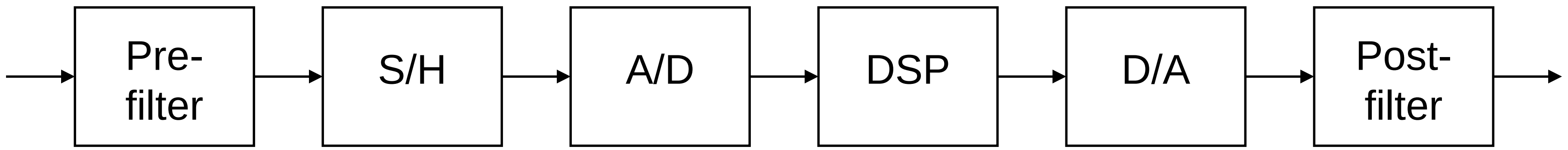
4 Digital Processing of Continuous-Time Signals

Introduction

- Analog-to-Digital (A/D) Converter and Digital-to Analog (D/A) Converter needed to interface the system with analog world
- Application examples:
  - Speech
  - Music
  - Images

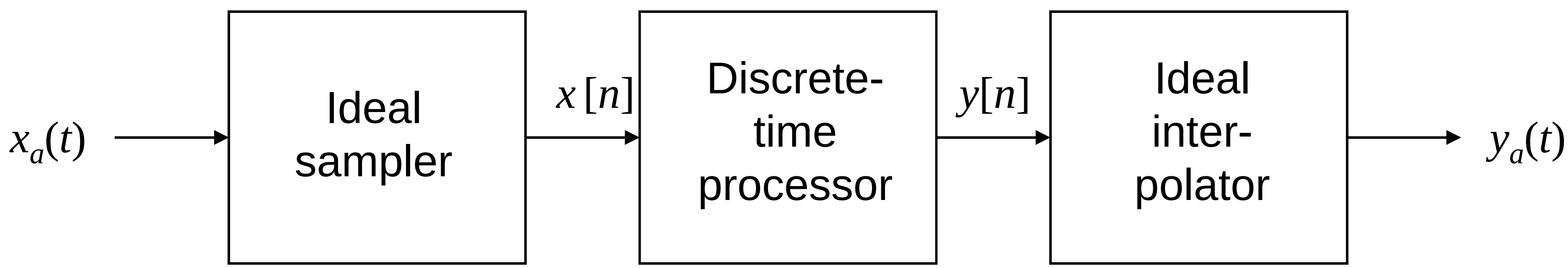
Building Blocks



- Anti-aliasing filter (pre-filter)
- Sample-and-hold (S/H) circuit
- A/D converter (ADC)
- Digital signal processor (DSP)
- D/A converter (DAC)
- Reconstruction (smoothing) filter (post-filter)

Ideal Interfaces

- Simplified block diagram with ideal CT-DT and DT-CT converters:



  - Finite precision A/D and D/A conversion is not considered here

Sampling of CT Signals

- Let $g_a(t)$ be a continuous-time signal that is uniformly sampled at $t=nT$

$$g[n] = g_a(nT), \quad -\infty < n < \infty$$

- T is the ***sampling period***
- $F_T=1/T$ is the ***sampling frequency***

Spectrum of CT and DT Signals

- Continuous-time Fourier transform of $g_a(t)$ is

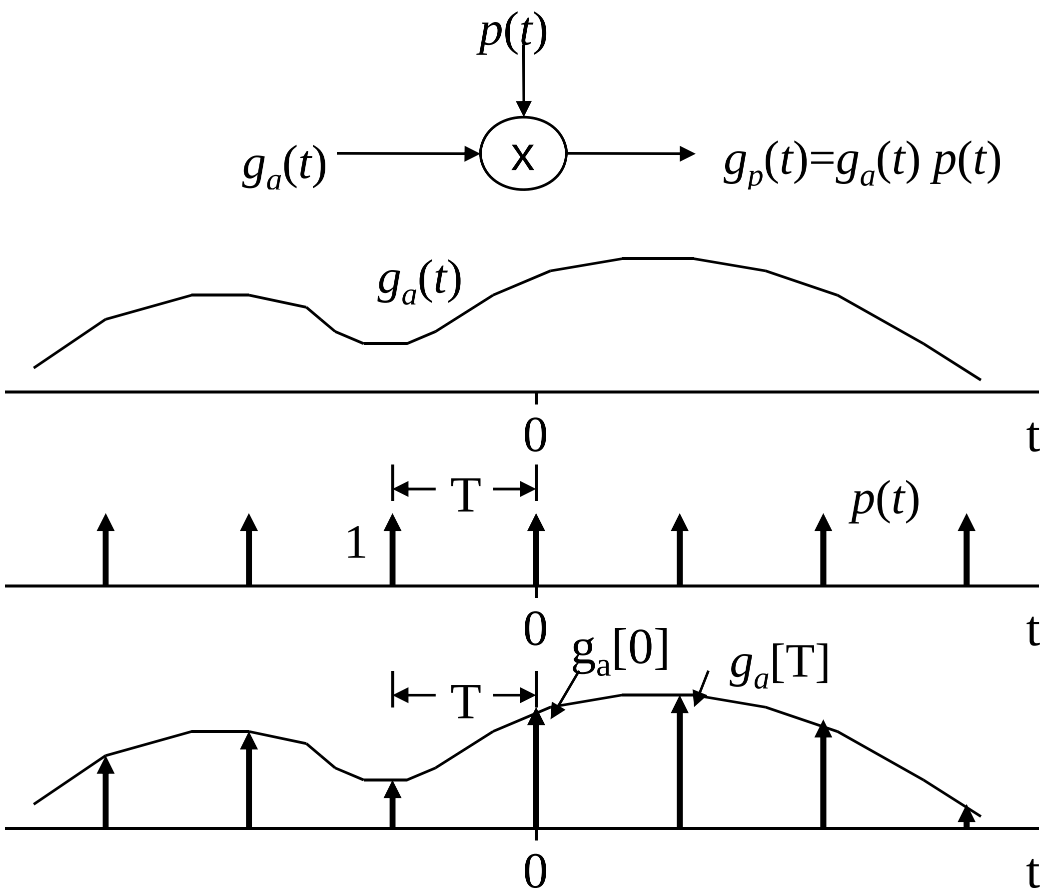
$$G_a(j\Omega) = \int_{-\infty}^{\infty} g_a(t) e^{-j\Omega t} dt$$

- Discrete-time Fourier transform of $g[n]$ is

$$G(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n] e^{-j\omega n}$$

- What is the difference between the two different types of Fourier spectra ?

Sampling Process



Continuous-time signal $g_a(t)$ is multiplied by an impulse train

Continuous-time signal $g_a(t)$

Impulse train $p(t)$

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Weighted impulse train

$g_p(t)=g_a(t)p(t)$

Impulse-Train Sampling

- The periodic impulse train $p(t)$ is the ***sampling function***
- In time-domain:

$$g_p(t) = g_a(t)\,p(t), \quad \text{where} \quad p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

- Multiplying $g_a(t)$ by a unit impulse, samples the value of the signal at the point at which the impulse is located, i.e.,

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$$

- Thus, $g_p(t)$ is an impulse train with the amplitudes of the impulses equal to the samples of $g_a(t)$ at intervals spaced by T, i.e.,

