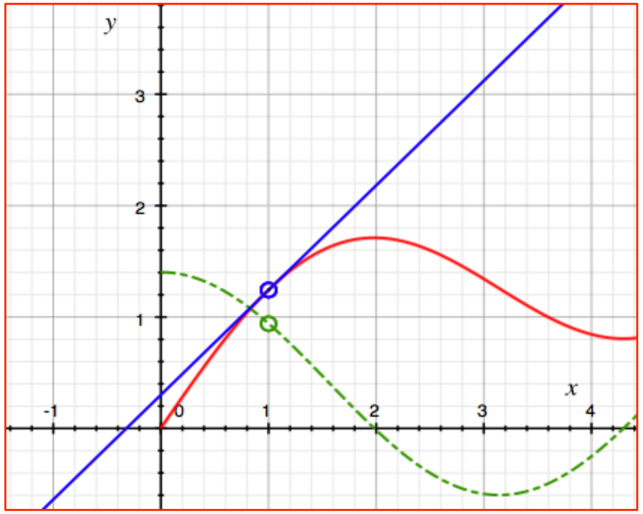
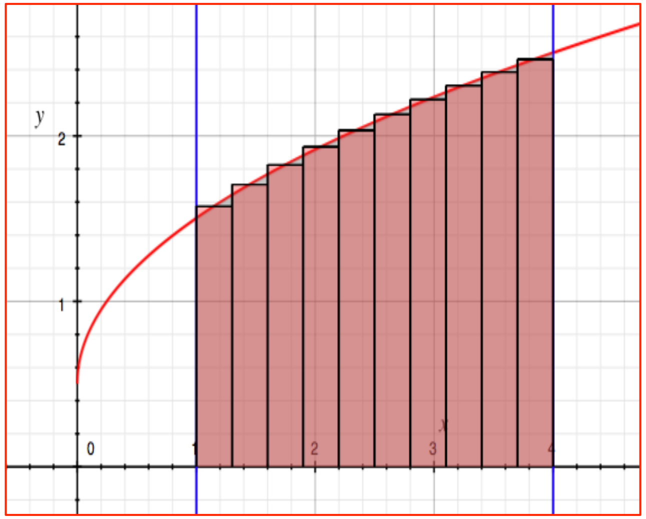
FYSIIKAN MATEMAATTISET MENETELMÄT

FYSP111, M1: Derivointi ja integrointi

Luentomateriaali, k. 2012

Markku Kataja

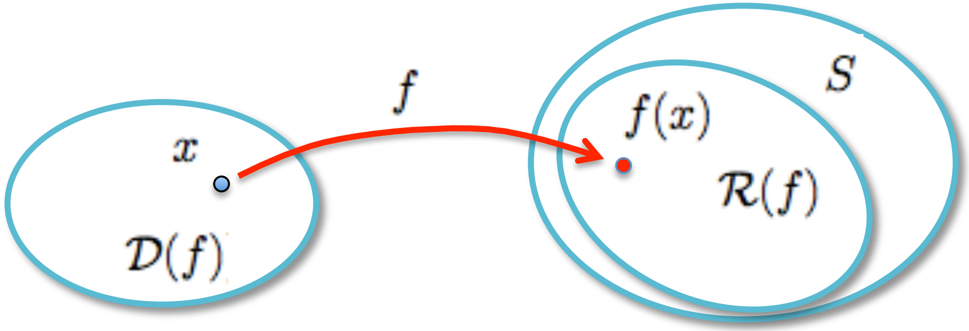


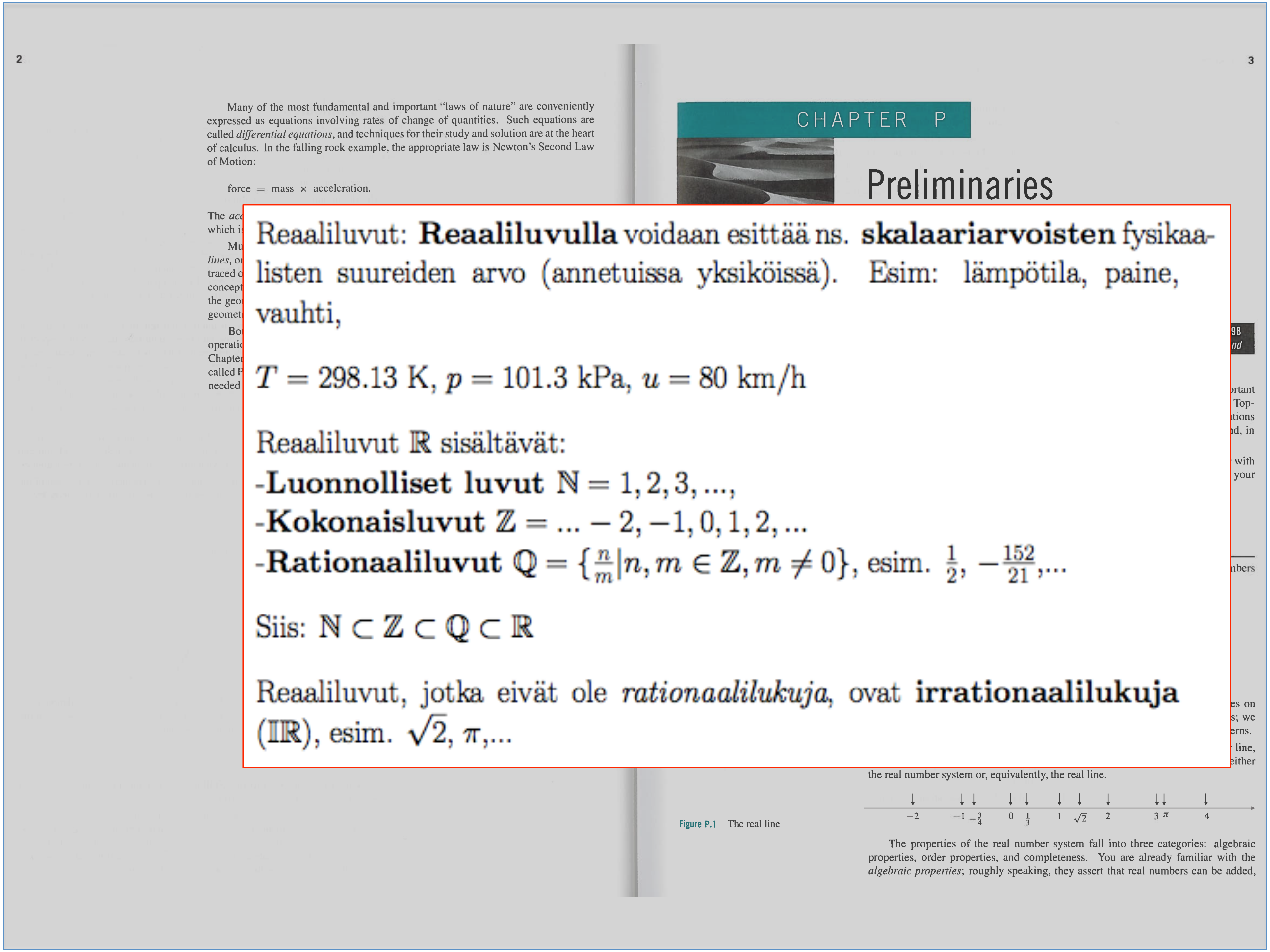


1: JOHDANTO

Reaaliluvut, koordinaatistot, yhtälöt, funktiot

Kurssikirjan luku P. 'Preliminaries', osin luku 3





Many of the most fundamental and important "laws of nature" are conveniently expressed as equations involving rates of change of quantities. Such equations are called *differential equations*, and techniques for their study and solution are at the heart of calculus. In the falling rock example, the appropriate law is Newton's Second Law of Motion:

force = mass × acceleration.

CHAPTER P

Preliminaries

the real number system or, equivalently, the real line.

Figure P.1 The real line

The properties of the real number system fall into three categories: algebraic properties, order properties, and completeness. You are already familiar with the *algebraic properties*; roughly speaking, they assert that real numbers can be added,

Reaaliluvut: **Reaaliluvulla** voidaan esittää ns. **skalaariarvoisten** fysikaalisten suureiden arvo (annetuissa yksiköissä). Esim: lämpötila, paine, vauhti,

$T = 298.13$ K, $p = 101.3$ kPa, $u = 80$ km/h

Reaaliluvut $\mathbb{R}$ sisältävät:
-**Luonnolliset luvut** $\mathbb{N} = 1, 2, 3, ...,$
-**Kokonaisluvut** $\mathbb{Z} = ... -2, -1, 0, 1, 2, ...$
-**Rationaaliluvut** $\mathbb{Q} = \{\frac{n}{m} | n, m \in \mathbb{Z}, m \neq 0\}$, esim. $\frac{1}{2}, -\frac{152}{21}, ...$

Siis: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$

Reaaliluvut, jotka eivät ole *rationaalilukuja*, ovat **irrationaalilukuja** ($\mathbb{IR}$), esim. $\sqrt{2}, \pi, ...$

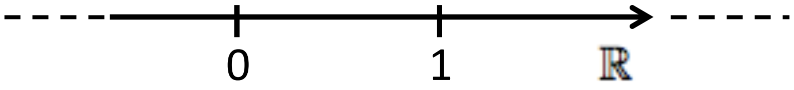
The symbol ⟹ means “implies.”

Reaalilukujoukkoon liittyviä määrittelyjä ja aksioomia

-ℝ:ssä on määritelty aritmeettiset binäärioperaatiot ”+” ja ”·”, jotka toteuttavat 'normaalit' laskulait (vaihdanta- liitäntä- ja osittelulait).
-ℝ:ssä on 0 -alkio ja 1 -alkio, jotka ovat + ja · -operaatioiden neutraalialkiot: $x + 0 = x$, $1 \cdot x = x$.
-ℝ:ssä on määritelty unaarioperaatiot ”$-$” (vastaluku) ja ”$\cdot^{-1}$” (käänteisluku): $x + (-x) = 0$ ja $x \cdot x^{-1} = 1$ $(x \neq 0)$
-ℝ on järjestetty joukko, ts. relaatiot $>$, $<$, $\leq$ ja $\geq$ on hyvin määritelty.
-ℝ toteuttaa *täydellisyysaksiooman*, jonka mukaan jokaisella ℝ:n ei-tyhjällä ylhäältä rajoitetulla osajoukolla on ns. *pienin yläraja* eli *supremum*.

Täydellisyysaksioomasta seuraa, että ℝ:ssä ei ole 'aukkoja' (kuten ℕ:ssa, ℤ:ssa ja ℚ:ssa). Jokaisen suppenevan reaalilukujonon raja-arvo on myös reaaliluku. Aritmeettisena joukkona ℝ muodostaa järjestetyn *lukukunnan*.

Graafisesti ℝ:aa kuvaa reaalilukusuora eli *reaaliakseli*:



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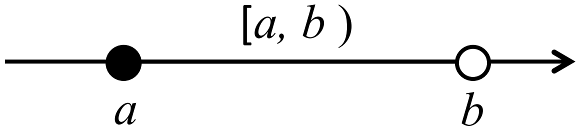
$\sqrt{2}$ is rational.
fraction m/n
$m^2/n^2 = 2$,
n odd integer is
$^2 = 2n^2$ and
t $\sqrt{2}$ could be
there can be no

Reaaliakselin **väli** on $\mathbb{R}$:n osajoukko joka:
-sisältää vähintään kaksi pistettä (reaalilukua)
-jos luvut a ja b kuuluvat ko. osajoukkoon ja $a < b$, niin jokainen luku x, jolle $a < x < b$ kuuluu siihen.

Välin päätepiste voi kuulua ko. väliin tai olla sen ulkopuolella. Puhutaan avoimesta, suljetusta ja puoliavoimesta välistä. Merk: suljettu väli $[a, b]$, avoin väli (a, b) ja puoliavoimet välit $[a, b)$ (vasemmalta suljettu, oikelta avoin väli) ja $(b, a]$ (vasemmalta avoin, oikealta suljettu väli). Huom: muunkinlaisia merkintätapoja käytetään.



subtracted, multiplied, and divided (except by zero) to produce more real numbers and that the usual rules of arithmetic are valid.

The *order properties* of the real numbers refer to the order in which the numbers appear on the real line. If x lies to the left of y, then we say that "x is less than y" or "y is greater than x." These statements are written symbolically as $x < y$ and $y > x$, respectively. The inequality $x \le y$ means that either $x < y$ or $x = y$. The order properties of the real numbers are summarized in the following *rules for inequalities*:

The symbol $\Longrightarrow$ means "implies."

(a) terminating, that is, ending with an infinite string of zeros, for example, $3/4 = 0.750000\ldots$, or
(b) repeating, that is, ending with a string of digits that repeats over and over, for example, $23/11 = 2.090909\ldots = 2.\overline{09}$. (The bar indicates the pattern of repeating digits.)

Real numbers that are not rational are called *irrational numbers*.

EXAMPLE 1 Show that each of the numbers (a) $1.323232\cdots = 1.\overline{32}$ and (b) $0.3405405405\ldots = 0.3\overline{405}$ is a rational number by expressing it as a quotient of two integers.

Solution

Figure P.3 Infinite intervals

is an interval, denoted by $(-\infty, \infty)$. The symbol ∞ ("infinity") does *not* denote a real number, so we never allow ∞ to belong to an interval.

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[1] How do we know that $\sqrt{2}$ is an irrational number? Suppose, to the contrary, that $\sqrt{2}$ is rational. Then $\sqrt{2} = m/n$, where m and n are integers and $n \ne 0$. We can assume that the fraction m/n has been "reduced to lowest terms"; any common factors have been cancelled out. Now $m^2/n^2 = 2$, so $m^2 = 2n^2$, which is an even integer. Hence m must also be even. (The square of an odd integer is always odd.) Since m is even, we can write $m = 2k$, where k is an integer. Thus $4k^2 = 2n^2$ and $n^2 = 2k^2$, which is even. Thus n is also even. This contradicts the assumption that $\sqrt{2}$ could be written as a fraction m/n in lowest terms; m and n cannot both be even. Accordingly, there can be no rational number whose square is 2.

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Rules for inequalities

If a, b, and c are real numbers, then:

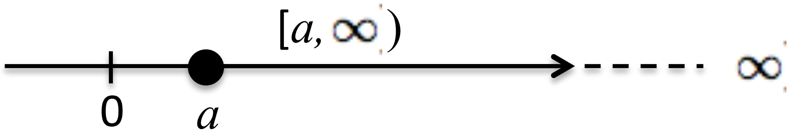
1. $a < b \quad \Longrightarrow \quad a + c < b + c$

Ääretön (merk. ∞):

Merkintä $x \to \infty$ tarkoittaa, että luku x kasvaa rajatta, ts. sitä voidaan pitää suurempana kuin mikä tahansa annettu reaaliluku. Samoin $x \to -\infty$ tarkoittaa, että luku x vähenee rajatta, ts. sitä voidaan pitää pienempänä kuin mikä tahansa annettu reaaliluku.

Huom: vaikka ∞ ei itse ole reaaliluku, merkitään usein esim. $[a, \infty)$ tarkoittaen (puoli)ääretöntä väliä $\{x | x \in \mathbb{R}, x \ge a\}$.

Väli $(-\infty, \infty) = \mathbb{R}$.



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Solution

(a) Let $x = 1.323232\ldots$ Then $x - 1 = 0.323232\ldots$ and

$$100x = 132.323232\ldots = 132 + 0.323232\ldots = 132 + x - 1.$$

Therefore, $99x = 131$ and $x = 131/99$.

(b) Let $y = 0.3405405405\ldots$ Then $10y = 3.405405405\ldots$ and

---

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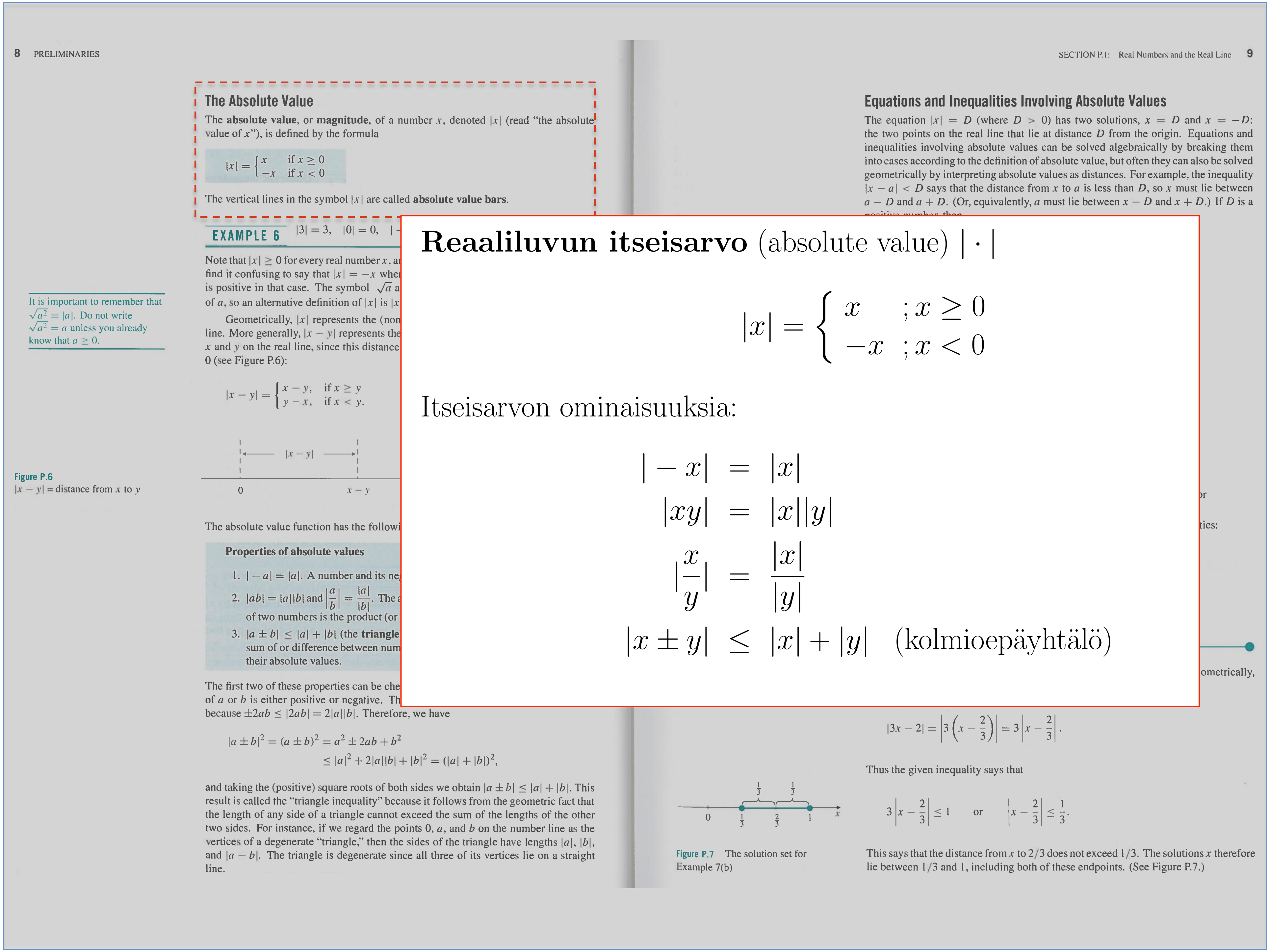


8 PRELIMINARIES

The Absolute Value

The **absolute value**, or **magnitude**, of a number x, denoted $|x|$ (read "the absolute value of x"), is defined by the formula

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The vertical lines in the symbol $|x|$ are called **absolute value bars**.

EXAMPLE 6 $|3| = 3, \quad |0| = 0, \quad |$

Note that $|x| \geq 0$ for every real number x, ar
find it confusing to say that $|x| = -x$ whe
is positive in that case. The symbol $\sqrt{a}$ a
of a, so an alternative definition of $|x|$ is $|x$

It is important to remember that $\sqrt{a^2} = |a|$. Do not write $\sqrt{a^2} = a$ unless you already know that $a \geq 0$.

