs}}[n]$	$X_{\mathrm{re}}(e^{j\omega})$
$x_{\mathrm{ca}}[n]$	$jX_{\mathrm{im}}(e^{j\omega})$

Note: $X_{\mathrm{cs}}(e^{j\omega})$ and $X_{\mathrm{ca}}(e^{j\omega})$ are the conjugate-symmetric and conjugate-antisymmetric parts of $X(e^{j\omega})$, respectively. Likewise, $x_{\mathrm{cs}}[n]$ and $x_{\mathrm{ca}}[n]$ are the conjugate-symmetric and conjugate-antisymmetric parts of $x[n]$, respectively.

$x[n]$: A complex sequence

Table 3.2: DTFT Properties: Symmetry Relations

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega}) = X_{re}(e^{j\omega}) + jX_{im}(e^{j\omega})$
$x_{ev}[n]$	$X_{re}(e^{j\omega})$
$x_{od}[n]$	$jX_{im}(e^{j\omega})$
Symmetry relations	$X(e^{j\omega}) = X^*(e^{-j\omega})$ $X_{re}(e^{j\omega}) = X_{re}(e^{-j\omega})$ $X_{im}(e^{j\omega}) = -X_{im}(e^{-j\omega})$ $\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ $\arg\{X(e^{j\omega})\} = -\arg\{X(e^{-j\omega})\}$

Note: $x_{ev}[n]$ and $x_{od}[n]$ denote the even and odd parts of $x[n]$, respectively.

$x[n]$: A real sequence

Important DTFT Theorems

- There are a number of important theorems of the DTFT that are useful in analysis and synthesis of discrete-time LTI systems
- Many algorithms in signal processing applications are based on these theorems
- Their proofs are straightforward based on the definitions
- Assume that:

$$g[n] \overset{F}{\leftrightarrow} G(e^{j\omega}) \quad \text{and} \quad h[n] \overset{F}{\leftrightarrow} H(e^{j\omega})$$

Table 3.4: General Properties of DTFT

Type of Property	Sequence	Discrete-Time Fourier Transform
	$g[n]$ $h[n]$	$G(e^{j\omega})$ $H(e^{j\omega})$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(e^{j\omega}) + \beta H(e^{j\omega})$
Time-shifting	$g[n - n_o]$	$e^{-j\omega n_o} G(e^{j\omega})$
Frequency-shifting	$e^{j\omega_o n} g[n]$	$G\left(e^{j(\omega-\omega_o)}\right)$
Differentiation in frequency	$ng[n]$	$j\dfrac{dG(e^{j\omega})}{d\omega}$
Convolution	$g[n] \circledast h[n]$	$G(e^{j\omega})H(e^{j\omega})$
Modulation	$g[n]h[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\theta})H(e^{j(\omega-\theta)})\,d\theta$
Parseval's relation	$\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega})H^*(e^{j\omega})\,d\omega$	

The Frequency Response of an LTI Discrete-Time System

- Time-Domain:
  An LTI discrete-time system is completely characterized by its impulse response sequence $\{h[n]\}$
- Transform-Domain:
  Alternative representations of an LTI discrete-time system using the DTFT (and the *z*-transform)

The Frequency Response - Definition

- An important property of an LTI system is that for certain types of input signals, called eigenfunctions, the output signal is the input signal multiplied by a complex constant
- We consider one such eigenfunction, the complex exponential sequence
- In general, for CT and DT systems:
  – Continuous-time: e^{sT} -> $H(s)\, e^{sT}$
  – Discrete-time: z^n -> $H(z)\, z^n$

The Frequency Response

Superposition property:

The response of an LTI system to a linear combination of complex exponential signals can be determined by knowing its response to a single complex exponential signal

The response of the LTI system to a complex exponential input is considered

Frequency Response is a transform-domain representation of the LTI discrete-time system

Complex Exponential Input

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = h[n] * x[n]$$

Input: $$x[n] = e^{j\omega n}, \quad -\infty < n < \infty$$

Output: $$y[n] = \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)}$$

$$= e^{j\omega n} \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} = x[n]\left(\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}\right)$$