$$g_p(t) = \sum_{n=-\infty}^{\infty} g_a(nT)\delta(t - nT)$$

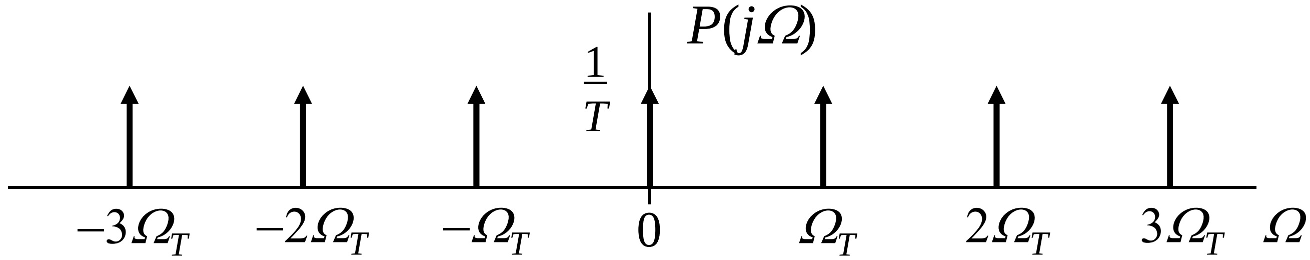
Impulse-Train Sampling

- Using the multiplication property of the convolution theorem

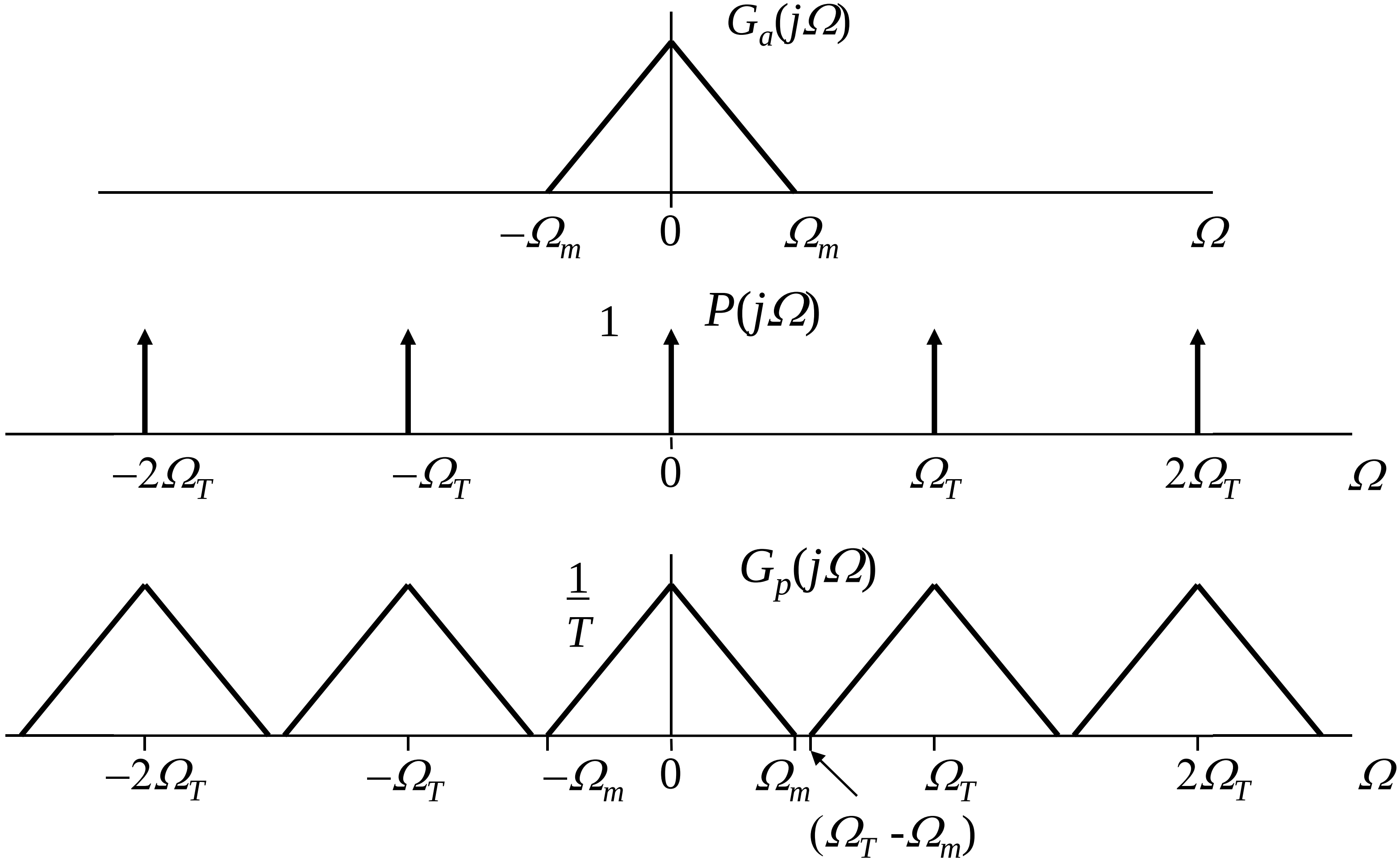
$$g_p(t) = g_a(t)\,p(t) \quad \Leftrightarrow \quad G_p(j\Omega) = G_a(j\Omega) * P(j\Omega)$$

- The Fourier transform of a periodic impulse train $p(t)$ is also a periodic impulse train in the frequency domain, i.e.,

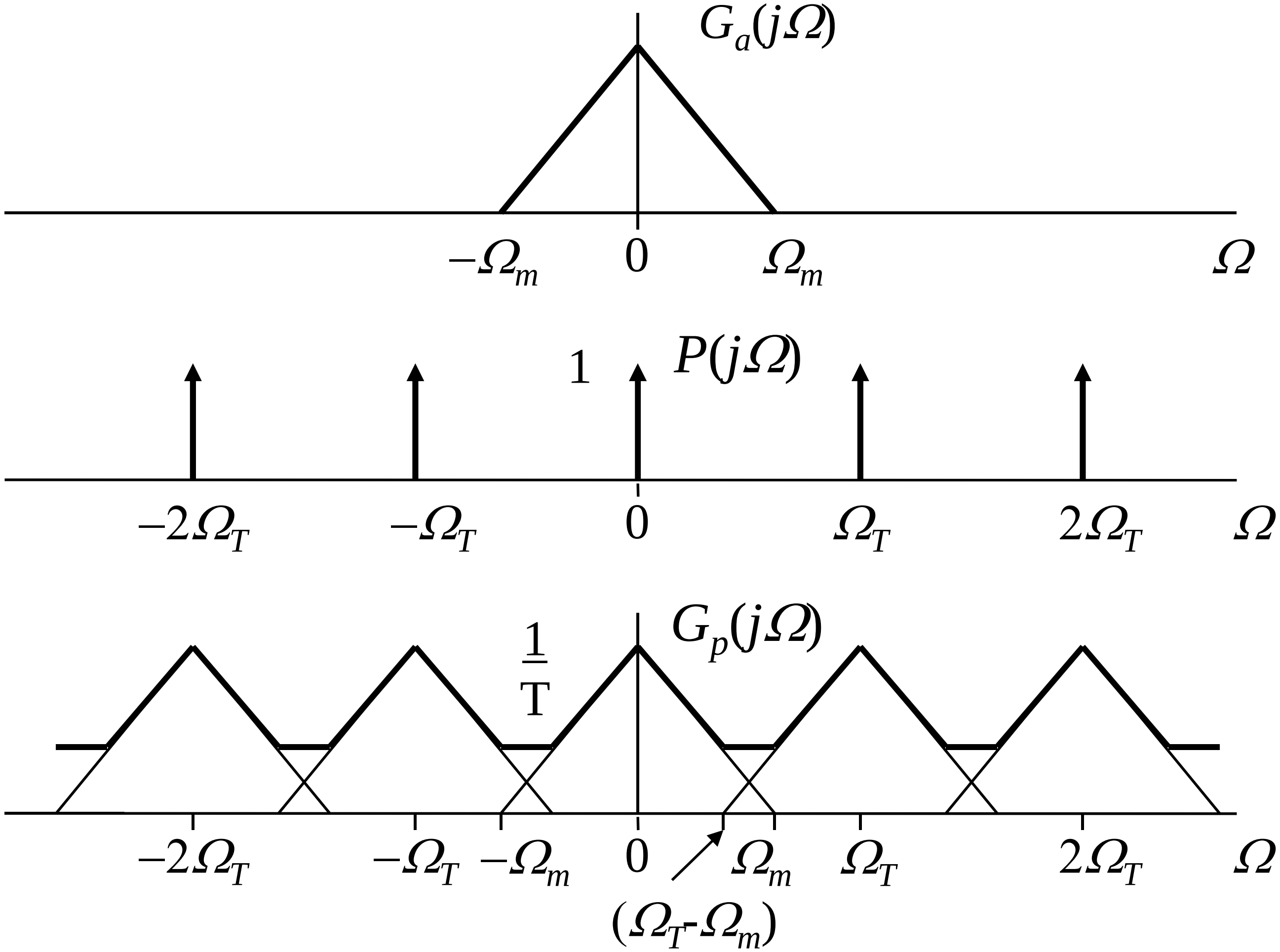
$$P(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty}\delta(\Omega - k\Omega_T)$$