Geometrically, $|x|$ represents the (non
line. More generally, $|x - y|$ represents the
x and y on the real line, since this distance
0 (see Figure P.6):

$$|x - y| = \begin{cases} x - y, & \text{if } x \geq y \\ y - x, & \text{if } x < y. \end{cases}$$

Figure P.6
$|x - y|$ = distance from x to y

The absolute value function has the followi

Properties of absolute values

1. $|-a| = |a|$. A number and its ne
2. $|ab| = |a||b|$ and $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$. The a
of two numbers is the product (or
3. $|a \pm b| \leq |a| + |b|$ (the **triangle**
sum of or difference between num
their absolute values.

The first two of these properties can be che
of a or b is either positive or negative. Th
because $\pm 2ab \leq |2ab| = 2|a||b|$. Therefore, we have

$$\begin{aligned} |a \pm b|^2 &= (a \pm b)^2 = a^2 \pm 2ab + b^2 \\ &\leq |a|^2 + 2|a||b| + |b|^2 = (|a| + |b|)^2, \end{aligned}$$

and taking the (positive) square roots of both sides we obtain $|a \pm b| \leq |a| + |b|$. This result is called the "triangle inequality" because it follows from the geometric fact that the length of any side of a triangle cannot exceed the sum of the lengths of the other two sides. For instance, if we regard the points 0, a, and b on the number line as the vertices of a degenerate "triangle," then the sides of the triangle have lengths $|a|$, $|b|$, and $|a - b|$. The triangle is degenerate since all three of its vertices lie on a straight line.

SECTION P.1: Real Numbers and the Real Line 9

Equations and Inequalities Involving Absolute Values

The equation $|x| = D$ (where $D > 0$) has two solutions, $x = D$ and $x = -D$: the two points on the real line that lie at distance D from the origin. Equations and inequalities involving absolute values can be solved algebraically by breaking them into cases according to the definition of absolute value, but often they can also be solved geometrically by interpreting absolute values as distances. For example, the inequality $|x - a| < D$ says that the distance from x to a is less than D, so x must lie between $a - D$ and $a + D$. (Or, equivalently, a must lie between $x - D$ and $x + D$.) If D is a

or

ties:

ometrically,

$$|3x - 2| = \left|3\left(x - \frac{2}{3}\right)\right| = 3\left|x - \frac{2}{3}\right|.$$

Thus the given inequality says that

$$3\left|x - \frac{2}{3}\right| \leq 1 \quad \text{or} \quad \left|x - \frac{2}{3}\right| \leq \frac{1}{3}.$$

Figure P.7 The solution set for Example 7(b)

This says that the distance from x to $2/3$ does not exceed $1/3$. The solutions x therefore lie between $1/3$ and 1, including both of these endpoints. (See Figure P.7.)

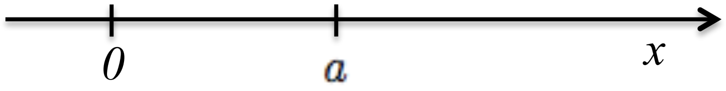
Reaaliluvun itseisarvo (absolute value) $|\cdot|$

$$|x| = \begin{cases} x & ; x \geq 0 \\ -x & ; x < 0 \end{cases}$$

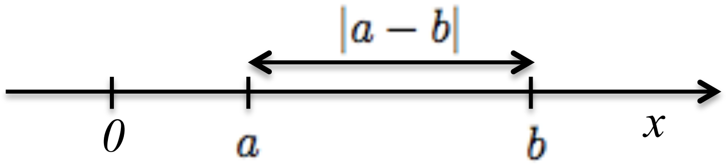
Itseisarvon ominaisuuksia:

$$\begin{aligned} |-x| &= |x| \\ |xy| &= |x||y| \\ \left|\frac{x}{y}\right| &= \frac{|x|}{|y|} \\ |x \pm y| &\leq |x| + |y| \quad \text{(kolmioepäyhtälö)} \end{aligned}$$

Reaalilukuakselia voidaan pitää yksiulotteisen (viivamaisen) avaruuden koordinaatistona, kun esim. 0 -alkion paikka l. *origo* ja akselin skaalaus (mittakaava) on kiinnitetty. Ko. avaruuden mv. pisteen paikka voidaan tällöin ilmoittaa yhdellä ainoalla luvulla, joka ilmoittaa pisteen etäisyyden origosta.



Kahden pisteen a ja b välinen etäisyys on $|a-b|$.



EXAMPLE 8 Solve the equation $|x+1| = |x-3|$.

Solution The equation says that x is equidistant from -1 and 3. Therefore, x is the point halfway between -1 and 3; $x = (-1+3)/2 = 1$. Alternatively, the given

EXERCISES P.1

In Exercises 1–2, express the given rational nun
repeating decimal. Use a bar to indicate the rep

1. $\frac{2}{9}$
2. $\frac{1}{11}$

In Exercises 3–4, express the given repeating de
quotient of integers in lowest terms.

3. $0.\overline{12}$
4. $3.2\overline{7}$
5. Express the rational numbers $1/7, 2/7, 3/7$
repeating decimals. (Use a calculator to giv
decimal digits as possible.) Do you see a pa
decimal expansions of $5/7$ and $6/7$ and che
6. Can two different decimals represent the sa
What number is represented by $0.999\ldots =$

In Exercises 7–12, express the set of all real nun
the given conditions as an interval or a union of

7. $x \geq 0$ and $x \leq 5$
8. $x < 2$ an
9. $x > -5$ or $x < -6$
10. $x \leq -1$
11. $x > -2$
12. $x < 4$ or $x \geq 2$

In Exercises 13–26, solve the given inequality, giving the solution set as an interval or union of intervals.

31. $|8-3s| = 9$
32. $\left|\frac{s}{2}-1\right| = 1$

In Exercises 33–40, write the interval defined by the given inequality.

33. $|x| < 2$
34. $|x| \leq 2$
35. $|s-1| \leq 2$
36. $|t+2| < 1$
37. $|3x-7| < 2$
38. $|2x+5| < 1$
39. $\left|\frac{x}{2}-1\right| \leq 1$
40. $\left|2-\frac{x}{2}\right| < \frac{1}{2}$

In Exercises 41–42, solve the given inequality by interpreting it

43. Do not fall into the trap $|-a| = a$. For what real numbers a is this equation true? For what numbers is it false?
44. Solve the equation $|x-1| = 1-x$.
45. Show that the inequality

$$|a-b| \geq \big||a|-|b|\big|$$

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Figure P.10 The four quadrants

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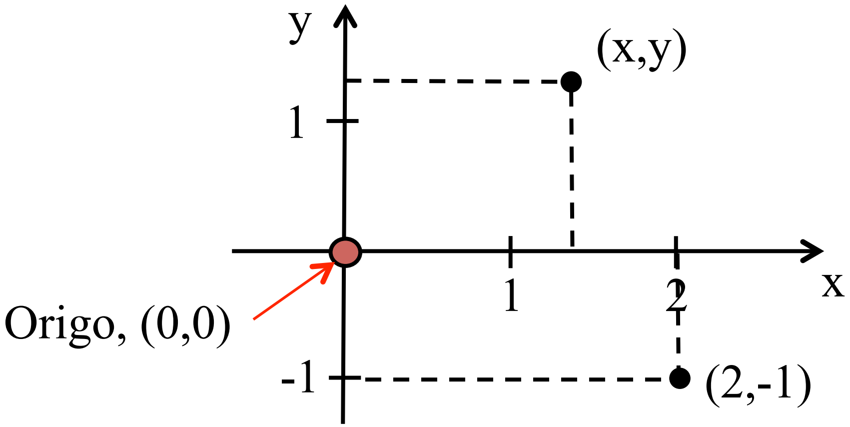
s. If, for
example, we plot height versus time for a falling rock, there is no reason to place the mark that shows 1 m on the height axis the same distance from the origin as the mark that shows 1 s on the time axis.

When we graph functions whose variables do not represent physical measurements and when we draw figures in the coordinate plane to study their geometry or trigonom-

Tason karteesiset koordinaatit

Tasoa puolestaan voidaan pitää kaksiulotteisena avaruutena, jolle voidaan muodostaa koordinaatisto kahdesta toisiaan vastaan kohtisuorassa olevasta reaalilukuakselista (kiinnittämällä origon paikka ja akselien mittakaavat). Jos akselit nimetään vaikkapa x- ja y -akseleiksi, voidaan ko avaruuden mielivaltainen piste ilmaista kahden luvun avulla lukuparina (x, y). Siten esim. lukupari $(2, -1)$ esittää pistettä, jonka paikan kohtisuora projektio x -akselille on 2 yksikön päässä origosta sen positiivisella puolella ja sen kohtisuora projektio y -akselille on 1 yksikön päässä origosta sen negatiivisella puolella. Luvut 2 ja -1 ovat ko. pisteen x- ja y *koordinaatit*. Näin muodostettua koordinaatistoa sanotaan *karteesiseksi koordinaatistoksi*.



etry, we usually make the scales identical. A vertical unit of distance then looks the same as a horizont that are supposed t supposed to be equ such as the relatio equal scales are use

Computer and scales on machine- tances, slopes, and rectangular or even Circumstances like High-quality comp to compensate for s to horizontal scale) range. When using configuration so tha be equal.

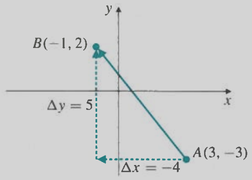


Figure P.11 Increments in x and y

Increments and

When a particle mo called increments. point from the coor change in the value x is $\Delta x = x_2 - x_1$.

EXAMPLE 1

Solution The inc

$\Delta x = -1 - 3$

If $P(x_1, y_1)$ and $Q($ the hypotenuse of a CQ of the triangle

$|\Delta x| = |x_2 -$

These are the *hori* Pythagorean Theor of these lengths.

Distance form

The distance D

$D = \sqrt{(\Delta$

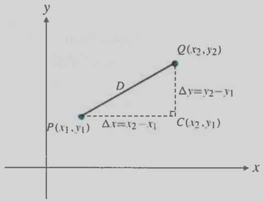


Figure P.12 The distance from P to Q is $D=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

EXAMPLE 2

$\sqrt{(-1-3)^2 +$

Kahden pisteen välinen etäisyys.

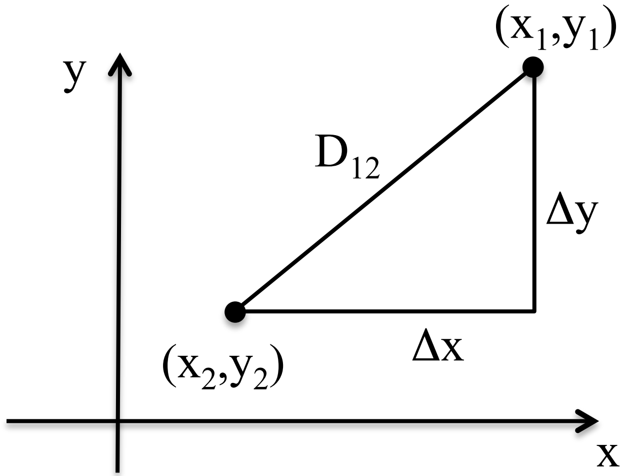
Olkoon tason pisteen P_1 koordinaatit (x_1, y_1) ja pisteen P_2 koordinaatit (x_2, y_2). Pisteiden P_1 ja P_2 välinen etäisyys

$$D_{12} = \sqrt{(\Delta x)^2 + (\Delta y)^2}, \tag{1}$$

missä

$$\begin{aligned}\Delta x &= x_2 - x_1\\ \Delta y &= y_2 - y_1\end{aligned}$$

ovat (koordinaattien) muutokset l. *siirtymät* pisteestä P_1 pisteeseen P_2.



etry, we usually make the scales identical. A vertical unit of distance then looks the same as a horizontal unit. As on a surveyor's map or a scale drawing, line segments that are supposed to have the same length will look as if they do, and angles that are supposed to be equal will look equal. Some of the geometric results we obtain later, such as the relationship between the slopes of perpendicular lines, are valid only if equal scales are used on the two axes.

Computer and calculator displays are another matter. The vertical and horizontal scales on machine-generated graphs usually differ, with resulting distortions in dis-
tances, slopes,
rectangular or e
Circumstances
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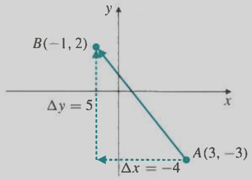

Figure P.11 Increments in x and y

Increments

When a particle
called incremen
point from the c
change in the v
x is $\Delta x = x_2 -$

EXAMPLE 1

Solution The

$\Delta x = -1$

If $P(x_1, y_1)$ and
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$|\Delta x| = |x_2$

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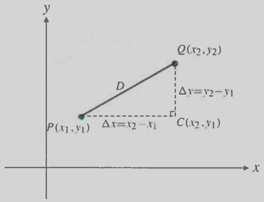


Figure P.12 The distance from P to Q is $D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Distance f

The distanc

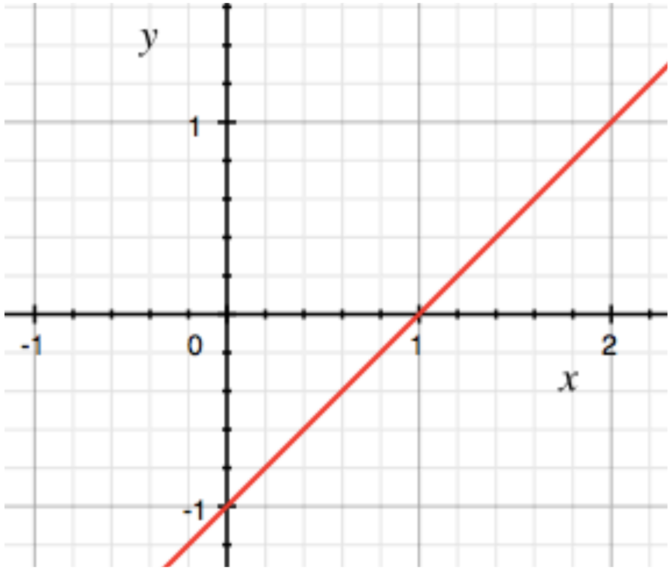
$D =$

EXAMPLE 2

$\sqrt{(-1-3)^2 + (2-(-3))^2} = \sqrt{(-4)^2 + 5^2} = \sqrt{41}$ units.

Koordinaatiston käyttö yhtälön ratkaisujoukon tai funktion kuvaajan esittämiseen.

Esim: Tason pisteet (x, y), jotka toteuttavat yhtälön $x + 1 = y + 2$.



Huom: Tämä on samalla funktion $f(x) = x - 1$ kuvaaja joka muodostetaan asettamalla $y = f(x)$.

EXAMPLE 3 The distance from the origin $O(0, 0)$ to a point $P(x, y)$ is

$$\sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}.$$

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has the *same value* for every choice of two distinct points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on the line. (See Figure P.15.) The constant $m = \Delta y/\Delta x$ is called the **slope** of the nonvertical line.

Tavallisimmat toisen asteen yhtälöt ja niiden kuvaajat.

a) Parabeli:

$$y = ax^2 + bx + c$$

b) R -säteinen ympyrä. Keskipiste (x_0, y_0):

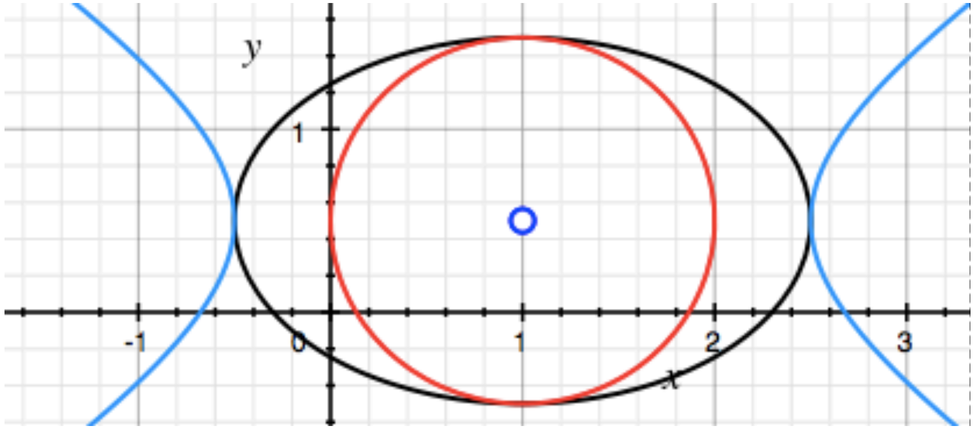
$$(x - x_0)^2 + (y - y_0)^2 = R^2$$

c) Ellipsi. Keskipiste (x_0, y_0):

$$(\frac{x - x_0}{a})^2 + (\frac{y - y_0}{b})^2 = 1$$

d) Hyperbeli. Keskipiste (symmetriapiste) (x_0, y_0):

$$(\frac{x - x_0}{a})^2 - (\frac{y - y_0}{b})^2 = 1$$



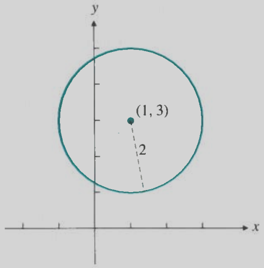


Figure P.20 Circle
$(x-1)^2+(y-3)^2=4$

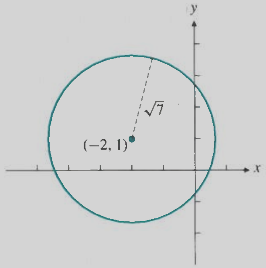


Figure P.21 Circle
$(x+2)^2+(y-1)^2=7$

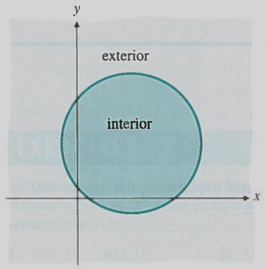


Figure P.22 The interior of a circle (darkly shaded) and the exterior (lightly shaded)

the equation of the circle of radius

$$\sqrt{(x-h)^2+(y-k)^2}=a.$$

A simpler form of this equation is

Standard equation of a circl

The circle with centre (h, k) a

$$(x-h)^2+(y-k)^2=a$$

In particular, the circle with ce

$$x^2+y^2=a^2.$$

EXAMPLE 1 The circle wit
equation $(x -$

EXAMPLE 2 The circle hav
the point $(-2$

If the squares in the standard equ
and all constant terms collected o

$$x^2-2hx+y^2-2ky=a^2 -$$

A quadratic equation of the form

$$x^2+y^2+2ax+2by=c$$

must represent a circle, a single p
complete the squares on the left si
terms of the square $(x+a)^2=x^2$
square of the x terms. (Note that a
add b^2 to both sides to complete th

$$(x+a)^2+(y+b)^2=c+a$$

If $c+a^2+b^2>0$, the graph is a c
If $c+a^2+b^2=0$, the graph cons
no points lie on the graph.