The Frequency Response

Define:

$$\boxed{H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}}$$

- $H(e^{j\omega})$ is called the frequency response of the LTI discrete-time system
- $H(e^{j\omega})$ is the DTFT of $h[n]$
- For a complex exponential input:

$$\boxed{y[n] = H(e^{j\omega})e^{j\omega n}}$$

The Response to a Complex Exponential

- For a fixed frequency $\omega = \omega_0$: $y[n] = H(e^{j\omega_0})e^{j\omega_0 n}$
- For a complex exponential input $x[n]$ of angular frequency ω_0, the output $y[n]$ is a complex exponential sequence of the same angular frequency ω_0 weighted by a complex constant $H(e^{j\omega_0})$
- In general, the frequency response $H(e^{j\omega})$ is a function of the angular frequency and can be evaluated at all input frequencies ω
- $H(e^{j\omega})$ completely characterizes the behavior of an LTI discrete-time system in frequency domain

The Frequency Response

- $H(e^{j\omega})$ is a complex function of ω with a period 2π

$$H(e^{j\omega}) = H_{re}(e^{j\omega}) + jH_{im}(e^{j\omega})$$
$$= \left|H(e^{j\omega})\right| e^{j\theta(\omega)}$$

where $\theta(\omega) = \arg\{H(e^{j\omega})\}$

- $|H(e^{j\omega})|$ is called the ***magnitude response***
- $\theta(\omega)$ is called the ***phase response***

The Frequency Response

- In some cases, the magnitude function is defined in decibels

$$\mathcal{G}(\omega) = 20\log_{10}\left|H(e^{j\omega})\right| \quad \text{dB}$$

- $\mathcal{G}(\omega)$ is called the ***gain*** function
- The negative of the gain function, $\mathcal{A}(\omega) = -\mathcal{G}(\omega)$ is called the ***attenuation*** or ***loss function***

Frequency-Domain Characterization of LTI Systems

- Input-output relation in frequency-domain

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) \;= \left(\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}\right) X(e^{j\omega})$$

- Convolution in the time-domain transforms into product in the frequency-domain

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

- The frequency response of an LTI discrete-time system is the ratio of $Y(e^{j\omega})$ and $X(e^{j\omega})$

Frequency Responses of LTI FIR Discrete-Time Systems

- Input-output relation of the LTI FIR discrete-time system

$$y[n] = \sum_{k=N_1}^{N_2} h[k]x[n-k], \quad N_1 < N_2$$

- Applying the discrete-time Fourier transform (DTFT) results in the transform-domain input-output relation

$$Y(e^{j\omega}) = \left(\sum_{k=N_1}^{N_2} h[k]\, e^{-j\omega k} \right) X(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

where $Y(e^{j\omega})$ and $X(e^{j\omega})$ are the DTFTs of the output and input sequences

Frequency Responses of LTI FIR Discrete-Time Systems

- The frequency response of the LTI FIR discrete-time system is thus

$$H\left(e^{j\omega}\right)=\sum_{k=N_1}^{N_2} h[k]e^{-j\omega k}$$

- The frequency response of the LTI FIR discrete-time system is a polynomial in $e^{-j\omega}$

Frequency Responses of LTI IIR Discrete-Time Systems

- Input-output relation of the LTI IIR discrete-time system

$$\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k]$$

- Applying the discrete-time Fourier transform (DTFT) results in the transform-domain input-output relation

$$\sum_{k=0}^{N} d_k e^{-j\omega k} Y(e^{j\omega}) = \sum_{k=0}^{M} p_k e^{-j\omega k} X(e^{j\omega})$$

Frequency Responses of LTI IIR Discrete-Time Systems

- The frequency-domain relation can be written in the form

$$\left(\sum_{k=0}^{N} d_k e^{-j\omega k}\right) Y(e^{j\omega}) = \left(\sum_{k=0}^{M} p_k e^{-j\omega k}\right) X(e^{j\omega})$$

- Solving the ratio $H(e^{j\omega}) = \dfrac{Y(e^{j\omega})}{X(e^{j\omega})} = \dfrac{\sum_{k=0}^{M} p_k e^{-j\omega k}}{\sum_{k=0}^{N} d_k e^{-j\omega k}}$