Spectrum of Sampled Signal with $\Omega_T > 2\Omega_m$

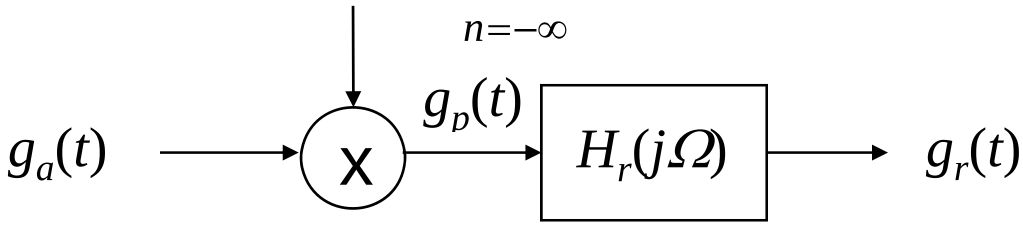


Spectrum of Sampled Signal with $\Omega_T < 2\Omega_m$



Sampling Process

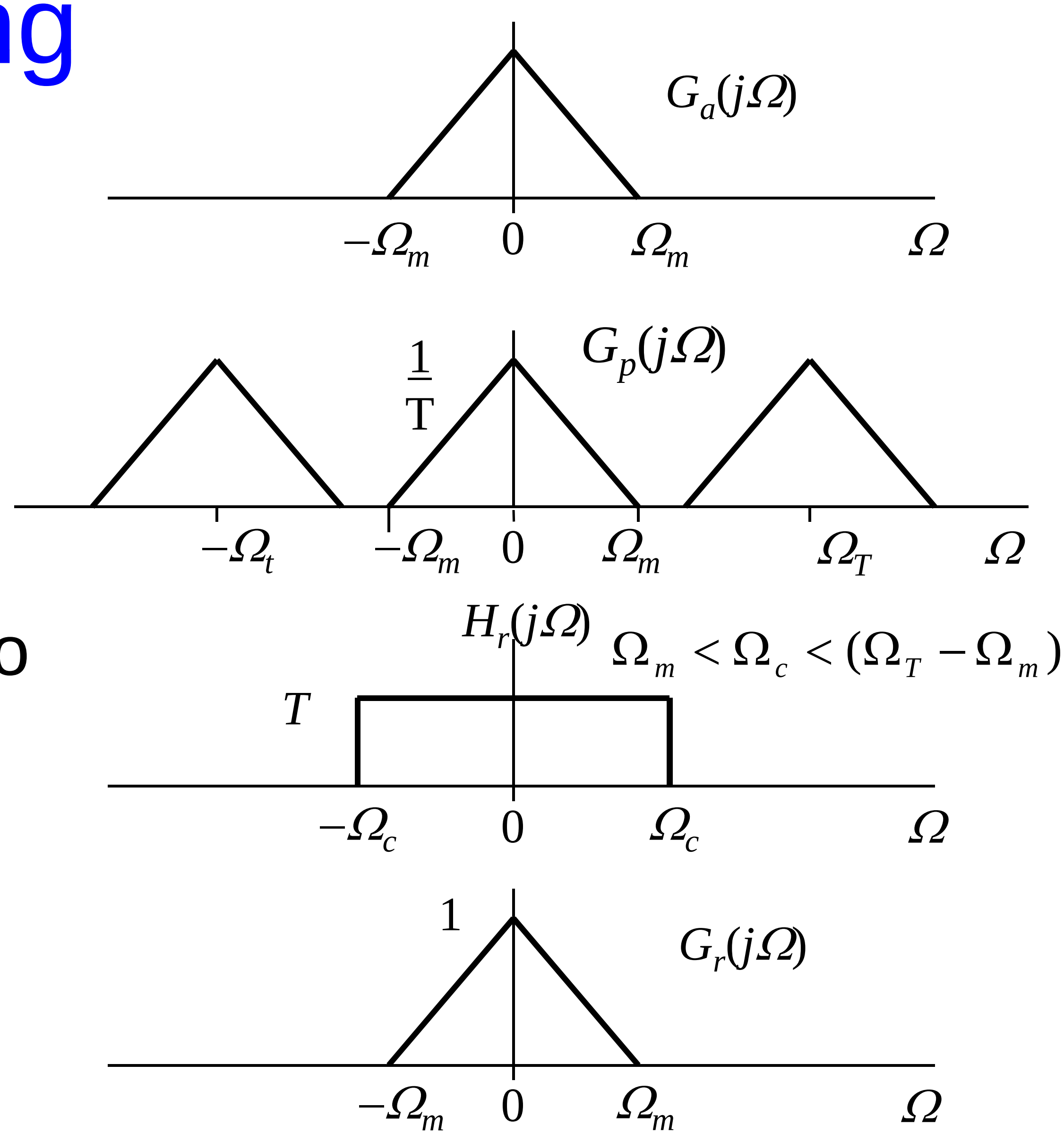
$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$



- Sampling process is modeled by multiplying the continuous-time signal $g_a(t)$ with a periodic impulse train $p(t)$
- The recovered signal $g_r(t)$ is obtained by lowpass filtering the sampled signal $g_p(t)$

Ideal Sampling

- Spectrum for $g_a(t)$
- Corresponding spectrum for $g_p(t)$
- Ideal lowpass filter to recover $H_r(j\Omega)$ from $G_p(j\Omega)$
- Spectrum of $g_r(t)$



Sampling Theorem

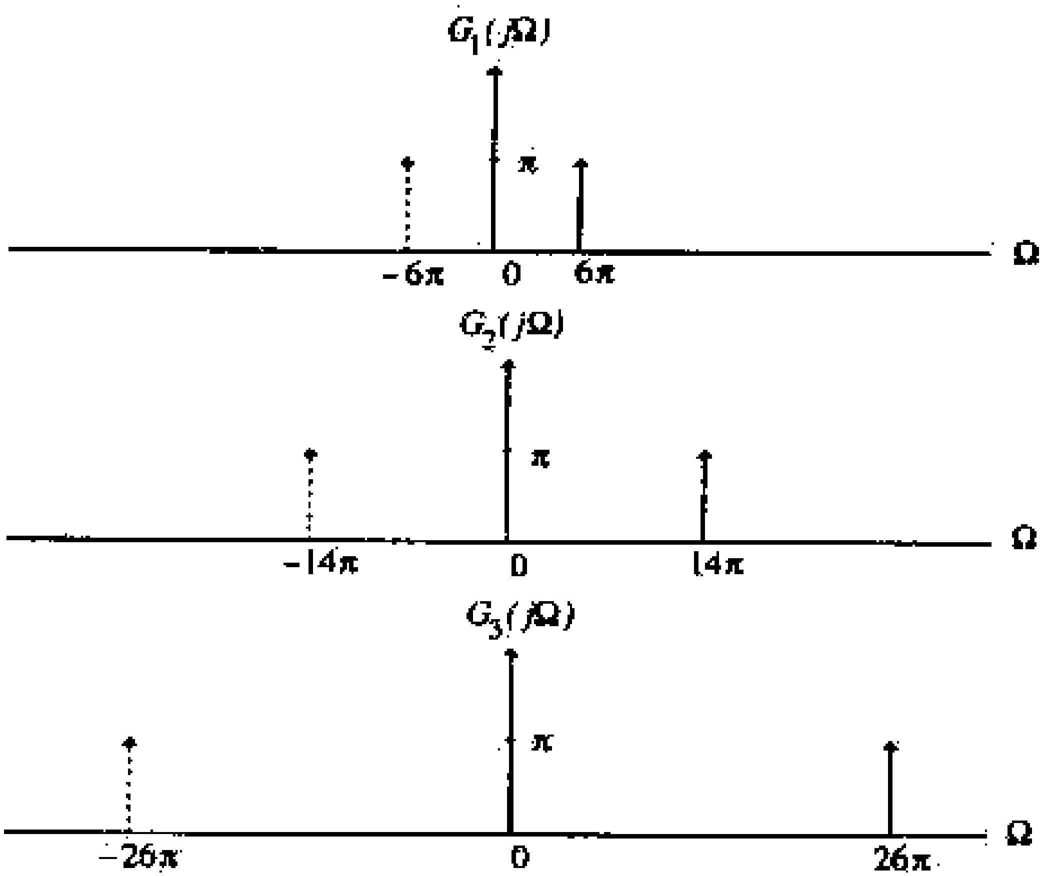
- If the sampling frequency at least twice as high as the highest frequency component of the bandlimited signal, i.e., $\Omega_T > 2\Omega_m$, then the original signal can be recovered from its samples
- If the above condition is not fulfilled, i.e., the frequency components above $\Omega_T/2$ will be ***aliased*** into the band of interest $|\Omega| < \Omega_m$

Sampling Theorem

- The highest frequency Ω_m contained in the signal is called the ***Nyquist frequency*** since it determines the minimum sampling frequency $\Omega_T = 2\Omega_m$, also called the ***Nyquist rate***
- The frequency $\Omega_T/2$ is referred to as the ***folding frequency***
- ***Critical sampling*** corresponds to $\Omega_T = 2\Omega_m$
- ***Undersampling*** corresponds to $\Omega_T < 2\Omega_m$
- ***Oversampling*** corresponds to $\Omega_T >> 2\Omega_m$