EXAMPLE 3 Find the centr

Solution Observe that x^2-4x ar
x^2-4x+4, and y^2+6y are the f
Hence we add 4 + 9 to both sides

$$x^2-4x+4+y^2+6y+9$$

This is the equation of a circle wi

The set of all points *inside* a circl
an **open disk**. The set of all points
(See Figure P.22.) The interior
closed disk, or simply a **disk**. Th

$$(x-h)^2+(y-k)^2\le a^2$$

represents the disk of radius $|a|$ c

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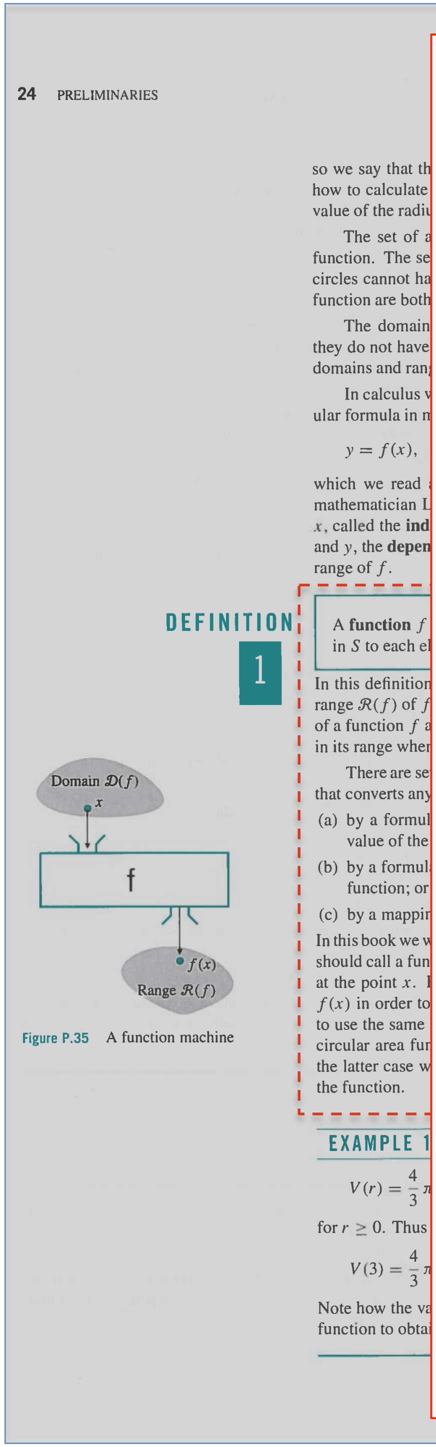


Funktio.

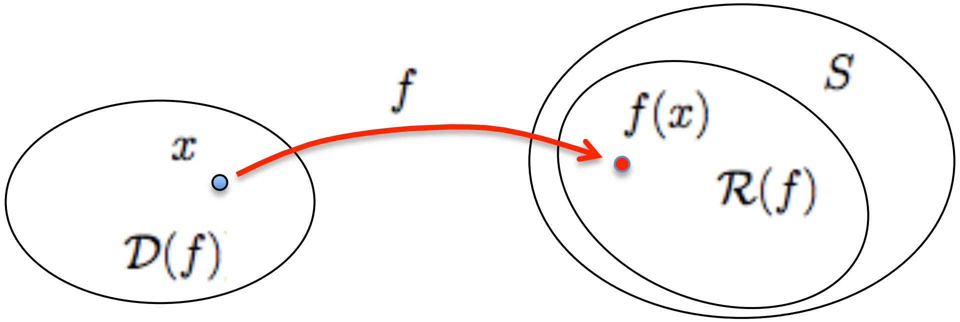
<u>Määritelmä</u>: Funktio f on joukolta D joukolle S määritelty kuvaus, joka liittää jokaiseen D:n alkioon x joukon S *yksikäsitteisen* alkion $f(x)$.

-Joukkoa D, jota merkitään usein $\mathcal{D}(f)$, kutsutaan funktion f **lähtö-** eli **määrittelyjoukoksi** (Engl. domain).
-Joukkoa S, kutsutaan funktion f **maalijoukoksi**.
-Funktion **arvojoukko** $\mathcal{R}(f)$ (Engl. range) on S:n osajoukko, joka koostuu kaikista f:n arvoista, ts. $\mathcal{R}(f) = \{f(x) | x \in \mathcal{D}(f)\}$

Funktion täydelliseen määrittelyyn kuuluu siis:

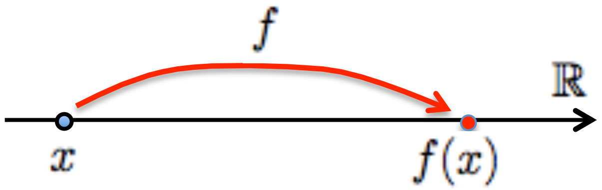
- Kuvaussääntö (joka voidaan antaa mv. tavalla, vaikkapa sanallisesti - yleensä se kuitenkin annetaan matemaattisena kaavana.)
- Määrittelyjoukko $\mathcal{D}(f)$
- Maalijoukko S (huom: määrittelyjoukko ja kuvaussääntö kiinnittävät arvojoukon $\mathcal{R}(f) \subset S$)

Huom: Määrittely- ja maalijoukkoja ei käytännössä aina erikseen kerrota funktion määrittelyn yhteydessä jos ne ovat asiayhteyden perusteella ilmeisiä.



Funktion kuvaaja.

Rajoitutaan seuraavassa yhden reaalimuuttujan reaaliarvoisiin funktioihin $f\colon D \to S;\ D \subset \mathbb{R},\ S \subset \mathbb{R}$.



Tulkitaan kuvaus seuraavaksi siten, että lähtöjoukkona on tason karteesisen koordinaatiston x -akseli ja maalijoukkona sen y -akseli. Funktion f kuvaaja on pistejoukko $\{(x,y) | x, y \in \mathbb{R}, y = f(x)\}$

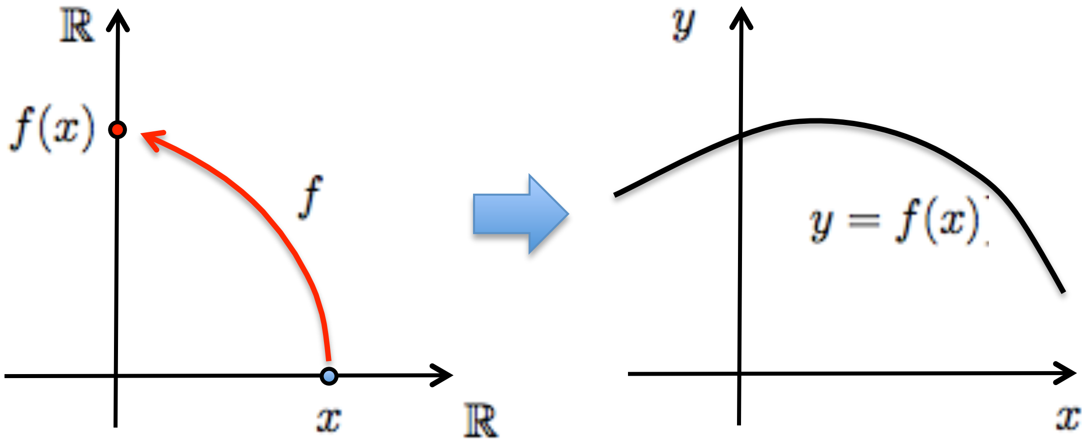


Table 1.

x	$y = f(x)$
-2	4
-1	1
0	0
1	1
2	4

Figure P.36
(a) Correct graph of $f(x) = x^2$
(b) Incorrect graph of $f(x) = x^2$

EXAMPLE 8 Sketch the graph of the function $f(x) = \frac{2-x}{x-1}$.

Figure P.49 The circle $x^2 + y^2 = 1$ is not the graph of a function

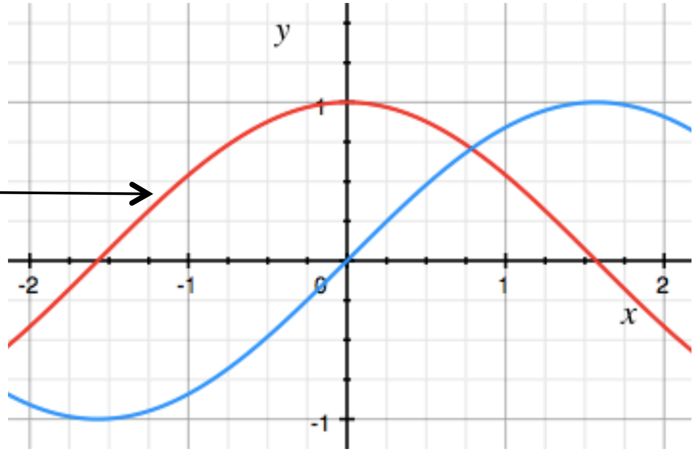
Funktion symmetrioista.

Funktio $f\colon \mathbb{R} \to \mathbb{R}$ on

- Symmetrinen eli parillinen jos $f(-x) = f(x)$
- Antisymmetrinen eli pariton jos $f(-x) = -f(x)$

Parillisen funktion kuvaaja on *peilaussymmetrinen* y -akselin suhteen.

$y = \cos(x)$ parillinen



$y = \sin(x)$ pariton.

Funktiolla voi olla myös muita symmetrioita, esim peilaussymmetria mv. suoran tai pisteen suhteen. Huom: Parittoman funktion kuvaaja on symmetrinen *origon* suhteen.

or $-5f(x)$. Sums (and differences) of even functions are even; sums (and differences) of odd functions are odd. For example, $f(x) = 3x^4 - 5x^2 - 1$ is even, since it is the sum of three even functions: $3x^4$, $-5x^2$, and $-1 = -x^0$. Similarly, $4x^3 - (2/x)$ is an odd function. The function $g(x) = x^2 - 2x$ is the sum of an even function and an odd function and is itself neither even nor odd.

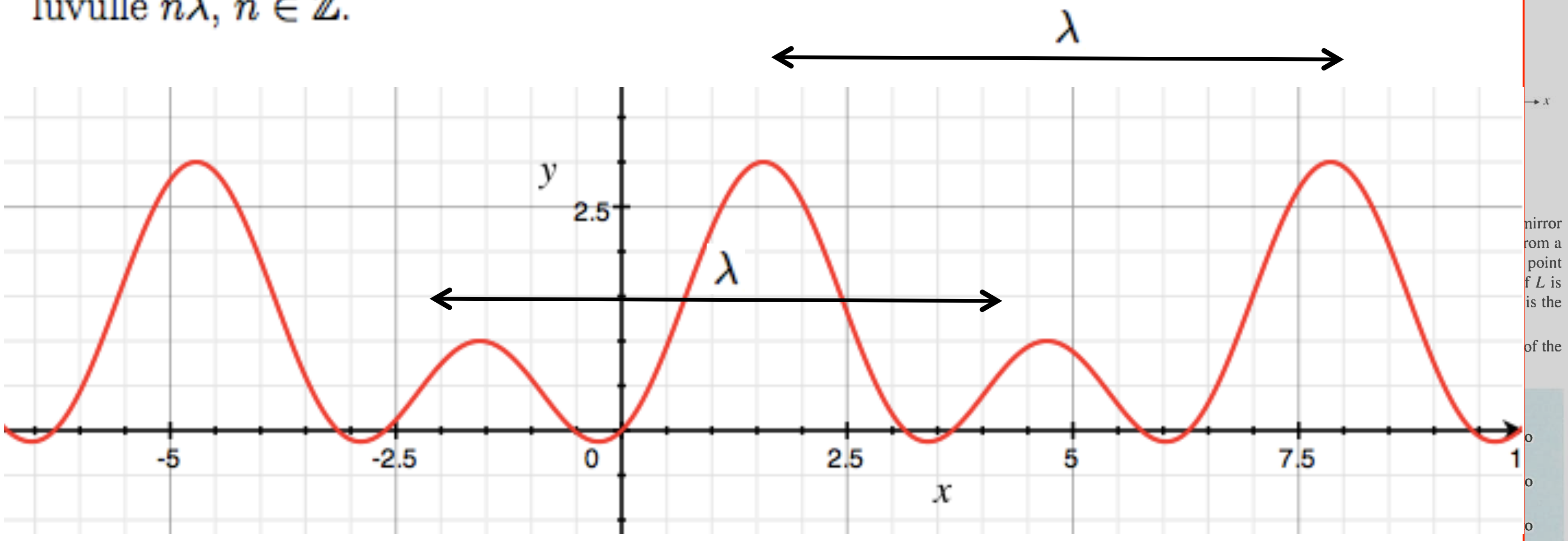
EXAMPLE 9 Describe and sketch the graph of $y = \sqrt{2-x} - 3$.

Solution The graph of $y = \sqrt{2-x}$ is the reflection of the graph of $y = \sqrt{x}$

Jaksollinen funktio.

Funktio $f\colon\ \mathbb{R} \to \mathbb{R}$ on **jaksollinen**, jos on olemassa luku λ jolle $f(x+\lambda) = f(x)$ kaikille x:n arvoille. Funktion *jakso* on pienin positiivinen luku λ, joka toteuttaa tämän ehdon.

Huom: jos y.o. jaksollisuusehto pätee luvulle λ, niin se pätee myös jokaiselle luvulle $n\lambda$, $n \in \mathbb{Z}$.



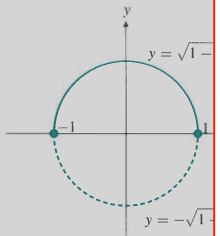


Figure P.49 The circle $x^2 + y^2$ the graph of a function

side of the origin, come to another point on the graph. If an odd function f is defined at $x = 0$, then its value must be zero there: $f(0) = 0$. (See Figure P.50(b).)

If $f(x)$ is even (or odd), then so is any constant multiple of $f(x)$ such as $2f(x)$ or $-5f(x)$. Sums (and differences) of even functions are even; sums (and differences) of odd functions are odd. For example, $f(x) = 3x^4 - 5x^2 - 1$ is even, since it is the sum of three even functions: $3x^4$, $-5x^2$, and $-1 = -x^0$. Similarly, $4x^3 - (2/x)$ is an odd function. The function $g(x) = x^2 - 2x$ is the sum of an even function and an odd function and is itself neither even nor odd.

reflecting the graph of the equation in the line $y = b/2$.

5. Interchanging x and y in an equation in x and y corresponds to reflecting the graph of the equation in the line $y = x$.

EXAMPLE 9 Describe and sketch the graph of $y = \sqrt{2-x} - 3$.

Solution The graph of $y = \sqrt{2-x}$ is the reflection of the graph of $y = \sqrt{x}$

and uses $g(x) = 0$ in the plot. This seems to happen between about -0.5×10^{-16} and 0.8×10^{-16} (the coloured horizontal line). As we move further away from the origin, Maple can tell the difference between $1 + x$ and 1, but loses most of the significant figures in the representation of x when it adds 1, and these remain lost when it subtracts 1 again. Thus the numerator remains constant over short intervals while the denominator increases as x moves away from 0. In those intervals the fraction behaves like *constant*/x so the arcs are hyperbolas, sloping downward away from the origin. The effect diminishes the farther x moves away from 0, as more of its significant figures are retained by Maple. It should be noted that the reason we used the absolute value of $1 + x$ instead of just $1 + x$ is that this forced Maple to add the x to the 1 before subtracting the second 1. (If we had used $(1+x) - 1$ as the numerator for $g(x)$, Maple would have simplified it algebraically and obtained $g(x) = 1$ before using any values of x for plotting.)

In later chapters we will encounter more such strange behaviour (which we call **numerical monsters**) in the context of calculator and computer calculations with floating point (i.e. real) numbers. They are a necessary consequence of the limitations of such hardware and software, and are not restricted to Maple, though they may show up somewhat differently with other software. It is necessary to be aware of how calculators and computers do arithmetic in order to be able to use them effectively without falling into errors that you do not recognize as such.

One final comment about Figure P.55: the graph of $y = g(x)$ was plotted as individual points, rather than a line as was $y = 1$, in order to make the jumps between consecutive arcs more obvious. Had we omitted the `style=[point,line]` option in the plot command, the default line style would have been used for both graphs and the arcs in the graph of g would have been connected with vertical line segments. Note how the command called for the plotting of two different functions by listing them within square brackets, and how the corresponding styles were correspondingly listed.

EXERCISES P.4

In Exercises 1–6, find the domain and range of each function.

1. $f(x) = 1 + x^2$

2. $f(x) = 1 - \sqrt{x}$

3. $G(x) = \sqrt{8 - 2x}$

4. $F(x) = 1/(x - 1)$

5. $h(t) = \dfrac{t}{\sqrt{2 - t}}$

6. $g(x) = \dfrac{1}{1 - \sqrt{x - 2}}$

7. Which of the graphs in Figure P.56 are graphs of functions $y = f(x)$? Why?

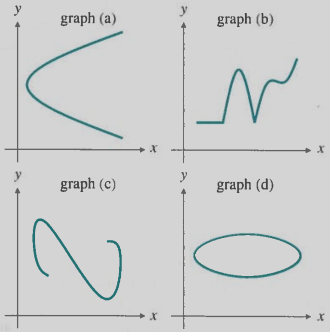


Figure P.56

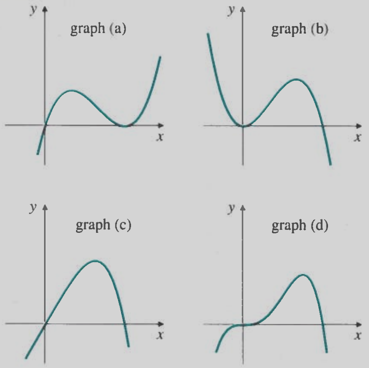


Figure P.57

8. Figure P.57 shows the graphs of the functions: (i) $x - x^4$, (ii) $x^3 - x^4$, (iii) $x(1 - x)^2$, (iv) $x^2 - x^3$. Which graph corresponds to which function?

In Exercises 9–10, sketch the graph of the function f by first making a table of values of $f(x)$ at $x = 0$, $x = \pm 1/2$, $x = \pm 1$, $x = \pm 3/2$, and $x = \pm 2$.

9. $f(x) = x^4$

10. $f(x) = x^{2/3}$

In Exercises 11–22, what (if any) symmetry does the graph of f possess? In particular, is f even or odd?

11. $f(x) = x^2 + 1$

12. $f(x) = x^3 + x$

13. $f(x) = \dfrac{x}{x^2 - 1}$

14. $f(x) = \dfrac{1}{x^2 - 1}$

15. $f(x) = \dfrac{1}{x - 2}$

16. $f(x) = \dfrac{1}{x + 4}$

17. $f(x) = x^2 - 6x$

18. $f(x) = x^3 - 2$

19. $f(x) = |x^3|$

20. $f(x) = |x + 1|$

21. $f(x) = \sqrt{2x}$

22. $f(x) = \sqrt{(x - 1)^2}$

Sketch the graphs of the functions in Exercises 23–38.

23. $f(x) = -x^2$

24. $f(x) = 1 - x^2$

25. $f(x) = (x - 1)^2$

26. $f(x) = (x - 1)^2 + 1$

27. $f(x) = 1 - x^3$

28. $f(x) = (x + 2)^3$

29. $f(x) = \sqrt{x} + 1$

30. $f(x) = \sqrt{x + 1}$

31. $f(x) = -|x|$

32. $f(x) = |x| - 1$

33. $f(x) = |x - 2|$

34. $f(x) = 1 + |x - 2|$

35. $f(x) = \dfrac{2}{x + 2}$

36. $f(x) = \dfrac{1}{2 - x}$

37. $f(x) = \dfrac{x}{x + 1}$

38. $f(x) = \dfrac{x}{1 - x}$

In Exercises 39–46, f refers to the function with domain [0, 2] and range [0, 1], whose graph is shown in Figure P.58. Sketch the graphs of the indicated functions and specify their domains and ranges.

39. $f(x) + 2$

40. $f(x) - 1$

41. $f(x + 2)$

42. $f(x - 1)$

43. $-f(x)$

44. $f(-x)$

45. $f(4 - x)$

46. $1 - f(1 - x)$

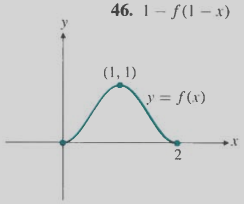


Figure P.58

It is often quite difficult to determine the range of a function exactly. In Exercises 47–48, use a graphing utility (calculator or computer) to graph the function f, and by zooming in on the graph determine the range of f with accuracy of 2 decimal places.

47. $f(x) = \dfrac{x + 2}{x^2 + 2x + 3}$

48. $f(x) = \dfrac{x - 1}{x^2 + x}$

In Exercises 49–52, use a graphing utility to plot the graph of the given function. Examine the graph (zooming in or out as necessary) for symmetries. About what lines and/or points are the graphs symmetric? Try to verify your conclusions algebraically.