- The frequency response of the LTI IIR discrete-time system is a polynomial in $e^{-j\omega}$

Example: Simple IIR Discrete-Time System

- Consider the first order recursive or infinite impulse response (IIR) filter

$$y[n]-\alpha y[n-1]=x[n], \quad \text{with} \ \ |\alpha|<1$$

- The frequency response of this system is obtained by the Fourier transform

$$Y(e^{j\omega})+\alpha Y(e^{j\omega})e^{-j\omega}=X(e^{j\omega})$$

- Solving the ratio: $$H(e^{j\omega})=\frac{Y(e^{j\omega})}{X(e^{j\omega})}=\frac{1}{1-\alpha e^{-j\omega}}$$

- The impulse response is: $$h[n]=\alpha^n\mu[n]$$

Response to a Causal Exponential Sequence

- In practice, the excitation to an LTI discrete-time system is usually a causal sequence applied at some finite sample index $n = n_0$
- The output for such an input when observed at sample instants beginning at $n = n_0$ will consist of a transient part along with a steady-state component
- Assume that the input is a causal exponential sequence applied at $n = 0$, i.e., $x[n] = e^{j\omega n}\mu[n]$

Response to a Causal Exponential Sequence

- For $n > 0$, the output is obtained using the convolution sum

$$y[n] = \sum_{k=0}^{\infty} h[k]\ e^{j\omega(n-k)} \mu[n-k] = \left(\sum_{k=0}^{n} h[k]\ e^{-j\omega k} \right) e^{j\omega n}$$

as $\mu[n\text{-}k] = 0$ for $k > n$

- Rewriting the last expression of the equation

$$y[n] = \left(\sum_{k=0}^{\infty} h[k]\ e^{-j\omega k} \right) e^{j\omega n} - \left(\sum_{k=n+1}^{\infty} h[k]\ e^{-j\omega k} \right) e^{j\omega n}$$

$$= H(e^{j\omega})\ e^{j\omega n} - \left(\sum_{k=n+1}^{\infty} h[k]\ e^{-j\omega k} \right) e^{j\omega n}, \quad n \geq 0$$

Response to a Causal Exponential Sequence

$$y[n] = \underbrace{H(e^{j\omega})\ e^{j\omega n}}_{\text{Steady-state response}} - \underbrace{\left(\sum_{k=n+1}^{\infty} h[k]\ e^{-j\omega k}\right) e^{j\omega n}}_{\text{Transient response}}, \quad n \geq 0$$

Steady-state response ***Transient response***

$$y_{sr}[n] = H(e^{j\omega})\ e^{j\omega n}$$

$$y_{tr}[n] = -\left(\sum_{k=n+1}^{\infty} h[k]\ e^{-j\omega k}\right) e^{j\omega n}$$

- The effect of the transient response on the output is

$$|\ y_{tr}[n]\ | = \left|\sum_{k=n+1}^{\infty} h[k]\ e^{-j\omega(k-n)}\right| \leq \sum_{k=n+1}^{\infty} |h[k]| \leq \sum_{k=0}^{\infty} |h[k]|$$

Response to a Causal Exponential Sequence

$$\left| y_{tr}[n] \right| = \left| \sum_{k=n+1}^{\infty} h[k]\, e^{-j\omega(k-n)} \right| \le \sum_{k=n+1}^{\infty} \left| h[k] \right| \le \sum_{k=0}^{\infty} \left| h[k] \right|$$

- For a causal and stable IIR LTI discrete-time system, the impulse response is absolutely summable
- As a result the transient response $y_{tr}[n]$ is a bounded sequence
- Moreover, as $n \rightarrow \infty$, $\sum_{k=n+1}^{\infty} \left| h[k] \right| \rightarrow 0$ the transient response decays to zero as n gets very large

Response to a Causal Exponential Sequence

- In most practical cases, the transient response becomes negligibly small after some finite amount of time, and the system can be assumed to be in a steady-state
- For a causal FIR LTI discrete-time system with an impulse response of length $N+1$, $h[n]=0$ for $n > N$ and, thus, $y_{tr}[n]=0$ for $n > N-1$
- It should be noted that transients will occur whenever an input is applied or changed