Example: Sampling on a Pure Cosine Signal

- Consider the three continuous-time sinusoidal signals



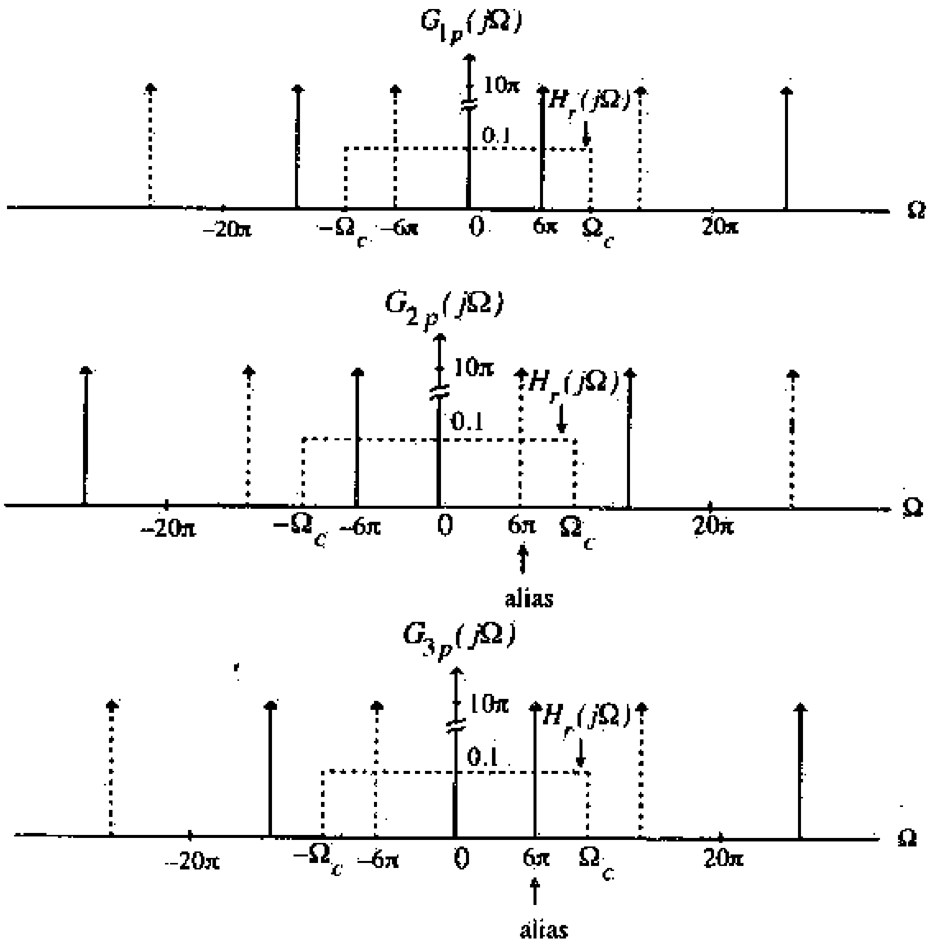
(a) Spectrum of $\cos(6\pi t)$

(b) Spectrum of $\cos(14\pi t)$

(c) Spectrum of $\cos(26\pi t)$

Example: Sampling on a Pure Cosine Signal

- The spectra of the sampled versions of the original cosine signals with the sampling frequency $\Omega_T = 20\pi$



(d) Spectrum of the sampled version of cos(6πt)

(e) Spectrum of the sampled version of cos(14πt)

(f) Spectrum of the sampled version of cos(26πt)

Recovery of the Analog Signal

- Ideal lowpass filter: $$H_r(j\Omega) = \begin{cases} T, & |\Omega| \le \Omega_c \\ 0, & |\Omega| > \Omega_c \end{cases}$$

$$h_r(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_r(j\Omega)e^{j\Omega t}d\Omega = \frac{T}{2\pi}\int_{-\Omega_c}^{\Omega_c} e^{j\Omega t}d\Omega$$

$$= \frac{\sin(\Omega_c t)}{\Omega_c t/2}, \quad -\infty < t < \infty$$

- Impulse train $g_p(t)$: $$g_p(t) = \sum_{n=-\infty}^{\infty} g_a(nT)\delta(t - nT)$$

- Output of the ideal lowpass filter is given by the convolution

Recovery of the Analog Signal

$$\hat{g}_a(t) = \sum_{n=-\infty}^{\infty} g[n] h_r(t - nT)$$

- Substituting $h_r(t)$ and assuming that $\Omega_c = \Omega_T/2 = \pi/T$

$$\hat{g}_a(t) = \sum_{n=-\infty}^{\infty} g[n] \frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T}$$

- $g_a(t)$ is obtained by shifting in time and scaling $h_r(t)$

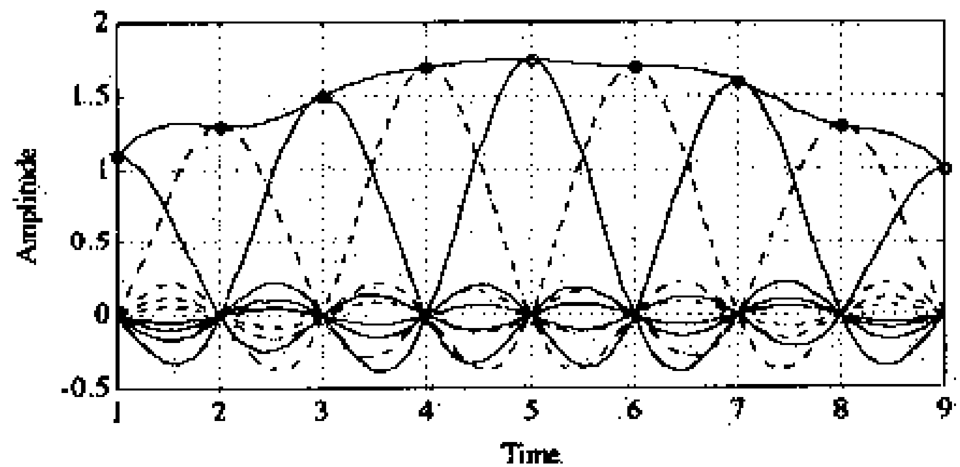


Illustration of the Sampling Process

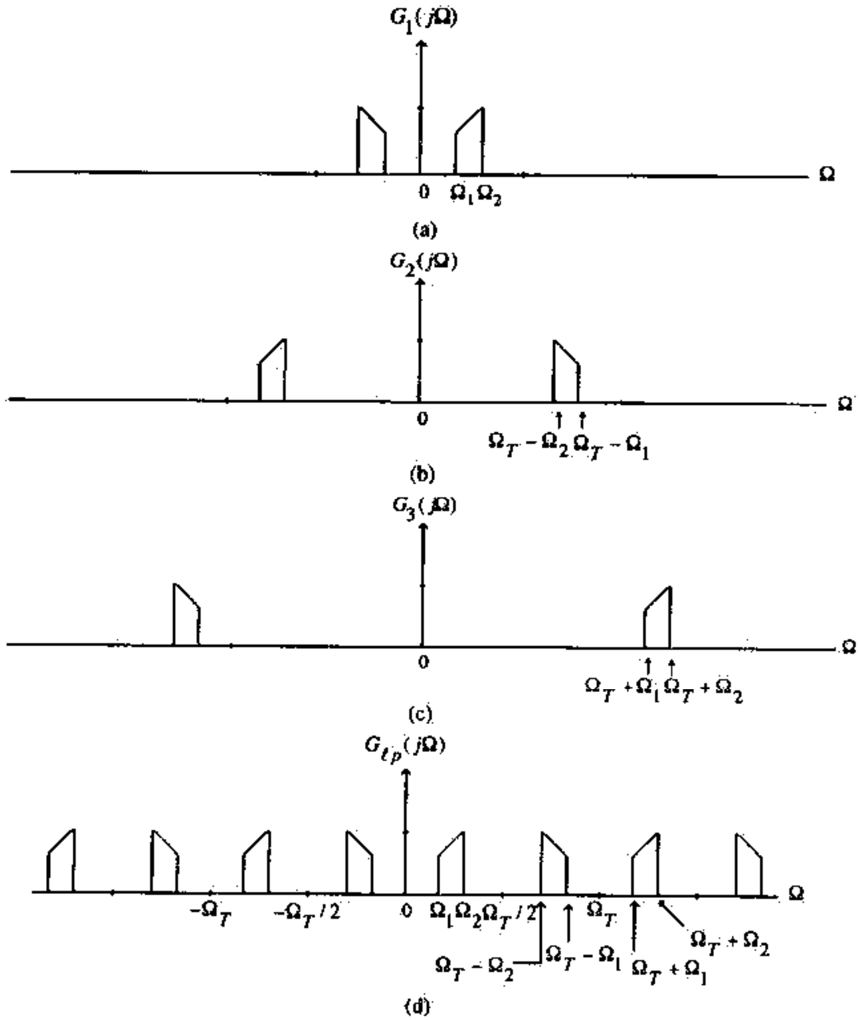
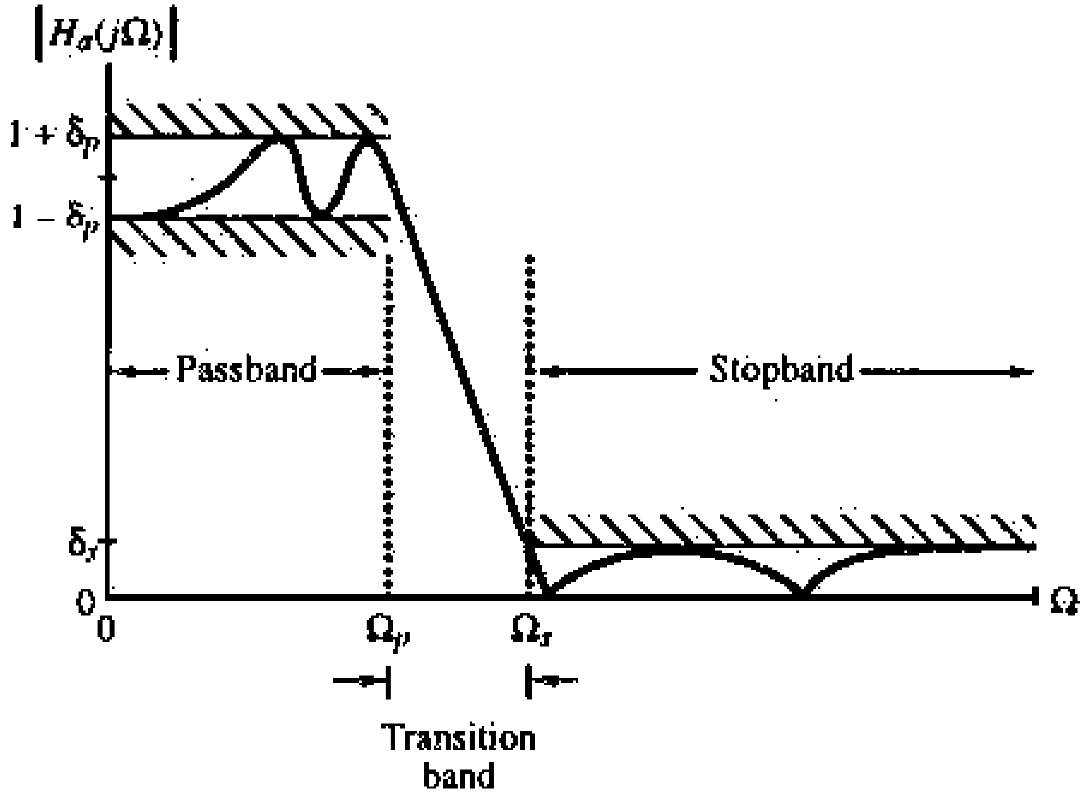