49. $f(x) = x^4 - 6x^3 + 9x^2 - 1$

50. $f(x) = \dfrac{3 - 2x + x^2}{2 - 2x + x^2}$

51. $f(x) = \dfrac{x - 1}{x - 2}$

52. $f(x) = \dfrac{2x^2 + 3x}{x^2 + 4x + 5}$

53. What function $f(x)$, defined on the real line $\mathbb{R}$, is both even and odd?

P.5 Combining Functions to Make New Functions

Functions can be combined in a variety of ways to produce new functions.

We begin by examining algebraic means of combining functions, that is, addition, subtraction, multiplication, and division..

Sums, Differences, Products, Quotients, and Multiples

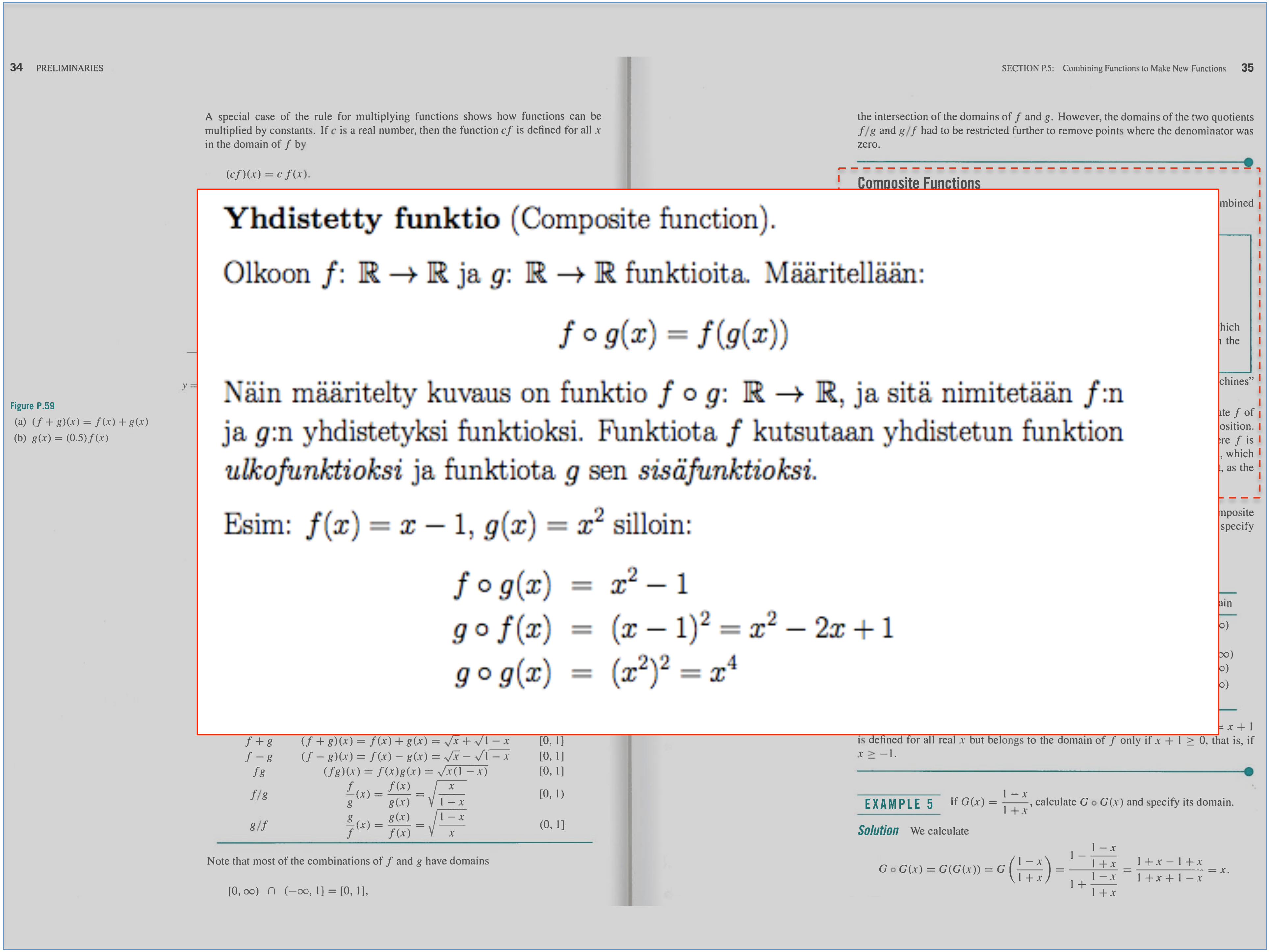
Like numbers, functions can be added, subtracted, multiplied, and divided (except where the denominator is zero) to produce new functions.

DEFINITION

3

If f and g are functions, then for every x that belongs to the domains of both f and g we define functions $f + g$, $f - g$, fg, and f/g by the formulas:

$$(f + g)(x) = f(x) + g(x)$$
$$(f - g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x)g(x)$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \qquad \text{where } g(x) \neq 0.$$



A special case of the rule for multiplying functions shows how functions can be multiplied by constants. If c is a real number, then the function cf is defined for all x in the domain of f by

$$(cf)(x) = c\,f(x).$$

Figure P.59
(a) $(f+g)(x) = f(x) + g(x)$
(b) $g(x) = (0.5)f(x)$

Yhdistetty funktio (Composite function).

Olkoon $f\colon \mathbb{R} \to \mathbb{R}$ ja $g\colon \mathbb{R} \to \mathbb{R}$ funktioita. Määritellään:

$$f \circ g(x) = f(g(x))$$

Näin määritelty kuvaus on funktio $f \circ g\colon \mathbb{R} \to \mathbb{R}$, ja sitä nimitetään f:n ja g:n yhdistetyksi funktioksi. Funktiota f kutsutaan yhdistetun funktion *ulkofunktioksi* ja funktiota g sen *sisäfunktioksi*.

Esim: $f(x) = x - 1$, $g(x) = x^2$ silloin:

$$\begin{aligned} f \circ g(x) &= x^2 - 1 \\ g \circ f(x) &= (x-1)^2 = x^2 - 2x + 1 \\ g \circ g(x) &= (x^2)^2 = x^4 \end{aligned}$$

$f+g$	$(f+g)(x) = f(x) + g(x) = \sqrt{x} + \sqrt{1-x}$	$[0, 1]$
$f-g$	$(f-g)(x) = f(x) - g(x) = \sqrt{x} - \sqrt{1-x}$	$[0, 1]$
fg	$(fg)(x) = f(x)g(x) = \sqrt{x(1-x)}$	$[0, 1]$
f/g	$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \sqrt{\frac{x}{1-x}}$	$[0, 1)$
g/f	$\frac{g}{f}(x) = \frac{g(x)}{f(x)} = \sqrt{\frac{1-x}{x}}$	$(0, 1]$

Note that most of the combinations of f and g have domains

$$[0, \infty) \cap (-\infty, 1] = [0, 1],$$

the intersection of the domains of f and g. However, the domains of the two quotients f/g and g/f had to be restricted further to remove points where the denominator was zero.

Composite Functions

$= x + 1$ is defined for all real x but belongs to the domain of f only if $x + 1 \geq 0$, that is, if $x \geq -1$.

EXAMPLE 5 If $G(x) = \dfrac{1-x}{1+x}$, calculate $G \circ G(x)$ and specify its domain.

Solution We calculate

$$G \circ G(x) = G(G(x)) = G\left(\frac{1-x}{1+x}\right) = \frac{1 - \dfrac{1-x}{1+x}}{1 + \dfrac{1-x}{1+x}} = \frac{1+x-1+x}{1+x+1-x} = x.$$

Because the resulting function, x, is defined for all real x, we might be tempted to say that the domain of $G \circ G$ is $\mathbb{R}$. This is wrong! To belong to the domain of $G \circ G$, x must satisfy two conditions:

(i) x must belong to the domain of G, and

(ii) $G(x)$ must belong to the domain of G.

The domain of G consists of all real numbers *except* $x = -1$. If we exclude $x = -1$ from the domain of $G \circ G$, condition (i) will be satisfied. Now observe that
$1 - x =$
main of G,
of $G \circ G$ is

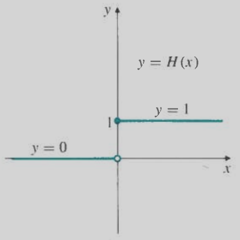


Figure P.61 The Heaviside function

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The functi
circuit by a

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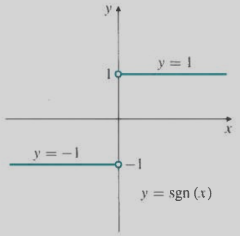


Figure P.62 The signum function

sgn (x

The name
whether x is positive or negative. Since 0 is neither positive nor negative, sgn (0) is not defined. The signum function is an odd function.

EXAMPLE 8 The function

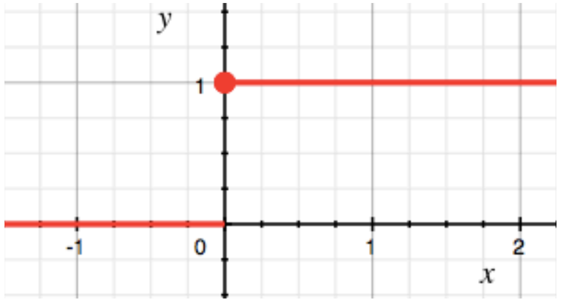
$$f(x) = \begin{cases} (x+1)^2 & \text{if } x < -1, \\ -x & \text{if } -1 \le x < 1, \\ \sqrt{x-1} & \text{if } x \ge 1, \end{cases}$$

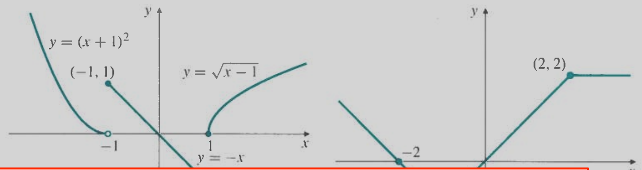
is defined on the whole real line but has values given by three different formulas depending on the position of x. Its graph is shown in Figure P.63(a).

Paloittain määritelty funktio.

Funktion määrittelyä ei aina voida tehdä yhdellä ainoalla matemaattisella lausekkeella, vaan määrittely on tehtävä erikseen kahdelle tai useammalle $\mathbb{R}$:n osavälille. Esimerkkinä jo aiemmin määritelty itseisarvofunktio $f(x) = |x|$ (ks. s. 8). Toinen yleinen esimerkki on ns. **askel-** l. **porrasfunktio** (kutsutaan myös Heavisiden funktioksi).

$$H(x) = \begin{cases} 1 & ; x \ge 0 \\ 0 & ; x < 0 \end{cases} \tag{1}$$





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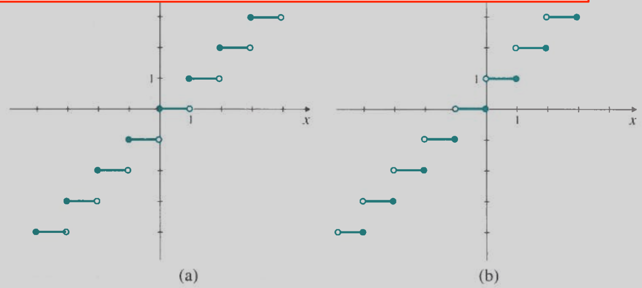
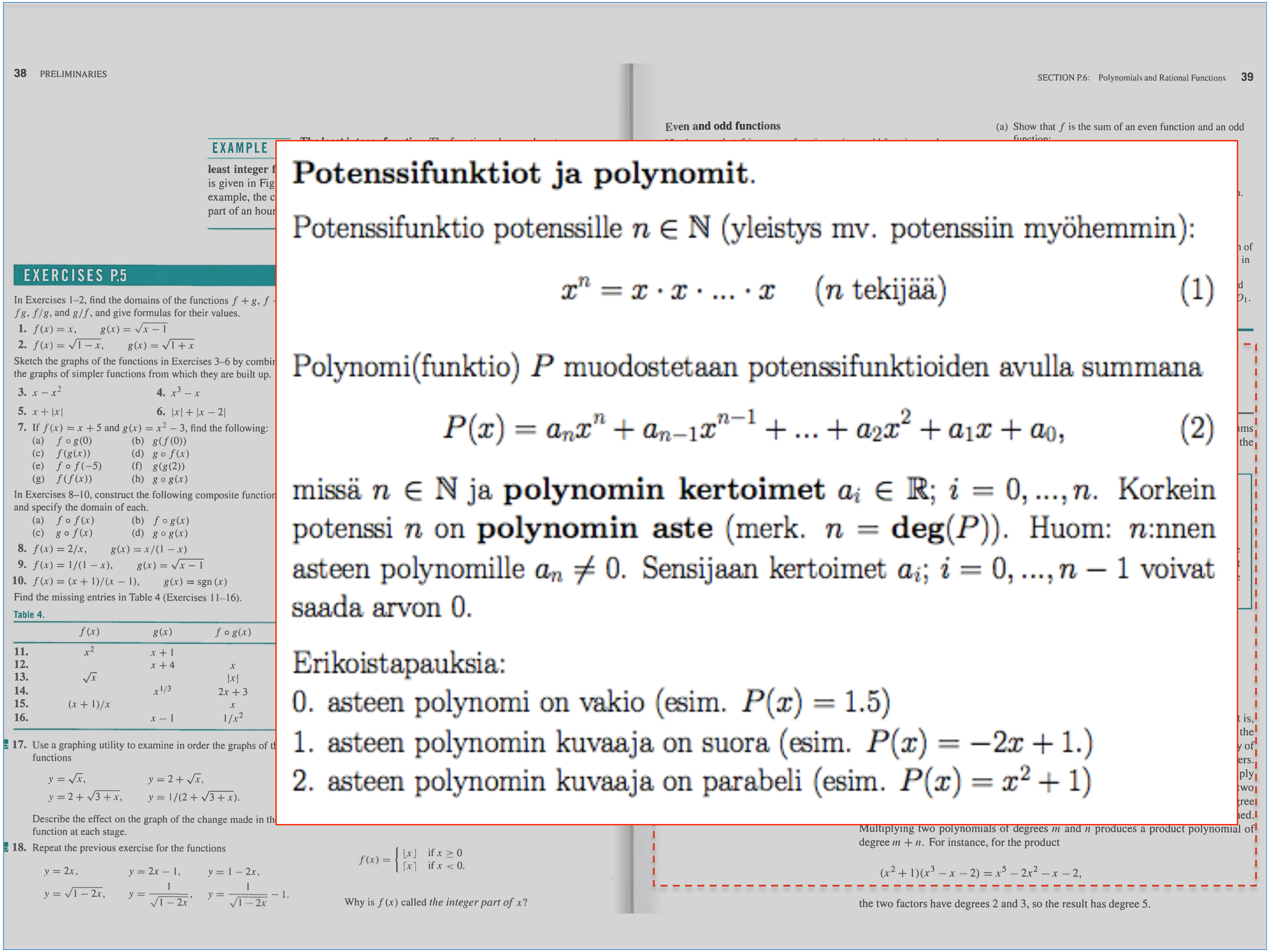


Figure P.64
(a) The greatest integer function $\lfloor x \rfloor$
(b) The least integer function $\lceil x \rceil$



Potenssifunktiot ja polynomit.

Potenssifunktio potenssille $n \in \mathbb{N}$ (yleistys mv. potenssiin myöhemmin):

$$x^n = x \cdot x \cdot \ldots \cdot x \quad (n \text{ tekijää}) \tag{1}$$

Polynomi(funktio) P muodostetaan potenssifunktioiden avulla summana

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0, \tag{2}$$

missä $n \in \mathbb{N}$ ja **polynomin kertoimet** $a_i \in \mathbb{R}$; $i = 0, \ldots, n$. Korkein potenssi n on **polynomin aste** (merk. $n = \mathbf{deg}(P)$). Huom: n:nnen asteen polynomille $a_n \neq 0$. Sensijaan kertoimet a_i; $i = 0, \ldots, n-1$ voivat saada arvon 0.

Erikoistapauksia:

0. asteen polynomi on vakio (esim. $P(x) = 1.5$)
1. asteen polynomin kuvaaja on suora (esim. $P(x) = -2x + 1$.)
2. asteen polynomin kuvaaja on parabeli (esim. $P(x) = x^2 + 1$)

EXERCISES P.5

In Exercises 1–2, find the domains of the functions $f+g$, f ... fg, f/g, and g/f, and give formulas for their values.

1. $f(x) = x$, $g(x) = \sqrt{x-1}$
2. $f(x) = \sqrt{1-x}$, $g(x) = \sqrt{1+x}$

Sketch the graphs of the functions in Exercises 3–6 by combin... the graphs of simpler functions from which they are built up.

3. $x - x^2$
4. $x^3 - x$
5. $x + |x|$
6. $|x| + |x-2|$
7. If $f(x) = x+5$ and $g(x) = x^2 - 3$, find the following:
   (a) $f \circ g(0)$ (b) $g(f(0))$
   (c) $f(g(x))$ (d) $g \circ f(x)$
   (e) $f \circ f(-5)$ (f) $g(g(2))$
   (g) $f(f(x))$ (h) $g \circ g(x)$

In Exercises 8–10, construct the following composite function... and specify the domain of each.

(a) $f \circ f(x)$ (b) $f \circ g(x)$
(c) $g \circ f(x)$ (d) $g \circ g(x)$

8. $f(x) = 2/x$, $g(x) = x/(1-x)$
9. $f(x) = 1/(1-x)$, $g(x) = \sqrt{x-1}$
10. $f(x) = (x+1)/(x-1)$, $g(x) = \operatorname{sgn}(x)$

Find the missing entries in Table 4 (Exercises 11–16).

Table 4.

	$f(x)$	$g(x)$	$f \circ g(x)$
11.	x^2	$x+1$	
12.		$x+4$	x
13.	$\sqrt{x}$		$\lvert x\rvert$
14.		$x^{1/3}$	$2x+3$
15.	$(x+1)/x$		x
16.		$x-1$	$1/x^2$

17. Use a graphing utility to examine in order the graphs of th... functions

$y = \sqrt{x}$, $y = 2 + \sqrt{x}$,
$y = 2 + \sqrt{3+x}$, $y = 1/(2 + \sqrt{3+x})$.

Describe the effect on the graph of the change made in th... function at each stage.

18. Repeat the previous exercise for the functions

$y = 2x$, $y = 2x - 1$, $y = 1 - 2x$,
$y = \sqrt{1-2x}$, $y = \dfrac{1}{\sqrt{1-2x}}$, $y = \dfrac{1}{\sqrt{1-2x}} - 1$.

$$f(x) = \begin{cases} \lfloor x \rfloor & \text{if } x \geq 0 \\ \lceil x \rceil & \text{if } x < 0. \end{cases}$$

Why is $f(x)$ called *the integer part of* x?