The Concept of Filtering

- A ***digital filter*** is a discrete-time system that passes certain frequency components in an input sequence without any distortion and blocks other frequency components
- The key to the filtering process is the inverse discrete-time Fourier transform which expresses an arbitrary sequence as a linear weighted sum of an infinite number of exponential (sinusoidal) sequences
- By appropriately choosing the frequency response (or its magnitude) of the LTI digital filter the individual sinusoidal components can be attenuated or amplified independent of each other

The Concept of Filtering

- Consider a real coefficient LTI discrete-time system characterized by a magnitude function

$$\left|H(e^{j\omega})\right| \cong \begin{cases} 1, & 0 \le |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

- An input sequence

$$x[n] = A\cos(\omega_1 n) + B\cos(\omega_2 n),$$

with $0 < \omega_1 < \omega_c < \omega_2 < \pi$

is applied to the system

The Concept of Filtering

- The output sequence is given by

$$y[n] = A\left|H(e^{j\omega_1})\right|\cos(\omega_1 n + \theta(\omega_1))$$
$$+ B\left|H(e^{j\omega_2})\right|\cos(\omega_2 n + \theta(\omega_2))$$

- Making use of $|H(e^{j\omega})|$ the output is

$$y[n] \cong A\left|H(e^{j\omega_1})\right|\cos(\omega_1 n + \theta(\omega_1))$$

- The LTI system is a lowpass filter

Response to a Sinusoidal Sequence

- Consider the sinusoidal input to an LTI discrete-time system with the frequency response $H(e^{j\omega}) = |H(e^{j\omega})| e^{j\theta(\omega)}$

$$x[n] = A\cos(\omega_0 n + \phi)$$

$$y[n] = A\left|H(e^{j\omega_0})\right|\cos(\omega_0 n + \theta(\omega_0) + \phi)$$

- The output signal $y[n]$ has the same sinusoidal waveform as the input $x[n]$ with two differences
  - The amplitude is multiplied by the constant value $\left|H(e^{j\omega_0})\right|$
  - The output has a phase ***lag*** by amount $\theta(\omega_0)$

Phase and Group Delays

- Let us rewrite the output to a sinusoidal input as

$$y[n] = A\left|H(e^{j\omega_0})\right|\cos\left(\omega_0\left(n + \frac{\theta(\omega_0)}{\omega_0}\right) + \phi\right)$$

$$= A\left|H(e^{j\omega_0})\right|\cos\big(\omega_0\big(n - \tau_p(\omega_0)\big) + \phi\big)$$

where $\tau(\omega_0) = -\dfrac{\theta(\omega_0)}{\omega_0}$ is called the ***phase delay***

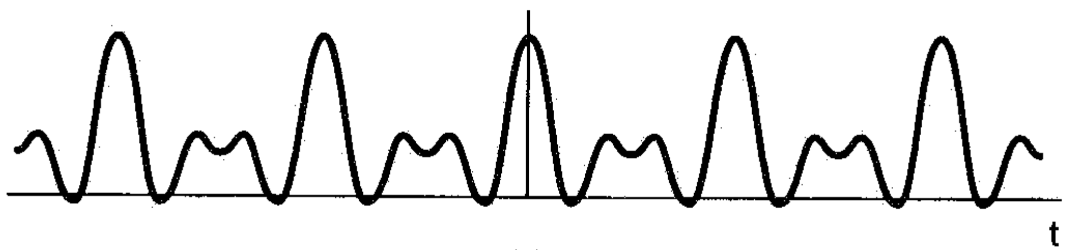
- The output $y[n]$ is a time-delayed version of the input $x[n]$

Example: Linear combination of sinusoidal signals

Consider the signal: $x(t) = 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t$