Figure 5.10 Further illustration of the effect of sampling.

- Three continuos-time signals with band-limited spectra
- Each of these signals is sampled at a sampling frequency of Ω_T
- The periodic frequency spectra of the sampled signals are identical

Analog Filter Design

- Magnitude response specifications for approximation of the ideal response



Analog Filter Specifications

- Passband: $1-\delta_p \le |H_a(j\Omega)| \le 1+\delta_p , \quad |\Omega| \le \Omega_p$

  Magnitude approximates unity within $\pm\delta_p$

- Stopband: $|H_a(j\Omega)| \le \delta_s , \quad \Omega_p \le |\Omega| < \infty$

  Magnitude approximates zero within $+\delta_s$

- Finite transition band between passband and stopband edge frequencies Ω_p and Ω_s
- The deviations, δ_p and δ_s , are called the ripples

Analog Filter Specifications

- The limits of the tolerances, δ_p and δ_s, i.e., the ripples can be defined in decibels
- The peak passband ripple α_p and the minimum stopband attenuation α_s, are defined as:

$$\alpha_p = -20\log_{10}(1-\delta_p)\ \text{dB}$$

$$\alpha_s = -20\log_{10}(\delta_s)\ \text{dB}$$

- The specifications can be given also as the loss or attenuation function $\alpha(j\Omega)$ in dB which is defined as the negative of the gain in dB, i.e.,

$$\alpha(j\Omega) = -20\log_{10}\left|H_a(j\Omega)\right|\ \text{dB}$$

Normalized Magnitude Specifications

- The maximum value of the magnitude is assumed to be unity

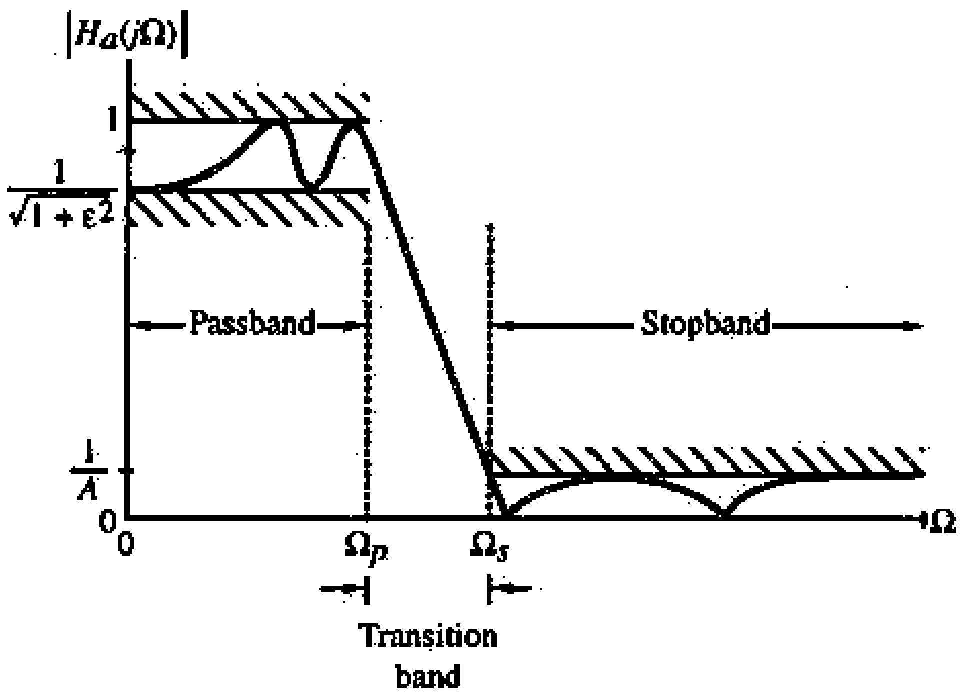


Figure 5.12 Normalized magnitude specifications for an analog lowpass filter.

Classical Filter Designs

- The classical filter designs
  - Butterworth,
  - Chebyshev, and
  - Elliptic

  satisfy the magnitude constraints of analog filters
- These approximation methods can be expressed using the closed form formulas
  - Extensive tables are available for analog filter design
  - The closed form formulas can be easily solved

Butterworth Approximation

- The magnitude response is required to be ***maximally flat*** in the passband
- For the lowpass filter, the first $2N$-1 derivatives of $|H(j\Omega)|^2$ are specified to equal to zero at $\Omega=0$
- The squared-magnitude response of an analog lowpass Butterworth filter is

$$|H_a(j\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}}$$

- The gain is: $\mathcal{G}(\Omega) = 10\log_{10}|H_a(j\Omega)|^2$ dB

Butterworth Approximation

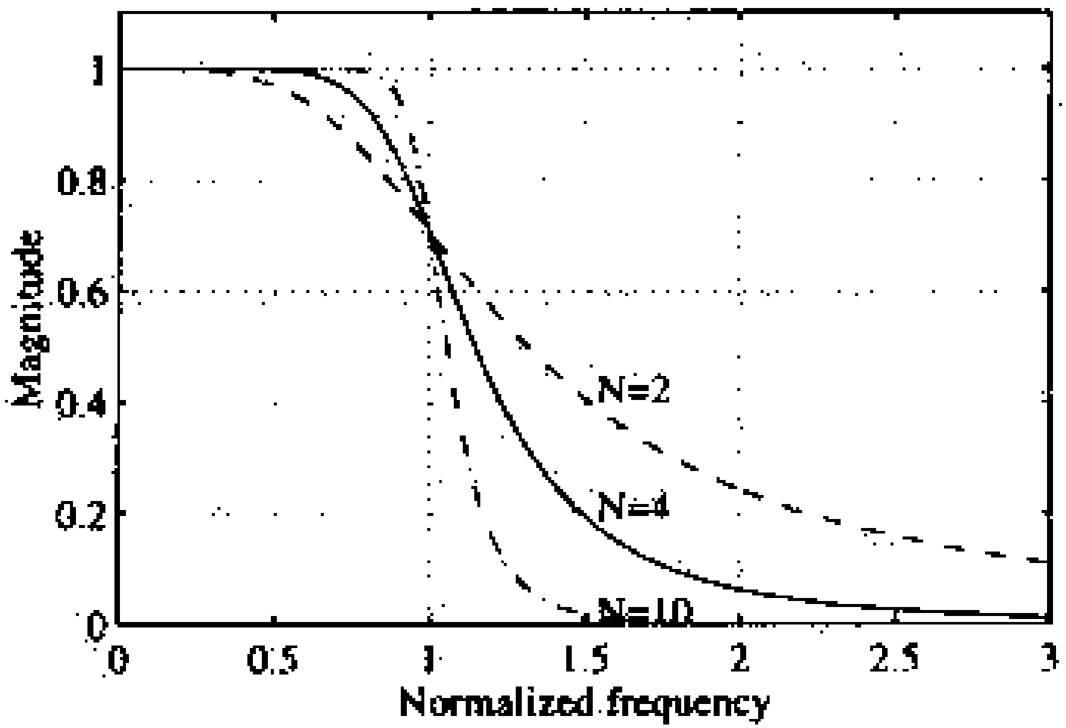
- Note that $\left|H_a(0)\right| = 1;$ and $\left|H_a(j\Omega_c)\right| = \frac{1}{\sqrt{2}}$
- At dc, i.e. at Ω=0, the gain in dB is equal to zero and at Ω=Ω_c, the gain is

$$\mathcal{G}(\Omega_c) = 10\log_{10}(½) = -3.0103 \cong -3\,\mathrm{dB}$$

- Therefore, Ω_c is called the ***3-dB cutoff frequency***
- Since the derivative of the squared-magnitude response is always negative for positive values of Ω, the magnitude response is monotonically decreasing with increasing Ω