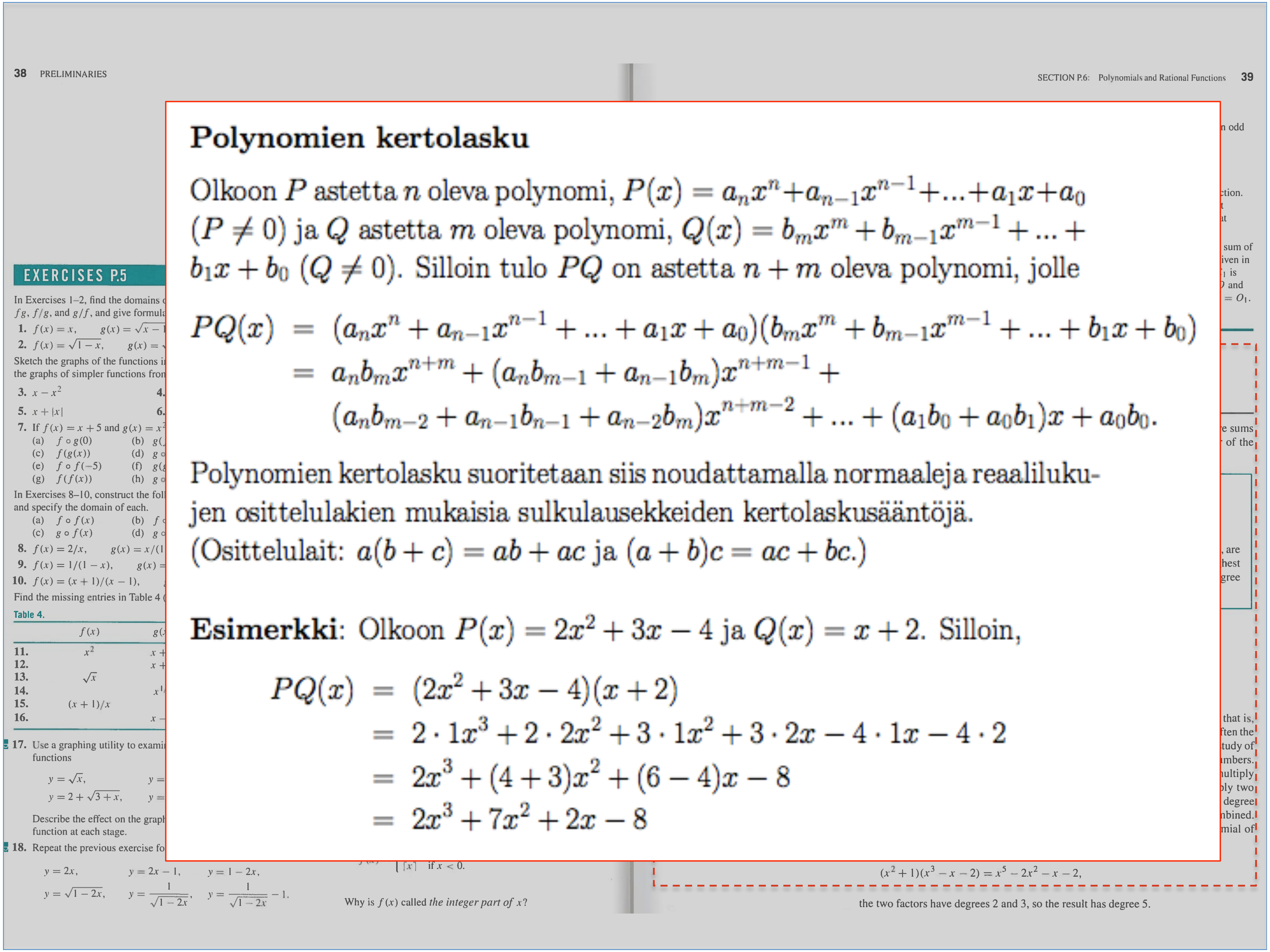
Even and odd functions

(a) Show that f is the sum of an even function and an odd function:

Multiplying two polynomials of degrees m and n produces a product polynomial of degree $m+n$. For instance, for the product

$$(x^2+1)(x^3-x-2) = x^5 - 2x^2 - x - 2,$$

the two factors have degrees 2 and 3, so the result has degree 5.



Polynomien kertolasku

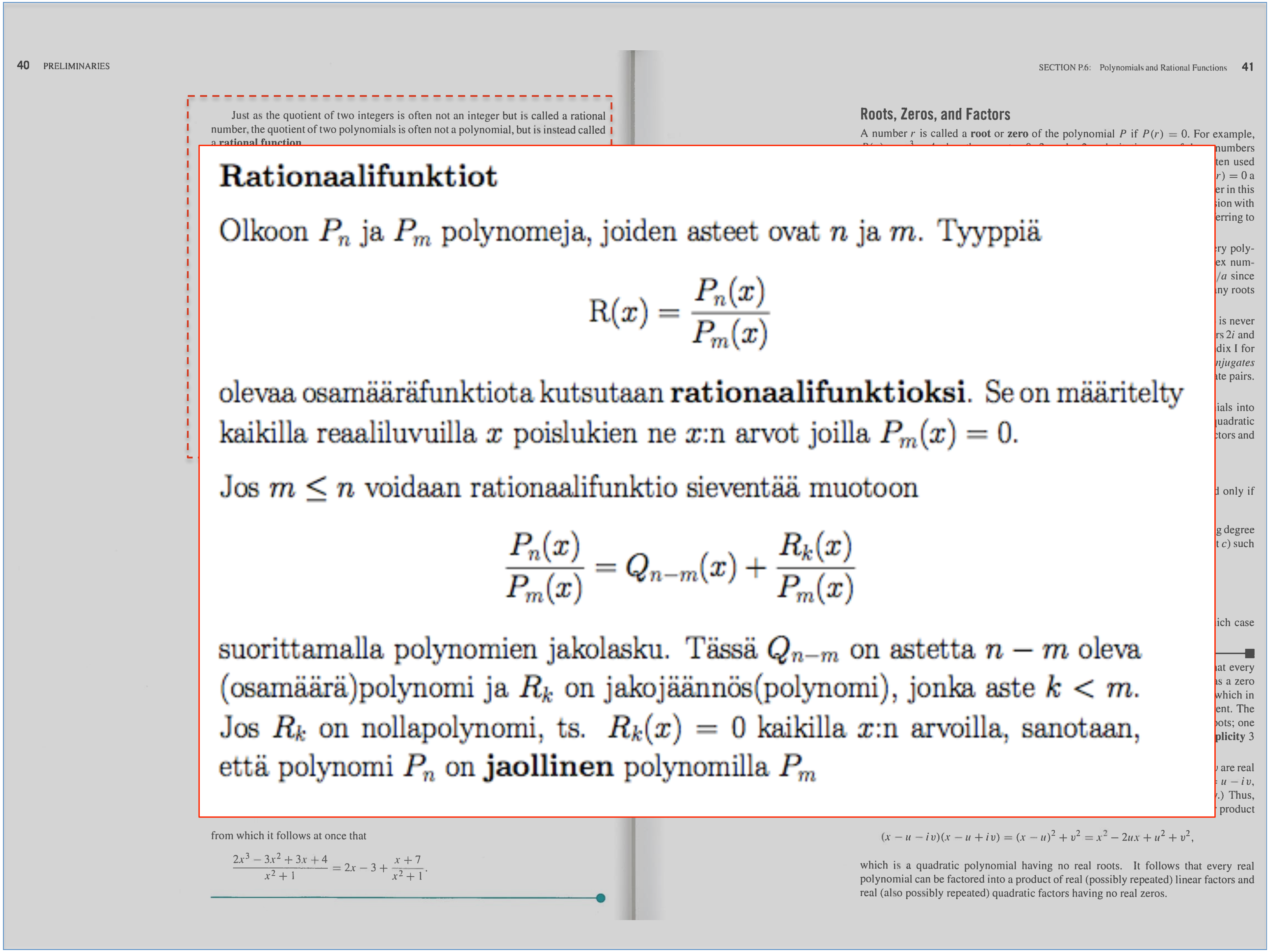
Olkoon P astetta n oleva polynomi, $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ ($P \neq 0$) ja Q astetta m oleva polynomi, $Q(x) = b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0$ ($Q \neq 0$). Silloin tulo PQ on astetta $n+m$ oleva polynomi, jolle

$$\begin{aligned} PQ(x) &= (a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0)(b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0) \\ &= a_n b_m x^{n+m} + (a_n b_{m-1} + a_{n-1} b_m) x^{n+m-1} + \\ &\quad (a_n b_{m-2} + a_{n-1} b_{n-1} + a_{n-2} b_m) x^{n+m-2} + \ldots + (a_1 b_0 + a_0 b_1) x + a_0 b_0. \end{aligned}$$

Polynomien kertolasku suoritetaan siis noudattamalla normaaleja reaalilukujen osittelulakien mukaisia sulkulausekkeiden kertolaskusääntöjä. (Osittelulait: $a(b+c) = ab + ac$ ja $(a+b)c = ac + bc$.)

Esimerkki: Olkoon $P(x) = 2x^2 + 3x - 4$ ja $Q(x) = x + 2$. Silloin,

$$\begin{aligned} PQ(x) &= (2x^2 + 3x - 4)(x + 2) \\ &= 2 \cdot 1x^3 + 2 \cdot 2x^2 + 3 \cdot 1x^2 + 3 \cdot 2x - 4 \cdot 1x - 4 \cdot 2 \\ &= 2x^3 + (4+3)x^2 + (6-4)x - 8 \\ &= 2x^3 + 7x^2 + 2x - 8 \end{aligned}$$



Just as the quotient of two integers is often not an integer but is called a rational number, the quotient of two polynomials is often not a polynomial, but is instead called a **rational function**.

from which it follows at once that

$$\frac{2x^3-3x^2+3x+4}{x^2+1} = 2x-3+\frac{x+7}{x^2+1}.$$

Roots, Zeros, and Factors

A number r is called a **root** or **zero** of the polynomial P if $P(r)=0$. For example,

$$(x-u-iv)(x-u+iv) = (x-u)^2+v^2 = x^2-2ux+u^2+v^2,$$

which is a quadratic polynomial having no real roots. It follows that every real polynomial can be factored into a product of real (possibly repeated) linear factors and real (also possibly repeated) quadratic factors having no real zeros.

Rationaalifunktiot

Olkoon P_n ja P_m polynomeja, joiden asteet ovat n ja m. Tyyppiä

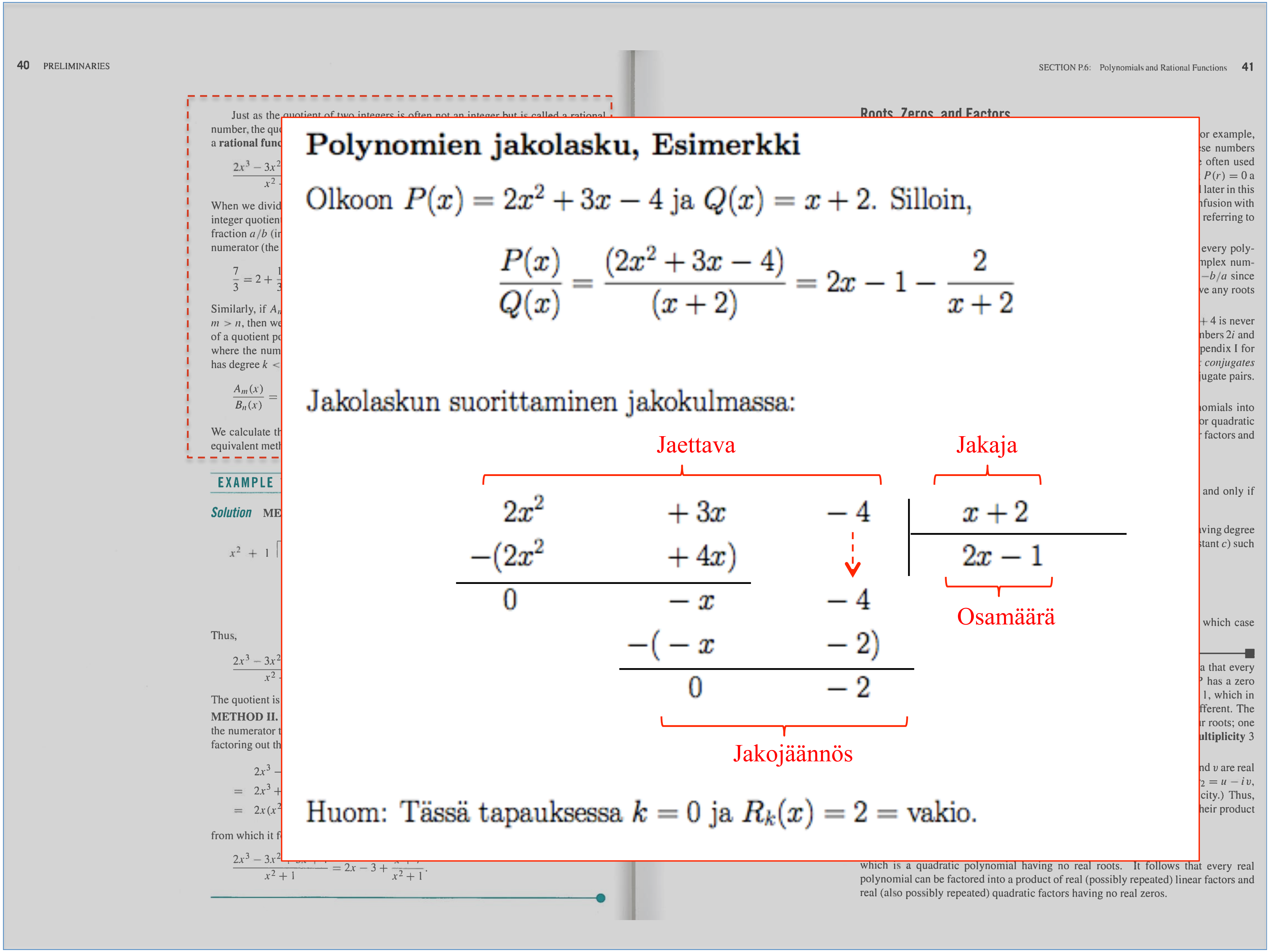
$$\mathrm{R}(x)=\frac{P_n(x)}{P_m(x)}$$

olevaa osamääräfunktiota kutsutaan **rationaalifunktioksi**. Se on määritelty kaikilla reaaliluvuilla x poislukien ne x:n arvot joilla $P_m(x)=0$.

Jos $m \le n$ voidaan rationaalifunktio sieventää muotoon

$$\frac{P_n(x)}{P_m(x)} = Q_{n-m}(x)+\frac{R_k(x)}{P_m(x)}$$

suorittamalla polynomien jakolasku. Tässä Q_{n-m} on astetta $n-m$ oleva (osamäärä)polynomi ja R_k on jakojäännös(polynomi), jonka aste $k<m$. Jos R_k on nollapolynomi, ts. $R_k(x)=0$ kaikilla x:n arvoilla, sanotaan, että polynomi P_n on **jaollinen** polynomilla P_m



Roots, Zeros, and Factors

which is a quadratic polynomial having no real roots. It follows that every real polynomial can be factored into a product of real (possibly repeated) linear factors and real (also possibly repeated) quadratic factors having no real zeros.

Polynomien jakolasku, Esimerkki

Olkoon $P(x) = 2x^2 + 3x - 4$ ja $Q(x) = x + 2$. Silloin,

$$\frac{P(x)}{Q(x)} = \frac{(2x^2 + 3x - 4)}{(x+2)} = 2x - 1 - \frac{2}{x+2}$$

Jakolaskun suorittaminen jakokulmassa:

$$
\begin{array}{rrr|l}
\multicolumn{3}{c|}{\overbrace{\hphantom{2x^2 \quad +3x \quad -4}}^{\text{Jaettava}}} & \overbrace{\hphantom{x+2}}^{\text{Jakaja}} \\
2x^2 & +3x & -4 & x+2 \\ \cline{4-4}
-(2x^2 & +4x) & \downarrow & \underbrace{2x-1}_{\text{Osamäärä}} \\ \cline{1-2}
0 & -x & -4 & \\
 & -(-x & -2) & \\ \cline{2-3}
 & 0 & -2 & \\
 & \multicolumn{2}{c}{\underbrace{\hphantom{0 \quad -2}}_{\text{Jakojäännös}}} &
\end{array}
$$

Huom: Tässä tapauksessa $k = 0$ ja $R_k(x) = 2 =$ vakio.

Polynomin nollakohdat ja jako alemman asteen tekijöihin

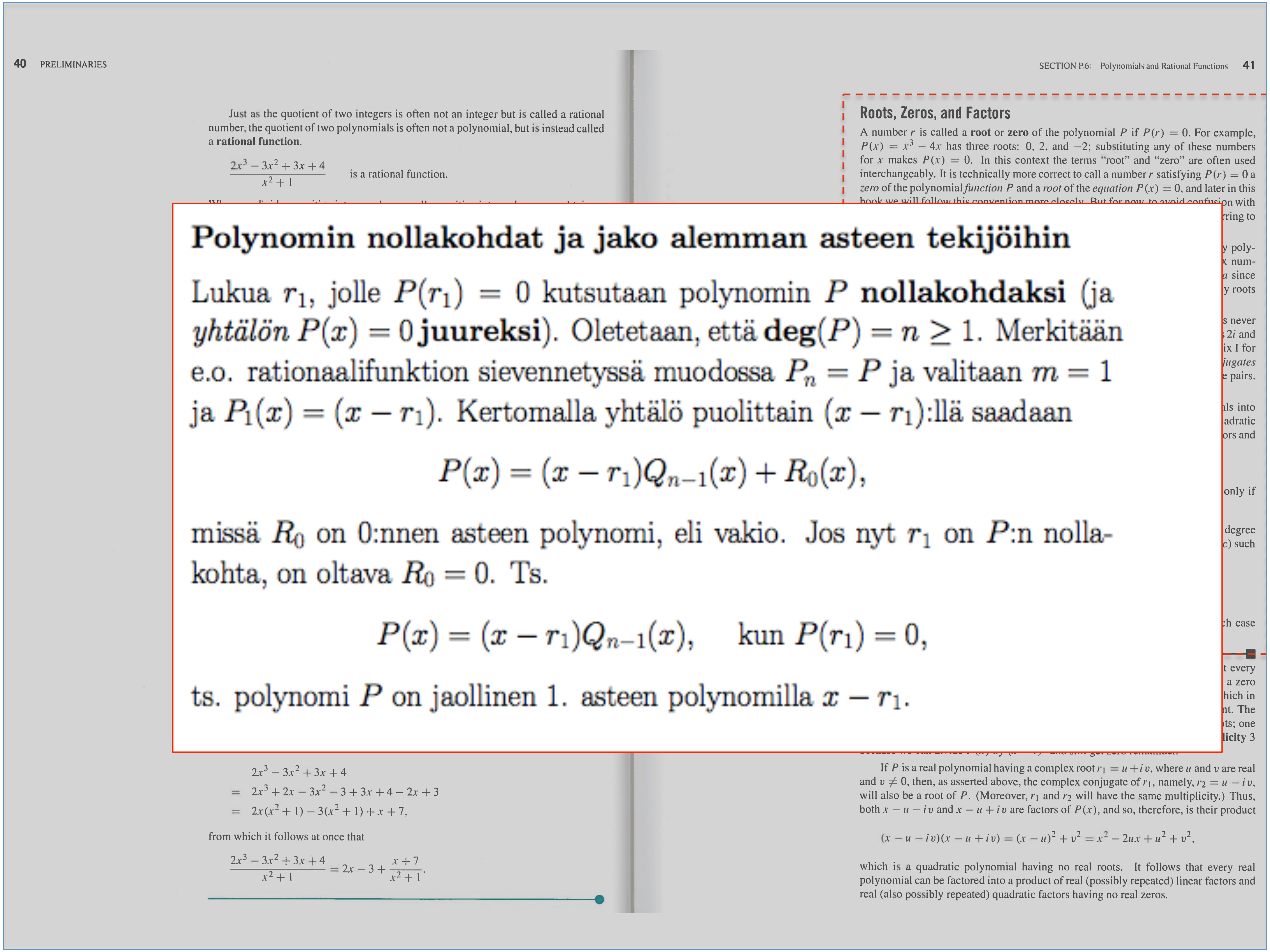
Lukua r_1, jolle $P(r_1) = 0$ kutsutaan polynomin P **nollakohdaksi** (ja *yhtälön* $P(x) = 0$ **juureksi**). Oletetaan, että $\mathbf{deg}(P) = n \geq 1$. Merkitään e.o. rationaalifunktion sievennetyssä muodossa $P_n = P$ ja valitaan $m = 1$ ja $P_1(x) = (x - r_1)$. Kertomalla yhtälö puolittain $(x - r_1)$:llä saadaan

$$P(x) = (x - r_1)Q_{n-1}(x) + R_0(x),$$

missä R_0 on 0:nnen asteen polynomi, eli vakio. Jos nyt r_1 on P:n nollakohta, on oltava $R_0 = 0$. Ts.

$$P(x) = (x - r_1)Q_{n-1}(x), \quad \text{kun } P(r_1) = 0,$$

ts. polynomi P on jaollinen 1. asteen polynomilla $x - r_1$.