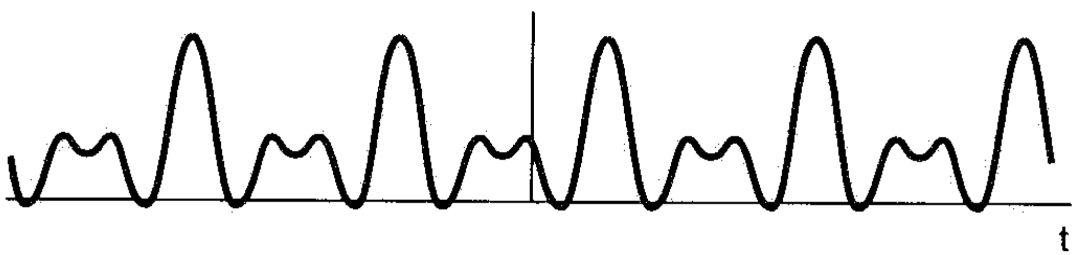
The same sinusoidal components with phase shifts:

$$x(t) = 1 + \frac{1}{2}\cos(2\pi t + \phi_1) + \cos(4\pi t + \phi_2) + \frac{2}{3}\cos(6\pi t + \phi_3)$$



(a)

(a) $\Phi_1 = \Phi_2 = \Phi_3 = 0$

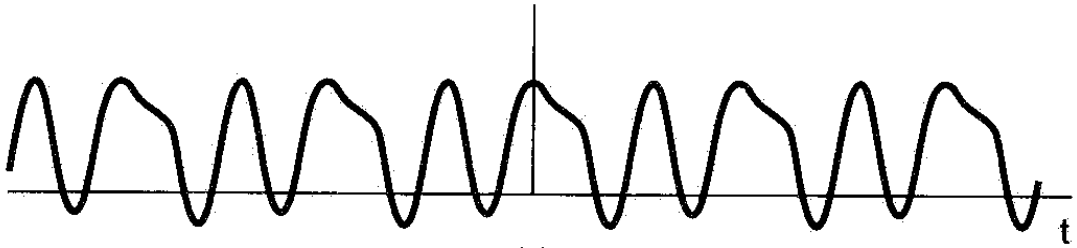


(b)

(b) $\Phi_1 = 4$, $\Phi_2 = 8$, and $\Phi_3 = 12$ rad

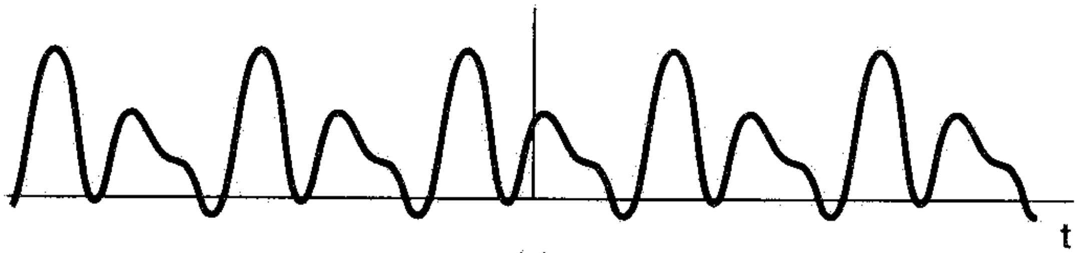
Example: Linear combination of sinusoidal signals

$$x(t) = 1 + \frac{1}{2}\cos(2\pi t + \phi_1) + \cos(4\pi t + \phi_2) + \frac{2}{3}\cos(6\pi t + \phi_3)$$



(c)

(c) $\Phi_1 = 6$, $\Phi_2 = -2.7$, $\Phi_3 = 0.93$ rad



(d)

(d) $\Phi_1 = 1.2$, $\Phi_2 = 4.1$, $\Phi_3 = -7.02$ rad

The resulting signals differ significantly for different relative phases

The Group Delay

- When the input signal contains many sinusoidal components with different frequencies that are not harmonically related, each component will go through different phase delays when processed by a frequency-selective LTI discrete-time system
- The delay is determined using a different parameter called the ***group delay*** defined as

$$\tau_g(\omega) = -\frac{d\theta(\omega)}{d\omega}$$

- Group delay has a physical interpretation in calculating the responses of discrete-time systems

The Group Delay

- Group delay function provides a measure of the linearity of the phase response

$$\tau_g(\omega) = -\frac{d\theta(\omega)}{d\omega}$$

- For a moving average filter of length M, the phase response is ***linear***

$$\theta(\omega) = -\frac{M-1}{2}\omega$$

and the group delay is ***constant***

$$\tau_g(\omega) = \frac{M-1}{2}$$