Butterworth Approximation

- Magnitude response of the normalized Butterworth lowpass filter with Ω_c=1



Butterworth Approximation

- The system function of the Butterworth filter is

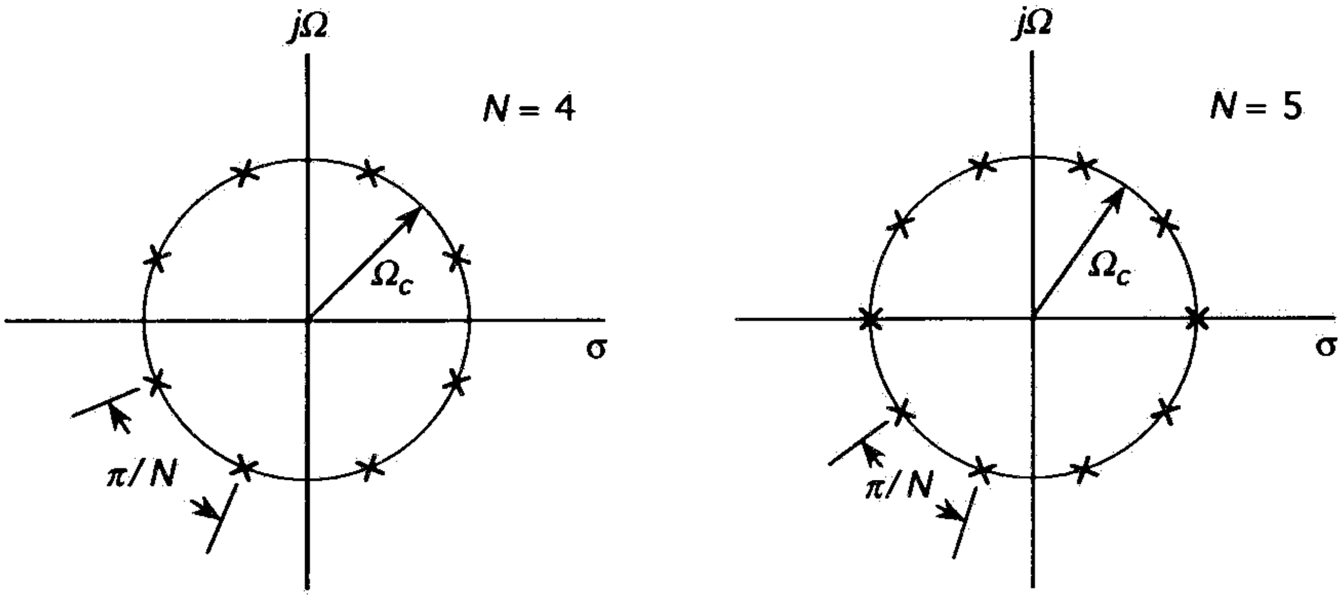
$$H_a(s)H_a(-s) = \frac{1}{1+(s/j\Omega_c)^{2N}}$$

and the poles of $H_a(s)H_a(-s)$ are

$$s_k = (-1)^{1/2N}(j\Omega_c)$$

- These $2N$ poles are uniformly distributed on circle of radius Ω_c in the s-plane and are symmetrically located with respect to both the real and imaginary axes

Butterworth Approximation



- The poles from left half s-plane are selected to the stable transfer function, an ***all-pole transfer function***

$$H(s) = \frac{1}{B_n(s)}$$

Chebyshev Approximation

- More rapid rolloff rate near the cutoff frequency than that of the Butterworth design can be achieved at the expense of a loss of monotonicity in the passband and/or the stopband
- The Chebyshev designs maintain monotonicity in one band but are equiripple in the other band

<u>Chebyshev Type I (normal Chebyshev):</u>

  - All-pole transfer function, i.e., all zeros at infinity

<u>Chebyshev Type II (inverse Chebyshev):</u>

  - Rational transfer function having zeros at finite frequencies

Chebyshev I Approximation

- The squared magnitude response for an analog Chebyshev I design is of the form

$$|H_a(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 T_N^2(\Omega/\Omega_p)}$$

where $T_N(\Omega)$ is the N^{th} order Chebyshev polynomial

$$T_N(\Omega) = \begin{cases} \cos(N\cos^{-1}\Omega), & |\Omega| \le 1 \\ \cosh(N\cosh^{-1}\Omega), & |\Omega| > 1 \end{cases}$$

- The recurrence relation for Chebyshev polynomials

$$T_r(\Omega) = 2\Omega T_{r-1}(\Omega) - T_{r-2}(\Omega)$$

with $T_0(\Omega)=1$ and $T_1(\Omega)=\Omega$

Chebyshev I Approximation

$$|H_a(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 T_N^2(\Omega/\Omega_p)}$$

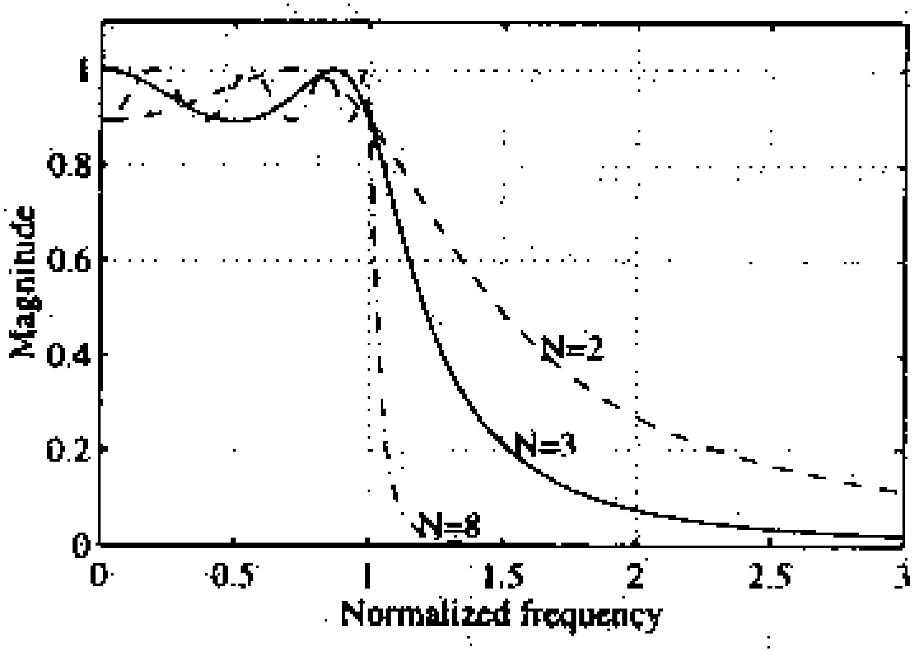
- In the passband, $\Omega \leq \Omega_p$, $T_N(\Omega)=\cos(N\cos^{-1}\Omega)$ varies between -1 and 1 and its square between 0 and 1
- Thus, $|H_a(j\Omega)|^2$ has equal ripple behaviour in the passband between 1 and $(1-\delta_1)^2$
- The deviation is determined by the ripple factor ε

$$(1-\delta_1)^2 = \frac{1}{1+\varepsilon^2} \quad \Rightarrow \quad \varepsilon^2 = \frac{1}{(1-\delta_1)^2} - 1$$