40 PRELIMINARIES

Just as the quotient of two integers is often not an integer but is called a rational number, the quotient of two polynomials is often not a polynomial, but is instead called a **rational function**.

$$\frac{2x^3 - 3x^2 + 3x + 4}{x^2 + 1} \quad \text{is a rational function.}$$

$$\begin{aligned} & 2x^3 - 3x^2 + 3x + 4 \\ = \ & 2x^3 + 2x - 3x^2 - 3 + 3x + 4 - 2x + 3 \\ = \ & 2x(x^2 + 1) - 3(x^2 + 1) + x + 7, \end{aligned}$$

from which it follows at once that

$$\frac{2x^3 - 3x^2 + 3x + 4}{x^2 + 1} = 2x - 3 + \frac{x + 7}{x^2 + 1}.$$

SECTION P.6: Polynomials and Rational Functions 41

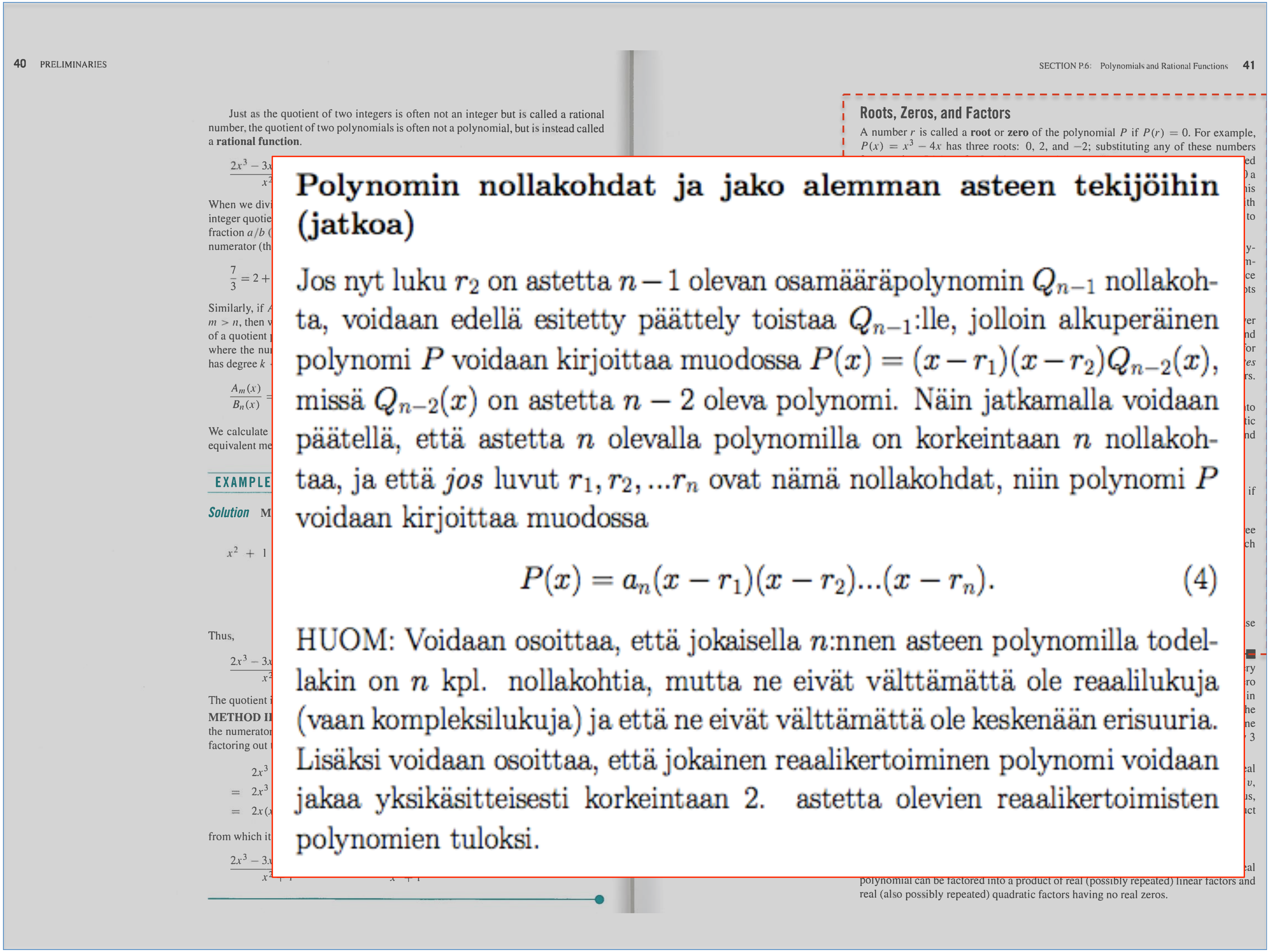
Roots, Zeros, and Factors

A number r is called a **root** or **zero** of the polynomial P if $P(r) = 0$. For example, $P(x) = x^3 - 4x$ has three roots: 0, 2, and -2; substituting any of these numbers for x makes $P(x) = 0$. In this context the terms "root" and "zero" are often used interchangeably. It is technically more correct to call a number r satisfying $P(r) = 0$ a *zero* of the polynomial *function* P and a *root* of the *equation* $P(x) = 0$, and later in this book we will follow this convention more closely. But for now, to avoid confusion with

If P is a real polynomial having a complex root $r_1 = u + iv$, where u and v are real and $v \neq 0$, then, as asserted above, the complex conjugate of r_1, namely, $r_2 = u - iv$, will also be a root of P. (Moreover, r_1 and r_2 will have the same multiplicity.) Thus, both $x - u - iv$ and $x - u + iv$ are factors of $P(x)$, and so, therefore, is their product

$$(x - u - iv)(x - u + iv) = (x - u)^2 + v^2 = x^2 - 2ux + u^2 + v^2,$$

which is a quadratic polynomial having no real roots. It follows that every real polynomial can be factored into a product of real (possibly repeated) linear factors and real (also possibly repeated) quadratic factors having no real zeros.



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Roots, Zeros, and Factors

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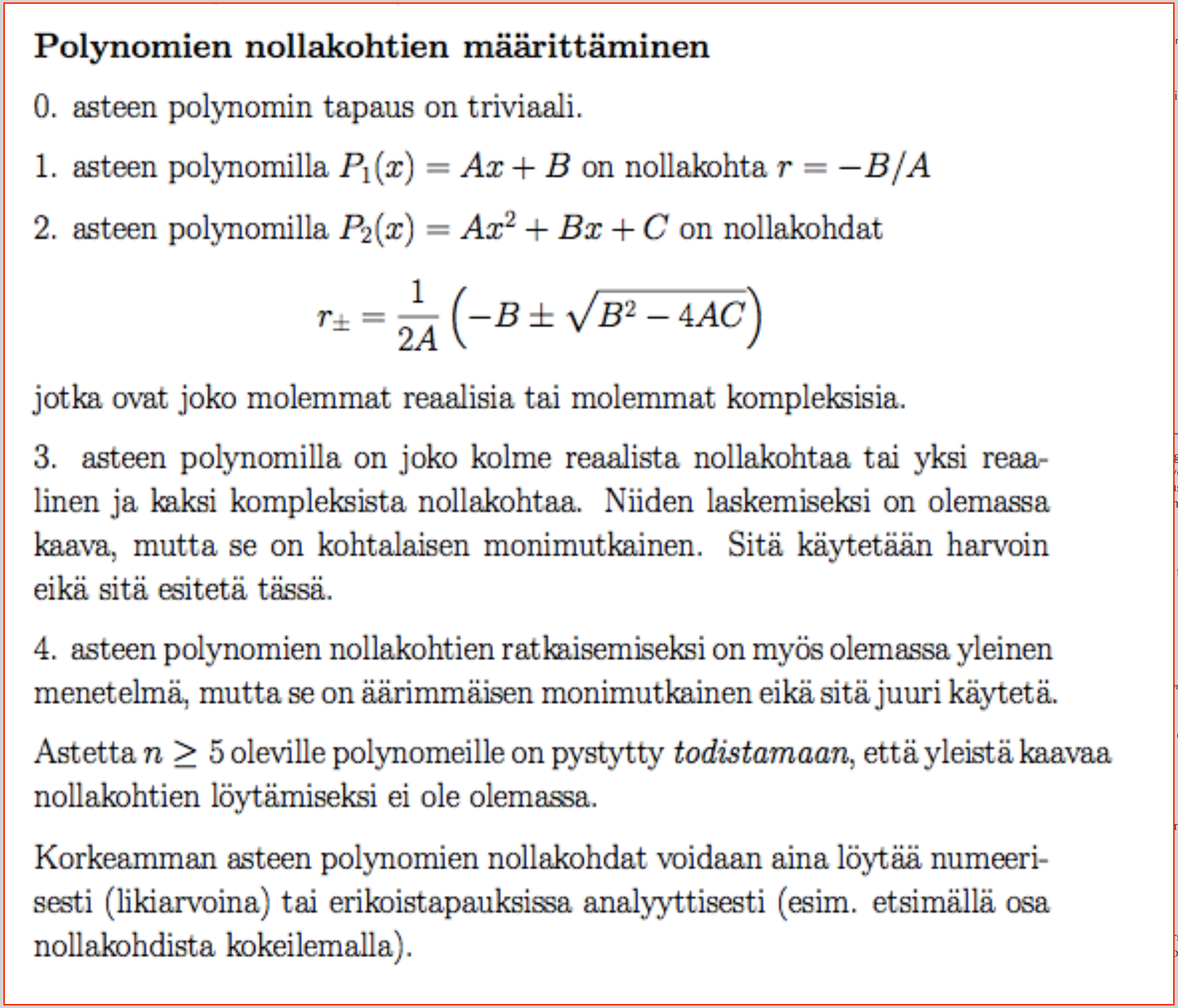
polynomial can be factored into a product of real (possibly repeated) linear factors and real (also possibly repeated) quadratic factors having no real zeros.

Polynomin nollakohdat ja jako alemman asteen tekijöihin (jatkoa)

Jos nyt luku r_2 on astetta $n-1$ olevan osamääräpolynomin Q_{n-1} nollakohta, voidaan edellä esitetty päättely toistaa Q_{n-1}:lle, jolloin alkuperäinen polynomi P voidaan kirjoittaa muodossa $P(x) = (x - r_1)(x - r_2)Q_{n-2}(x)$, missä $Q_{n-2}(x)$ on astetta $n-2$ oleva polynomi. Näin jatkamalla voidaan päätellä, että astetta n olevalla polynomilla on korkeintaan n nollakohtaa, ja että *jos* luvut $r_1, r_2, ...r_n$ ovat nämä nollakohdat, niin polynomi P voidaan kirjoittaa muodossa

$$P(x) = a_n(x - r_1)(x - r_2)...(x - r_n). \qquad (4)$$

HUOM: Voidaan osoittaa, että jokaisella n:nnen asteen polynomilla todellakin on n kpl. nollakohtia, mutta ne eivät välttämättä ole reaalilukuja (vaan kompleksilukuja) ja että ne eivät välttämättä ole keskenään erisuuria. Lisäksi voidaan osoittaa, että jokainen reaalikertoiminen polynomi voidaan jakaa yksikäsitteisesti korkeintaan 2. astetta olevien reaalikertoimisten polynomien tuloksi.



Polynomien nollakohtien määrittäminen

0. asteen polynomin tapaus on triviaali.

1. asteen polynomilla $P_1(x) = Ax + B$ on nollakohta $r = -B/A$

2. asteen polynomilla $P_2(x) = Ax^2 + Bx + C$ on nollakohdat

$$r_{\pm} = \frac{1}{2A}\left(-B \pm \sqrt{B^2 - 4AC}\right)$$

jotka ovat joko molemmat reaalisia tai molemmat kompleksisia.

3. asteen polynomilla on joko kolme reaalista nollakohtaa tai yksi reaalinen ja kaksi kompleksista nollakohtaa. Niiden laskemiseksi on olemassa kaava, mutta se on kohtalaisen monimutkainen. Sitä käytetään harvoin eikä sitä esitetä tässä.

4. asteen polynomien nollakohtien ratkaisemiseksi on myös olemassa yleinen menetelmä, mutta se on äärimmäisen monimutkainen eikä sitä juuri käytetä.

Astetta $n \geq 5$ oleville polynomeille on pystytty *todistamaan*, että yleistä kaavaa nollakohtien löytämiseksi ei ole olemassa.

Korkeamman asteen polynomien nollakohdat voidaan aina löytää numeerisesti (likiarvoina) tai erikoistapauksissa analyyttisesti (esim. etsimällä osa nollakohdista kokeilemalla).

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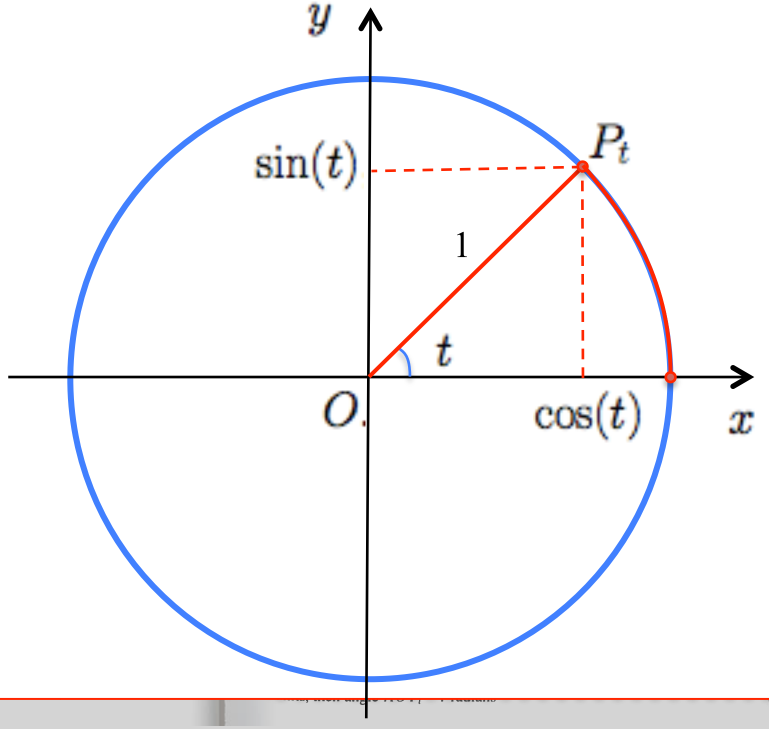
Trigonometriset funktiot

Tarkastellaan *yksikköympyrää* jonka keskipiste on origossa O. Ympyrän yhtälö on $x^2 + y^2 = 1$. Olkoon P_t sellainen piste yksikköympyrän kehällä, että jana (ympyrän eräs säde) OP_t muodostaa x -akselin kanssa kulman t (ks. kuva).

Määritellään funktiot 'cos' (**kosinifunktio**) ja 'sin' (**sinifunktio**) siten että ko. pisteen x -koordinaatti on $\cos(t)$ ja y -koordinaatti on $\sin(t)$, ts. $P_t = (\cos(t), \sin(t))$. Huom: sekä sini- että kosinifunktio ovat kuvauksia $\mathbb{R} \to [-1, 1]$

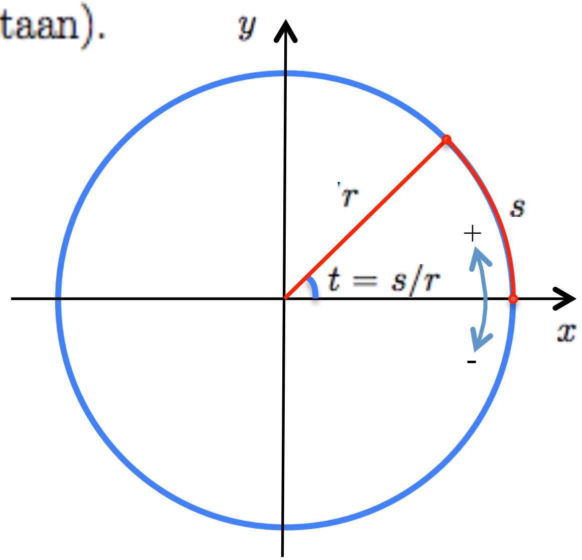


EXERCISES P.6

Find the roots of the polynomials in Exercises 1–12. If a ro
repeated, give its multiplicity. Also, write each polynomial
product of linear factors.

1. $x^2 + 7x + 10$
2. $x^2 - 3x - 10$
3. $x^2 + 2x + 2$
4. $x^2 - 6x + 13$
5. $16x^4 - 8x^2 + 1$
6. $x^4 + 6x^3 + 9x^2$
7. $x^3 + 1$
8. $x^4 - 1$
9. $x^6 - 3x^4 + 3x^2 - 1$
10. $x^5 - x^4 - 16x + 16$

Huom: kulmaa voidaan mitata useammalla eri asteikolla, joista tavallisimmat ovat *aste* ja *radiaani*. Aste määriltellään kulmaksi joka on 1/360 kertaa täyden ympyrän kulma (ts. täysi ympyrä on 360°). Kulma radiaaniyksiköissä määritellään ympyrän kehän pituuden s ja ympyrän säteen r välisenä suhteena, siis $t = s/r$ (rad), missä kulma t on ympyräsegmentin keskuskulma (ks. kuva). Huom: saman muotoisille ympyräsegmenteille suhde s/r ei riipu ympyrän koosta, joten myöskään kulman määrittely ei riipu siitä. Täyden ympyrän kulma $= 2\pi$. Koska kulma on määritelty kahden pituuden suhteena, se on itseasiassa dimensioton suure. Yleisen käytännön mukaan, jos kulma ilmoitetaan asteina, asteluku merkitään symbolilla °, esim. $\alpha = 30°$. Jos kulma ilmoitetaan radiaaneina, se merkitään dimensiottomana lukuna, esim. $\alpha = \pi/6$. Kulma määritellään positiivisena, jos se on referenssisuunnasta (esim. x -akselista) vastapäivään (ns. positiiviseen kiertosuuntaan) ja negatiivisena jos se on siitä myötäpäivään (negatiiviseen kiertosuuntaan).



EXERCISES P.6

Find the roots of the polynomials in E
repeated, give its multiplicity. Also, w
product of linear factors.

1. $x^2 + 7x + 10$
2. x
3. $x^2 + 2x + 2$
4. x
5. $16x^4 - 8x^2 + 1$
6. x
7. $x^3 + 1$
8. x
9. $x^6 - 3x^4 + 3x^2 - 1$
10. x

DEFINITION 6

The **radian measure** of angle AOP_t is t radians:

$$\angle AOP_t = t \text{ radians.}$$

We are more used
halfway (π units of dis

$$\pi \text{ radians} = 180°.$$

To convert degrees to
multiply by $180/\pi$.

Angle convention

In calculus it is ass
or other units are
mean $\pi/3$ radians

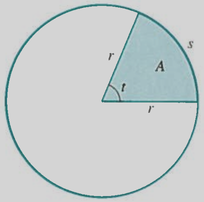


Figure P.68 Arc length $s = rt$
Sector area $A = r^2t/2$

EXAMPLE 1 Ar
an
and the area A of the se

Solution The length
circle that the angle t is

$$s = \frac{t}{2\pi}(2\pi r) = r$$

Similarly, the area A of
πr^2 of the whole circle

$$A = \frac{t}{2\pi}(\pi r^2) =$$

(We will show that the

Using the procedure de
real number t, positive
P_t. (See Figure P.69.)

DEFINITION 7

Cosine and sine

For any real t, the co
$\sin t$) are the x- and

$$\cos t = \text{the } x\text{-coordinate of } P_t$$
$$\sin t = \text{the } y\text{-coordinate of } P_t$$

Because they are defined this way, cosine and sine are often called the **circular functions**. Note that these definitions agree with the ones given earlier for an acute angle. (See formulas (∗) at the beginning of this section.) The triangle involved is P_tOQ_t in Figure P.69.

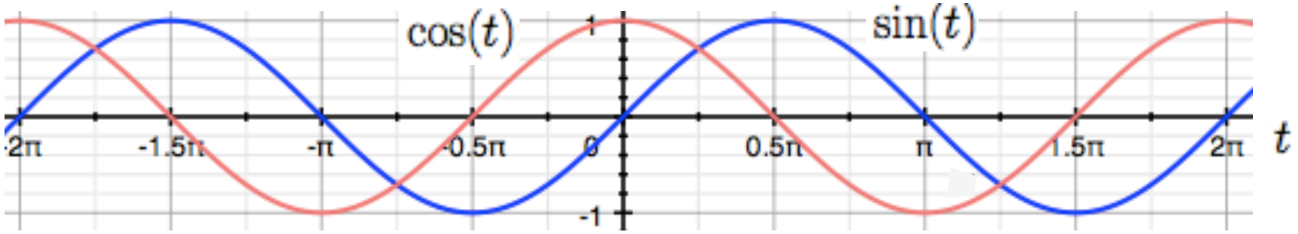
Sini- ja kosinifunktioiden ominaisuuksia

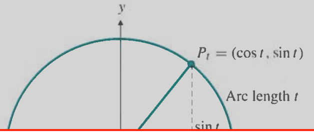
$$\sin^2(t) + \cos^2(t) = 1;$$ Pythagoraan lause!
Huom: $\sin^2(t)$ tarkoittaa $(\sin(t))^2$

$$\sin(t + 2\pi) = \sin(t);$$
$$\cos(t + 2\pi) = \cos(t);$$
Sini ja kosini ovat jaksollisia funktioita, jakso $= 2\pi$

$$\cos(-t) = \cos(t);$$ Kosini on parillinen funktio

$$\sin(-t) = -\sin(t);$$ Sini on pariton funktio





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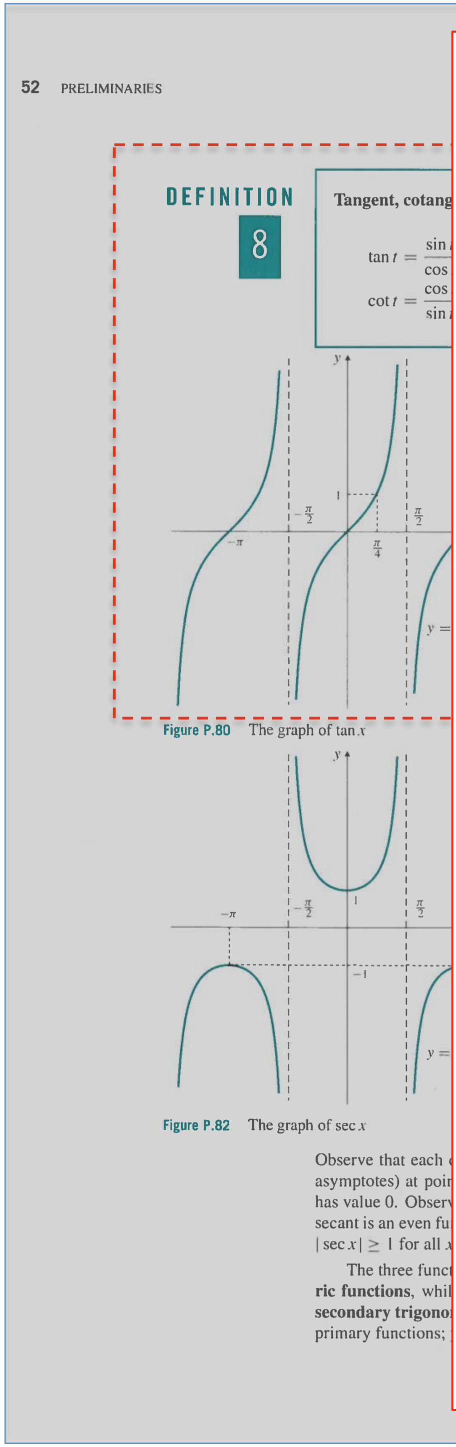
$$\cos^2 t + \sin^2 t = 1.$$

(Note that $\cos^2 t$ means $(\cos t)^2$, not $\cos(\cos t)$. This is an unfortunate notation, but it is used everywhere in technical literature, so you have to get used to it!)

Periodicity. Since C has circumference 2π, adding 2π to t causes the point P_t to go one extra complete revolution around C and end up in the same place: $P_{t+2\pi} = P_t$. Thus, for every t,


52 PRELIMINARIES

DEFINITION 8

Tangent, cotang

Figure P.80 The graph of tan x

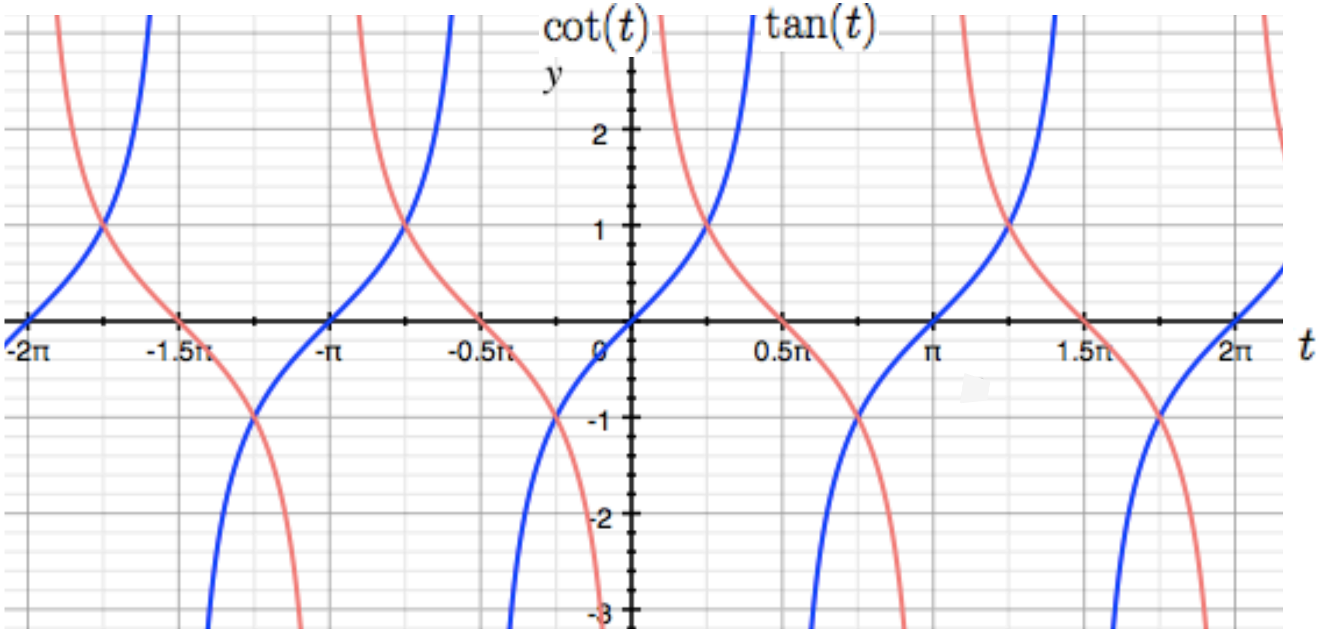
Figure P.82 The graph of sec x

Tangentti- ja kotangenttifunktiot

$$\begin{aligned}\tan(t) &= \frac{\sin(t)}{\cos(t)};\\ \cot(t) &= \frac{\cos(t)}{\sin(t)} = \frac{1}{\tan(t)};\end{aligned}$$

Tangentti ja kotangentti ovat molemmat antisymmetrisiä ja jaksollisia, jaksona π (!)

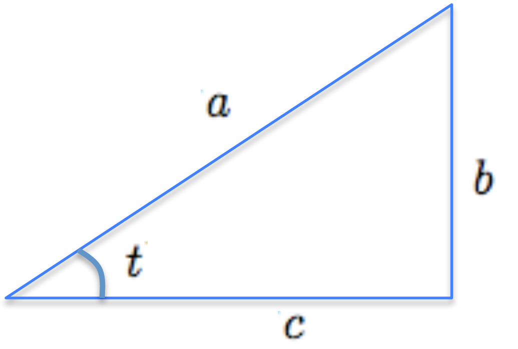
Huom: Tangentti ei ole määritelty pisteissä, joissa $\cos(t) = 0$, ts. kun $t = \frac{\pi}{2} + n\pi; n \in \mathbb{Z}$. Kotangentti ei ole määritelty pisteissä, joissa $\sin(t) = 0$, ts. kun $t = n\pi; n \in \mathbb{Z}$.



Trigonometriset funktiot ja kolmiogeometria

Trigonometriset funktiot liittyvät suorakulmaisten kolmioiden mittasuhteisiin

$$\begin{aligned}\sin(t) &= b/a\\ \cos(t) &= c/a\\ \tan(t) &= b/c\\ \cot(t) &= c/b\end{aligned}$$



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54 PRELIMINARIES

Note that the constant Pi (with an uppercase P) is known to Maple. The `evalf()` function converts its argument to a number expressed as a floating point decimal with 10 significant digits. (This precision can be changed by defining a new value for the variable `Digits`.) Without it, the sine of 30 radians would have been left unexpanded because it is not an integer.

```
> Digits := 20; evalf(100*Pi); sin(30);
```

$$Digits := 20$$

$$314.15926535897932385$$

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beginning of this section, if θ is one of the acute angles in a right-angled triangle, we can refer to the three sides of the triangle as adj (side adjacent θ), opp (side opposite θ), and hyp (hypotenuse). (SeeFigure P.85.) The trigonometric functions of θ can then be expressed as ratios of these sides, in particular:

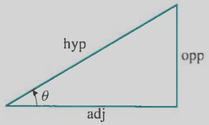


Figure P.85

$$\sin\theta = \frac{\text{opp}}{\text{hyp}}, \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}}, \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}.$$

EXAMPLE 8 Find the unknown sides x and y of the triangle in Figure P.86.

Solution Here, x is the side opposite and y is the side adjacent the 30° angle. The hypotenuse of the triangle is 5 units. Thus,

$$\frac{x}{5} = \sin 30° = \frac{1}{2} \quad \text{and} \quad \frac{y}{5} = \cos 30° = \frac{\sqrt{3}}{2},$$

so $x = \frac{5}{2}$ units and $y = \frac{5\sqrt{3}}{2}$ units.

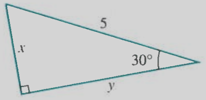


Figure P.86

SECTION P.7: The Trigonometric Functions 55

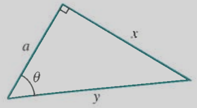


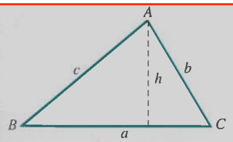
Figure P.87

EXAMPLE 9 For the triangle in Figure P.87, express sides x and y in terms of side a and angle θ.

Solution The side x is opposite the angle θ, and y is the hypotenuse. The side adjacent θ is a. Thus,

$$\frac{x}{a} = \tan\theta \quad \text{and} \quad \frac{a}{y} = \cos\theta.$$

Hence, $x = a\ \tan\theta$ and $y = \frac{a}{\quad} = a\ \sec\theta$.



$$\begin{aligned}&= h^2 + (a - b\cos C)^2 \qquad (\text{since } \cos(\pi - C) = -\cos C)\\ &= b^2\sin^2 C + a^2 - 2ab\cos C + b^2\cos^2 C\\ &= a^2 + b^2 - 2ab\cos C.\end{aligned}$$

The other versions of the Cosine Law can be proved in a similar way.

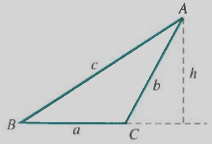


Figure P.89

EXAMPLE 10 A triangle has sides $a = 2$ and $b = 3$ and angle $C = 40°$. Find side c and the sine of angle B.

Solution From the third version of the Cosine Law:

$$c^2 = a^2 + b^2 - 2ab\cos C = 4 + 9 - 12\cos 40° \approx 13 - 12 \times 0.766 = 3.808.$$

Side c is about $\sqrt{3.808} = 1.951$ units in length. Now using Sine Law we get

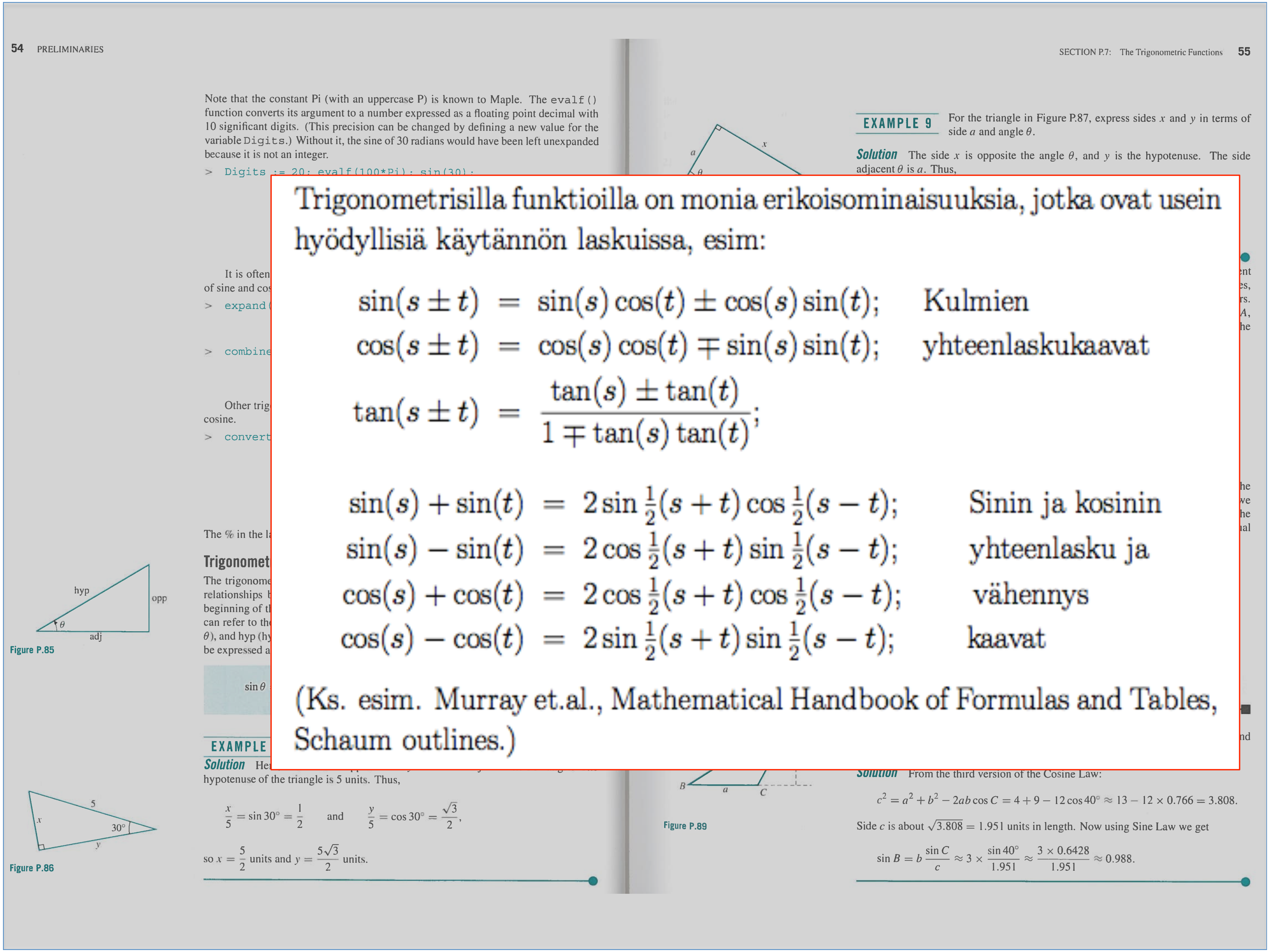
$$\sin B = b\frac{\sin C}{c} \approx 3 \times \frac{\sin 40°}{1.951} \approx \frac{3 \times 0.6428}{1.951} \approx 0.988.$$

Trigonometrisilla funktioilla on monia erikoisominaisuuksia, jotka ovat usein hyödyllisiä käytännön laskuissa, esim:

$$\begin{array}{rcll} \sin(s\pm t) &=& \sin(s)\cos(t)\pm\cos(s)\sin(t); & \text{Kulmien} \\ \cos(s\pm t) &=& \cos(s)\cos(t)\mp\sin(s)\sin(t); & \text{yhteenlaskukaavat} \\ \tan(s\pm t) &=& \dfrac{\tan(s)\pm\tan(t)}{1\mp\tan(s)\tan(t)}; & \end{array}$$

$$\begin{array}{rcll} \sin(s)+\sin(t) &=& 2\sin\frac{1}{2}(s+t)\cos\frac{1}{2}(s-t); & \text{Sinin ja kosinin} \\ \sin(s)-\sin(t) &=& 2\cos\frac{1}{2}(s+t)\sin\frac{1}{2}(s-t); & \text{yhteenlasku ja} \\ \cos(s)+\cos(t) &=& 2\cos\frac{1}{2}(s+t)\cos\frac{1}{2}(s-t); & \text{vähennys} \\ \cos(s)-\cos(t) &=& 2\sin\frac{1}{2}(s+t)\sin\frac{1}{2}(s-t); & \text{kaavat} \end{array}$$

(Ks. esim. Murray et.al., Mathematical Handbook of Formulas and Tables, Schaum outlines.)



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> Digits := 20; evalf(100*Pi); sin(30);
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Figure P.85

EXAMPLE

Solution He
hypotenuse of the triangle is 5 units. Thus,

$$\frac{x}{5}=\sin 30^\circ=\frac{1}{2} \quad\text{and}\quad \frac{y}{5}=\cos 30^\circ=\frac{\sqrt{3}}{2},$$

so $x=\frac{5}{2}$ units and $y=\frac{5\sqrt{3}}{2}$ units.