- The transfer function is an all-pole function in the s-plane

Chebyshev I Approximation

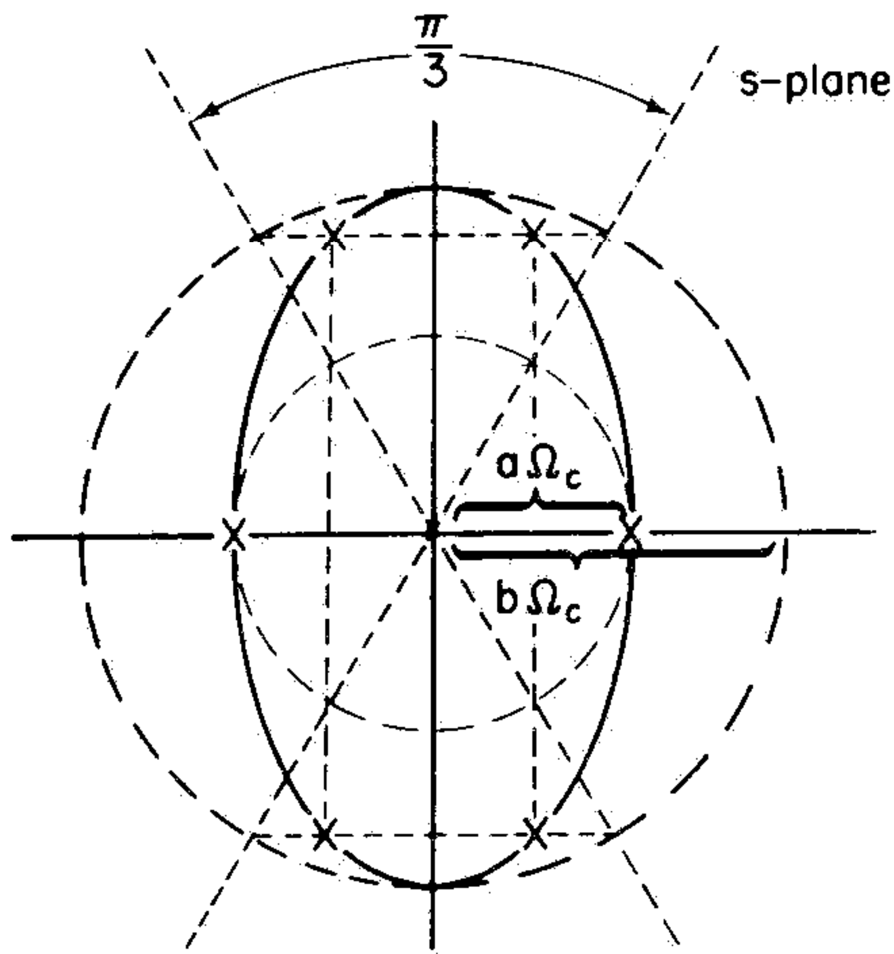
- The squared magnitude response of a lowpass Chebyshev I filter for different values of N
- The behavior is determined by the cutoff frequency Ω_p, the passband ripple factor ε, and the order N
- For the stopband specifications δ_2 and Ω_s the order N can be determined from:



$$N \approx \frac{\cosh^{-1}(1/\delta_2\varepsilon)}{\cosh^{-1}(\Omega_s/\Omega_p)}$$

Chebyshev I Approximation

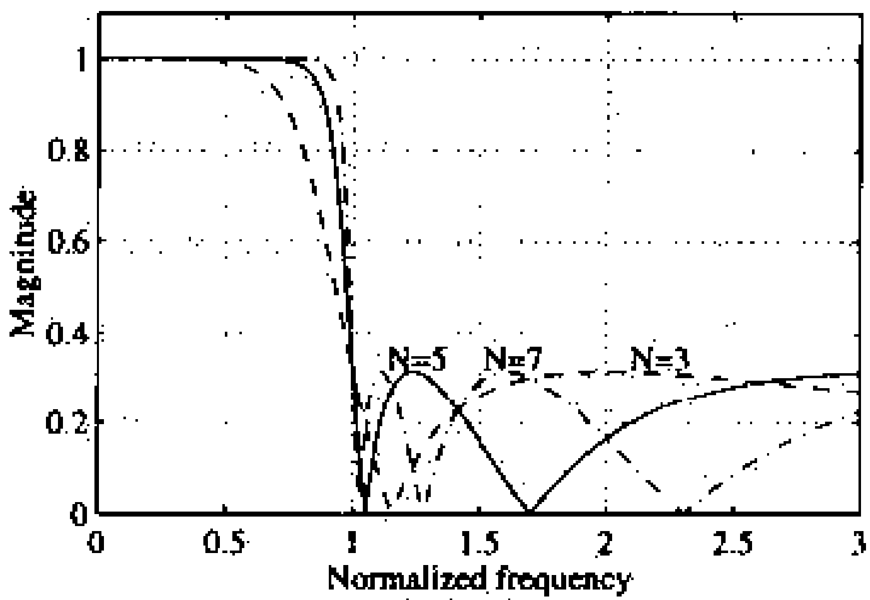
- The poles of the Chebyshev I filter lie on an ellipse in the s-plane
- The equiripple behavior in the passband can be explained by considering the locations of the poles (and comparing them to those of the Butterworth filter)



Chebyshev II Approximation

- The squared magnitude response is of the form

$$|H_a(j\Omega)|^2 = \frac{1}{1+\varepsilon^2\left[\dfrac{T_N^2(\Omega_s/\Omega_p)}{T_N^2(\Omega_s/\Omega)}\right]^2}$$



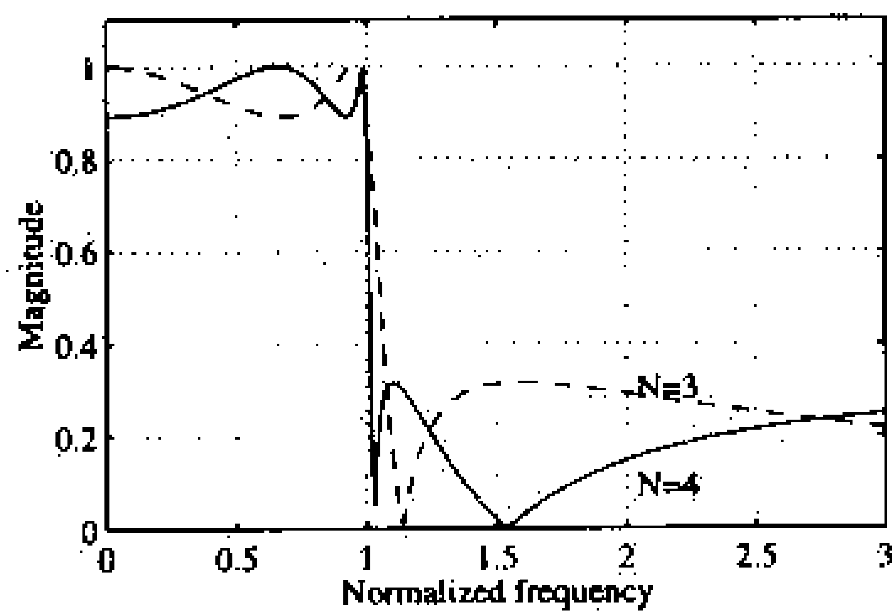
- The transfer function has equal ripple behavior in the stopband due to zeros at finite frequencies, i.e., it is not an all-pole transfer function

Elliptic Approximation

- The squared magnitude response is of the form

$$\left|H_a(j\Omega)\right|^2 = \frac{1}{1+\varepsilon^2 R_N^2(\Omega/\Omega_p)}$$

where $R_N(\Omega)$ is a rational function with $R_N(1/\Omega) = 1/R_N(\Omega)$

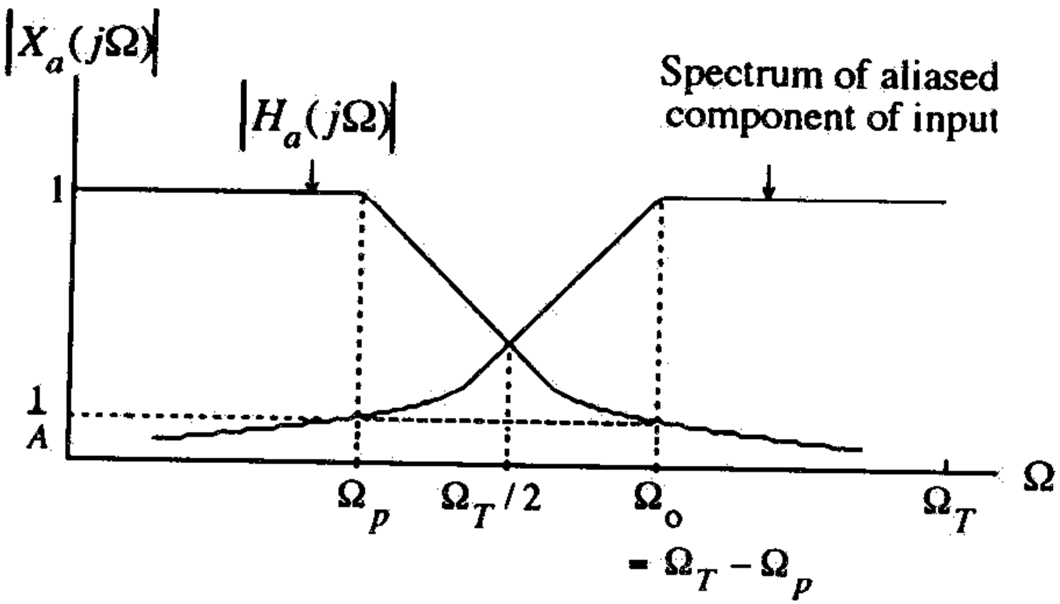


- The transfer function has equal ripple behavior both in the passband and in the stopband
- Elliptic approximation has the narrowest transition band

Anti-Aliasing Filter Design

- Ideally, the anti-aliasing filter $H_a(s)$ should have a lowpass frequency response $H_a(j\Omega)$ given by

$$H_a(j\Omega) = \begin{cases} 1, & |\Omega| < \Omega_T/2 \\ 0, & |\Omega| \geq \Omega_T/2 \end{cases}$$



- In practice, it is necessary to filter out those frequencies that will be aliased to the band of interest

Reconstruction Filter Design

- Reconstruction or smoothing filter is used to eliminate all the replicas of the spectrum outside the baseband
- If the cutoff frequency Ω_c of the reconstruction filter is chosen as $\Omega_T/2$, where Ω_T is the sampling frequency, the corresponding frequency response is given by

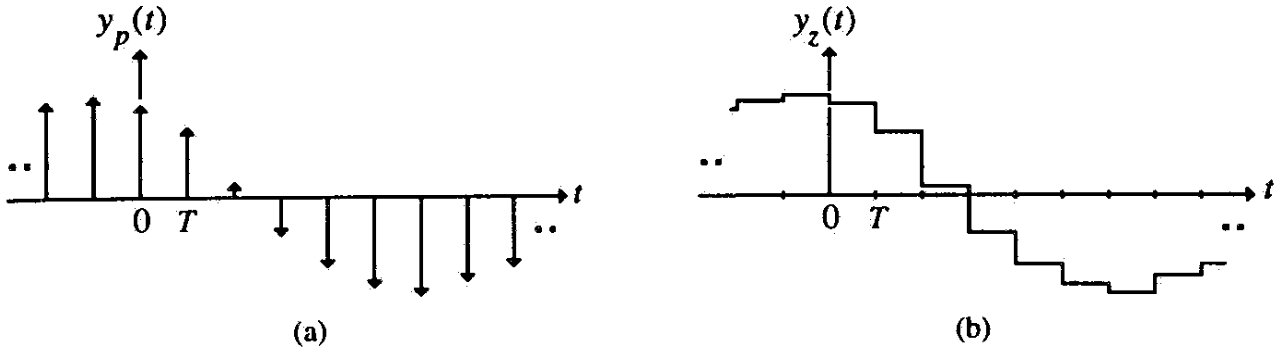
$$H_r(j\Omega) = \begin{cases} T, & |\Omega| \le \Omega_T/2 \\ 0, & |\Omega| > \Omega_T/2 \end{cases}$$

- The reconstruction filter is not causal!
- The reconstructed analog signal is

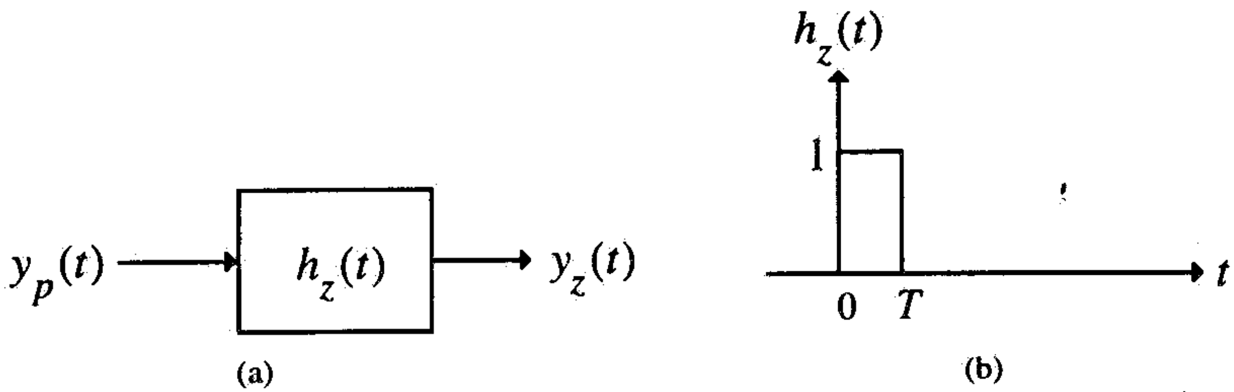
$$y_a(t) = \sum_{n=-\infty}^{\infty} y[n] \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

Zero-Order Hold

- The analog signal is approximated by the staircase-like waveform



- The zero-order hold circuit has the impulse response $h_z(t)$



Zero-Order Hold

- Fourier transform of the output of the zero-order hold is

$$Y_z(j\Omega) = H_z(j\Omega)Y_p(j\Omega)$$

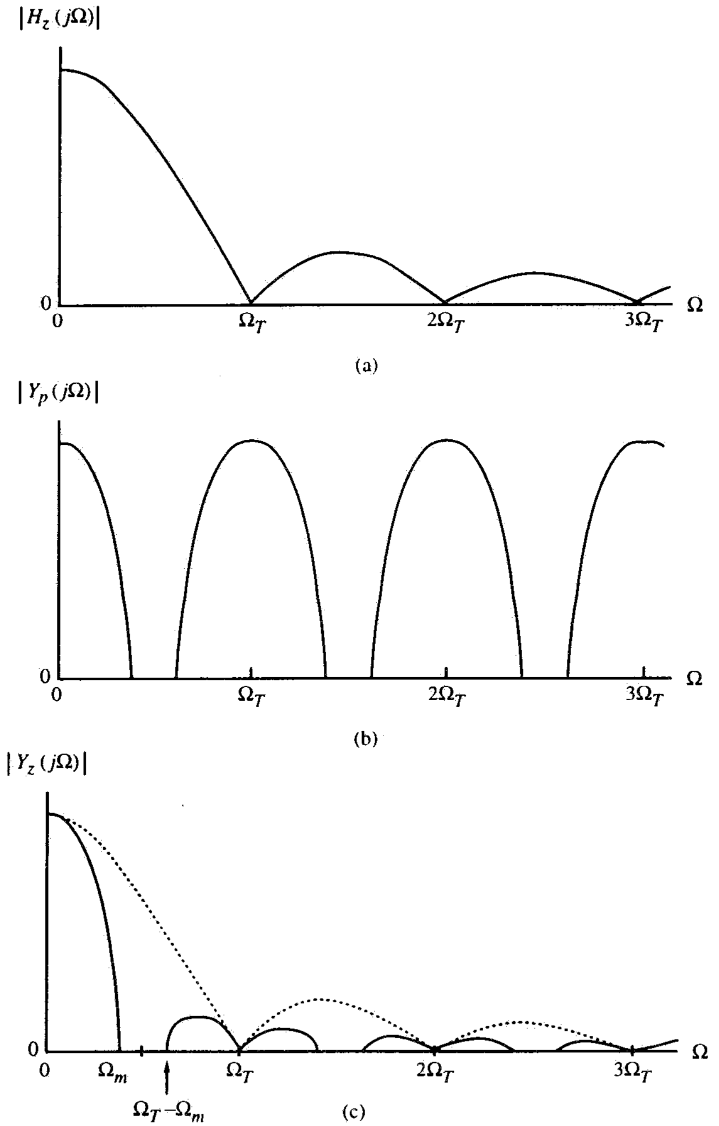
where

$$H_z(j\Omega) = \int_0^T e^{-j\Omega t}dt = -\frac{e^{-j\Omega t}}{j\Omega}\Bigg|_0^T = \frac{1-e^{-j\Omega t}}{j\Omega}$$

$$= e^{-j\frac{\Omega t}{2}}\left[\frac{\sin(\Omega T/2)}{\Omega/2}\right]$$

- The magnitude response of the zero-order hold has a lowpass characteristic with zeros at $\pm\Omega_T$, $\pm 2\Omega_T$,..., where Ω_T=1/T
- The zero-order hold somewhat attenuates the unwanted replicas of the periodic digital signal at multiples of Ω_T

Zero-Order Hold



- The zero-order hold circuit also distorts the magnitude in the band of interest (close to Ω_m)
  a) Zero-order hold
  b) Output of the ideal D/A converter
  c) Output of the practical D/A converter

Zero-Order Hold

- The distortion of the zero-order hold can be compensated, e.g., digitally prior to D/A converter

- FIR filter:

$$H_{FIR}(z) = -\frac{1}{16} + \frac{9}{8}z^{-1} - \frac{1}{16}z^{-2}$$

- IIR filter:

$$H_{IIR}(z) = \frac{9}{8 + z^{-1}}$$