Figure P.86

Figure P.89

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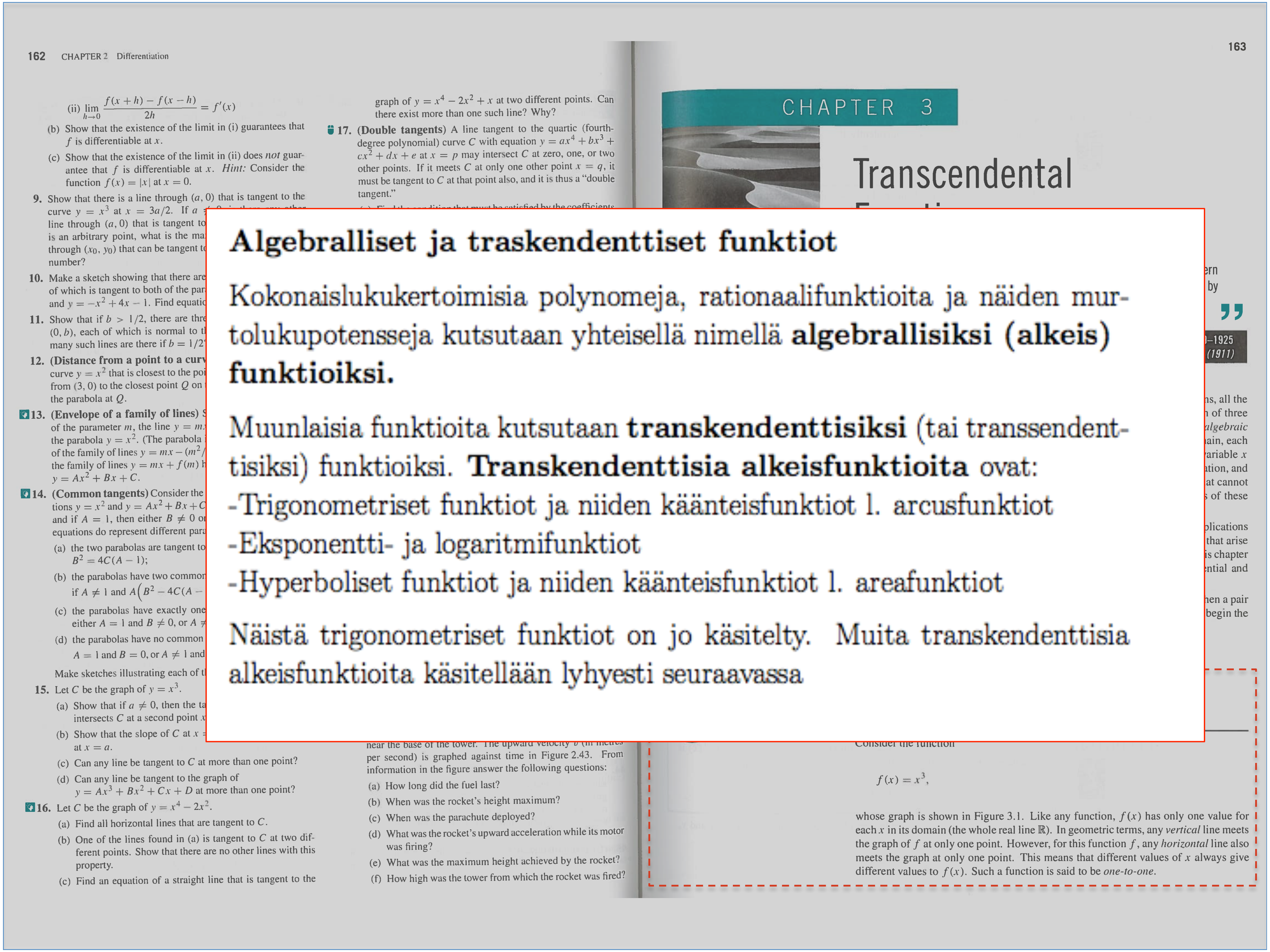
Solution From the third version of the Cosine Law:

$$c^2=a^2+b^2-2ab\cos C=4+9-12\cos 40^\circ\approx 13-12\times 0.766=3.808.$$

Side c is about $\sqrt{3.808}=1.951$ units in length. Now using Sine Law we get

$$\sin B=b\,\frac{\sin C}{c}\approx 3\times\frac{\sin 40^\circ}{1.951}\approx\frac{3\times 0.6428}{1.951}\approx 0.988.$$



Algebralliset ja traskendenttiset funktiot

Kokonaislukukertoimisia polynomeja, rationaalifunktioita ja näiden murtolukupotensseja kutsutaan yhteisellä nimellä **algebrallisiksi (alkeis) funktioiksi.**

Muunlaisia funktioita kutsutaan **transkendenttisiksi** (tai transsendenttisiksi) funktioiksi. **Transkendenttisia alkeisfunktioita** ovat:

-Trigonometriset funktiot ja niiden käänteisfunktiot l. arcusfunktiot

-Eksponentti- ja logaritmifunktiot

-Hyperboliset funktiot ja niiden käänteisfunktiot l. areafunktiot

Näistä trigonometriset funktiot on jo käsitelty. Muita transkendenttisia alkeisfunktioita käsitellään lyhyesti seuraavassa

DEFINITION 1

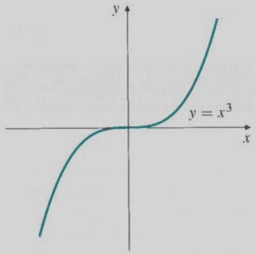


Figure 3.1 The graph of $f(x) = x^3$

Do not confuse the -1 in f^{-1} with an exponent. The inverse f^{-1} is *not* the reciprocal $1/f$. If we want to denote the reciprocal $1/f(x)$ with an exponent we can write it as $\big(f(x)\big)^{-1}$.

Figure 3.2

(a) f is one-to-one and has an inverse. $y = f(x)$ means the same thing as $x = f^{-1}(y)$

(b) g is not one-to-one

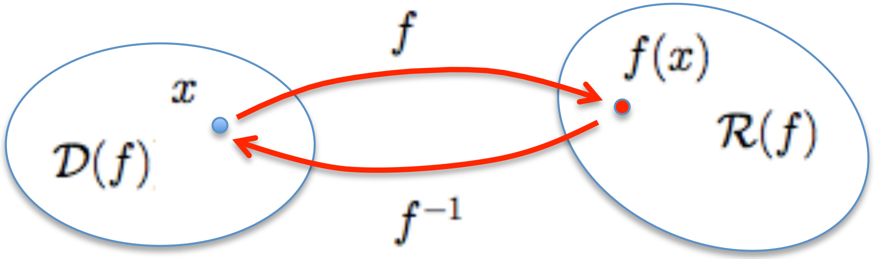
DEFINITION 2

Käänteisfunktio.

Funktio $f : \mathcal{D}(f) \to S$ on **injektio** eli **yksi-yhteen kuvaus** jos mitkään kaksi määrittelyjoukon $\mathcal{D}(f)$ eri alkiota eivät kuvaudu samaksi maalijoukon alkioksi, ts. $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.

Funktio on **surjektio** jos jokainen maalijoukon alkio on jonkin määrittelyjoukon alkion kuva, ts. jos $\mathcal{R}(f) = S$.

Funktio on **bijektio**, jos se on surjektio ja injektio. Tässä tapauksessa funktiolla on **käänteisfunktio** $f^{-1} : \mathcal{R}(f) \to \mathcal{D}(f)$ (ts. $\mathcal{D}(f^{-1}) = \mathcal{R}(f)$ ja $\mathcal{R}(f^{-1}) = \mathcal{D}(f)$)



Käänteisfunktion yleisiä ominaisuuksia.

$$\begin{aligned} y &= f(x) \Leftrightarrow x = f^{-1}(y) \\ (f^{-1})^{-1} &= f \\ f \circ f^{-1}(x) &= f(f^{-1}(x)) = x \quad \text{ja} \quad f^{-1} \circ f(x) = f^{-1}(f(x)) = x \end{aligned}$$

(ts. $f \circ f^{-1}$ ja $f^{-1} \circ f$ ovat **identtisiä kuvauksia**)

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Jos funktio $f : \mathbb{R} \to \mathbb{R}$ on bijektio ja jatkuva (funktion jatkuvuus määritellään tarkemmin myöhemmin), sen kuvaaja $y = f(x)$ on aidosti monotoninen (aidosti kasvava tai vähenevä). Käänteisfunktion f^{-1} kuvaaja $y = f^{-1}(x)$ saadaan tällöin alkuperäisen funktion kuvaajasta peilaamalla se suoran $y = x$ suhteen.

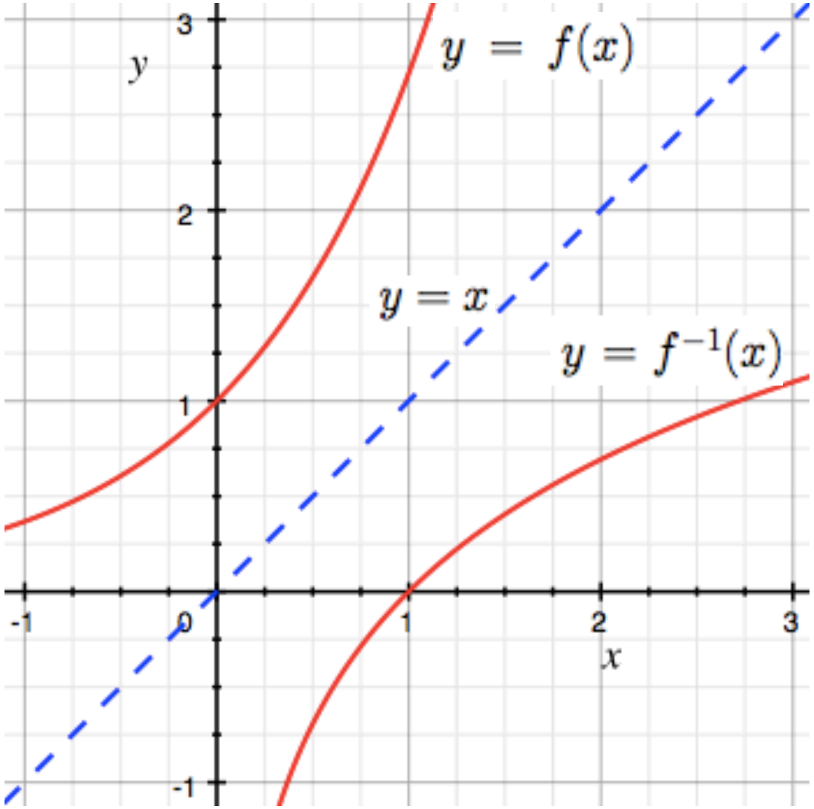
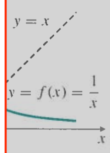


Figure 3.3 The graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ i the line $y = x$



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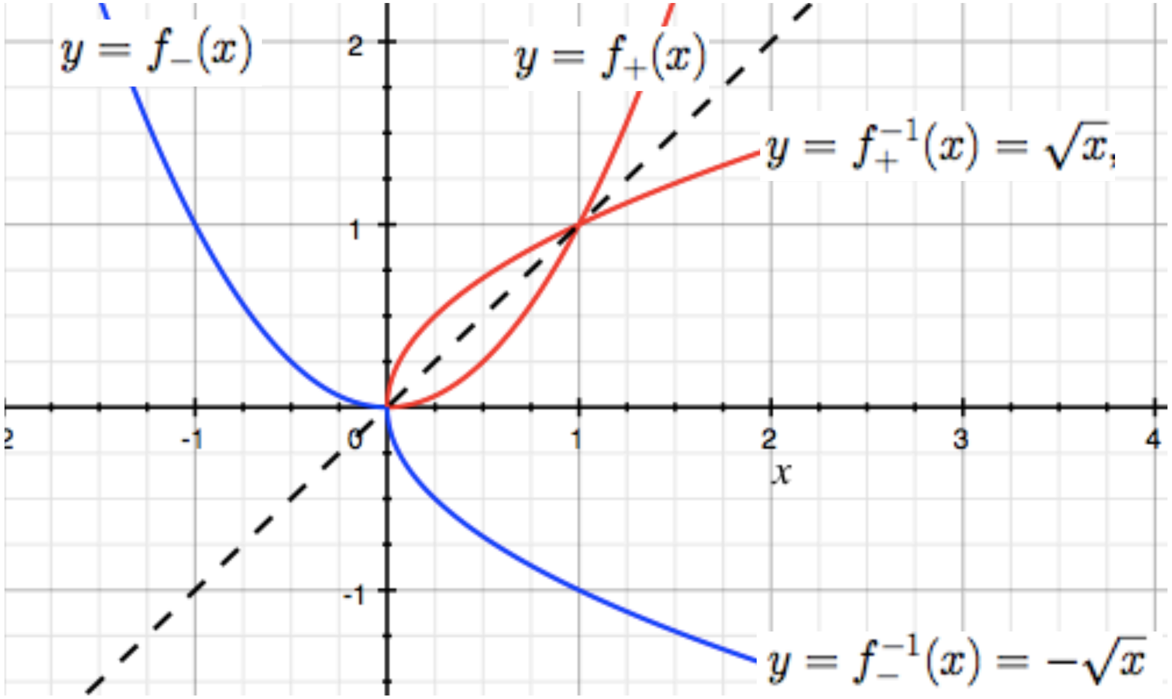
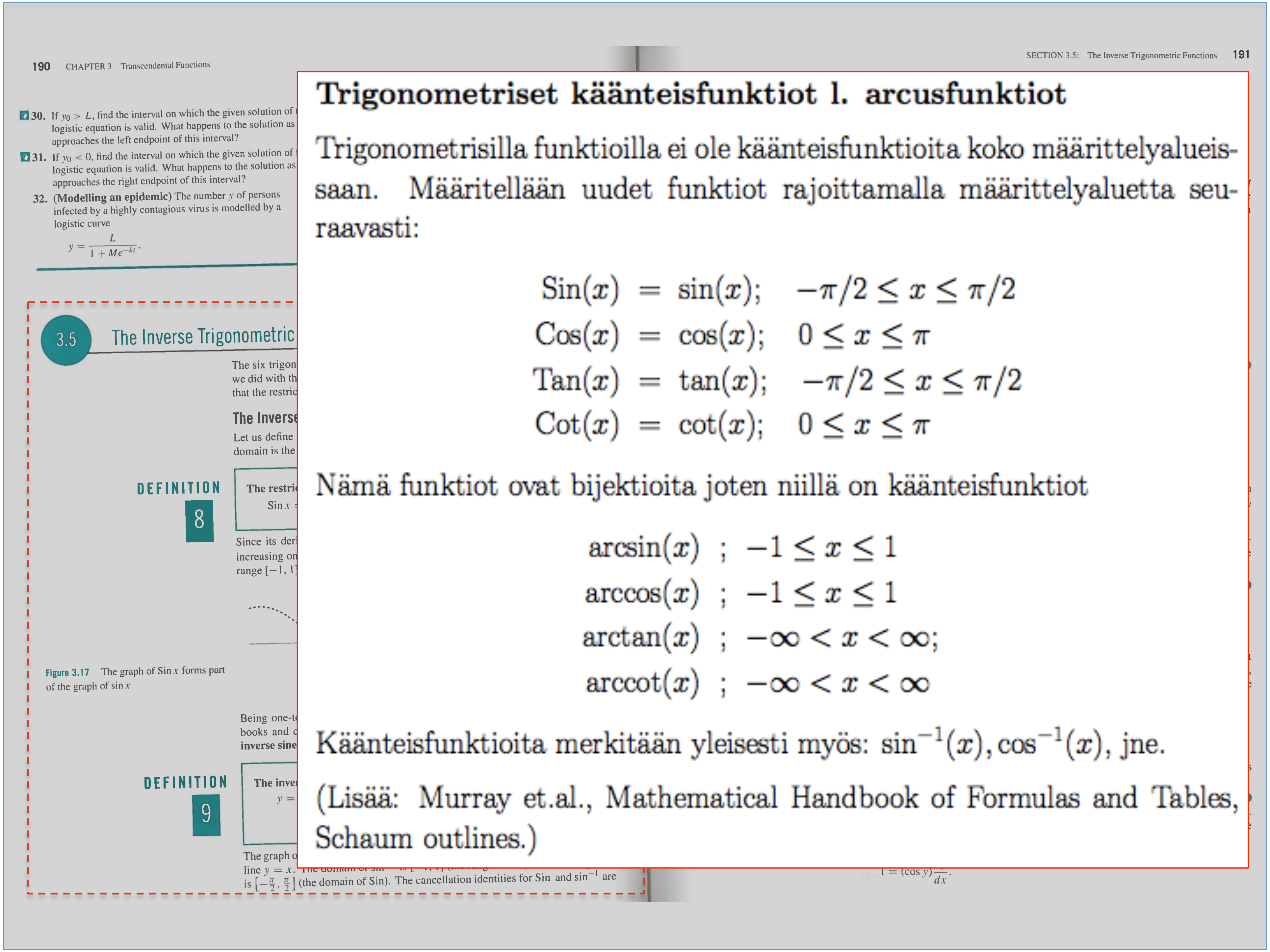
graphs of g and g^{-1}.

Figure 3.3 The graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ in the line $y = x$

Huom: monet tavanomaiset funktiot eivä ole bijektioita koko määrittelyalueessaan, jolloin niillä ei myöskään ole käänteisfunktiota. Rajoittamalla määrittely- ja maalijoukkoa sopivasti, päästään kuitenkin usein tilanteeseen, jossa näin ('lievästi') uudelleen määritelty funktio on bijektio ja sillä siis on käänteisfunktio. Esim. funktiolla $f(x) = x^2$, jonka määrittelyjoukko $\mathcal{D}(f) = \mathbb{R}$ ja arvojoukko $\mathcal{R}(f) = \mathbb{R}_+ \equiv \{x \in \mathbb{R} | x \geq 0\}$ ei ole käänteisfunktiota koko $\mathbb{R}$:ssa. Funktiolla $f_+(x) = x^2 : \mathbb{R}_+ \to \mathbb{R}_+$ sensijaan on käänteisfunktio $f_+^{-1}(x) = \sqrt{x}$. Samoin funktiolla $f_-(x) = x^2 : \mathbb{R}_- \to \mathbb{R}_+$, missä $\mathbb{R}_- = \{x \in \mathbb{R} | x \leq 0\}$ on käänteisfunktio $f_-^{-1}(x) = -\sqrt{x}$.





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30. If $y_0 > L$, find the interval on which the given solution of
logistic equation is valid. What happens to the solution as
approaches the left endpoint of this interval?

31. If $y_0 < 0$, find the interval on which the given solution of
logistic equation is valid. What happens to the solution as
approaches the right endpoint of this interval?

32. **(Modelling an epidemic)** The number y of persons infected by a highly contagious virus is modelled by a logistic curve

$$y = \frac{L}{1+Me^{-kt}},$$

3.5 The Inverse Trigonometric

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DEFINITION 8

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Figure 3.17 The graph of Sin x forms part of the graph of sin x

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DEFINITION 9

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line $y = x$. The domain of sin
is $[-\frac{\pi}{2}, \frac{\pi}{2}]$ (the domain of Sin). The cancellation identities for Sin and $\sin^{-1}$ are

SECTION 3.5: The Inverse Trigonometric Functions 191

$1 = (\cos y)\frac{dy}{dx}$.

Trigonometriset käänteisfunktiot l. arcusfunktiot

Trigonometrisilla funktioilla ei ole käänteisfunktioita koko määrittelyalueissaan. Määritellään uudet funktiot rajoittamalla määrittelyaluetta seuraavasti:

$$\begin{aligned}
\mathrm{Sin}(x) &= \sin(x); \quad -\pi/2 \leq x \leq \pi/2 \\
\mathrm{Cos}(x) &= \cos(x); \quad 0 \leq x \leq \pi \\
\mathrm{Tan}(x) &= \tan(x); \quad -\pi/2 \leq x \leq \pi/2 \\
\mathrm{Cot}(x) &= \cot(x); \quad 0 \leq x \leq \pi
\end{aligned}$$

Nämä funktiot ovat bijektioita joten niillä on käänteisfunktiot

$$\begin{aligned}
&\arcsin(x) \;\; ; \;\; -1 \leq x \leq 1 \\
&\arccos(x) \;\; ; \;\; -1 \leq x \leq 1 \\
&\arctan(x) \;\; ; \;\; -\infty < x < \infty; \\
&\mathrm{arccot}(x) \;\; ; \;\; -\infty < x < \infty
\end{aligned}$$

Käänteisfunktioita merkitään yleisesti myös: $\sin^{-1}(x), \cos^{-1}(x)$, jne.

(Lisää: Murray et.al., Mathematical Handbook of Formulas and Tables, Schaum outlines.)

Since we are assuming that the graph $y = f(x)$ has a *nonhorizontal* tangent line at any x in (a, b), its reflection, the graph $y = f^{-1}(x)$, has a *nonvertical* tangent line at any x in the interval between $f(a)$ and $f(b)$. Therefore, f^{-1} is differentiable at any such x. (See Figure 3.6.)

Let $y = f^{-1}(x)$. We want to find dy/dx. Solve the equation $y = f^{-1}(x)$ for $x = f(y)$ and differentiate implicitly with respect to x to obtain

$$1 = f'(y)\frac{dy}{dx}, \qquad \text{so} \qquad \frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'\big(f^{-1}(x)\big)}.$$

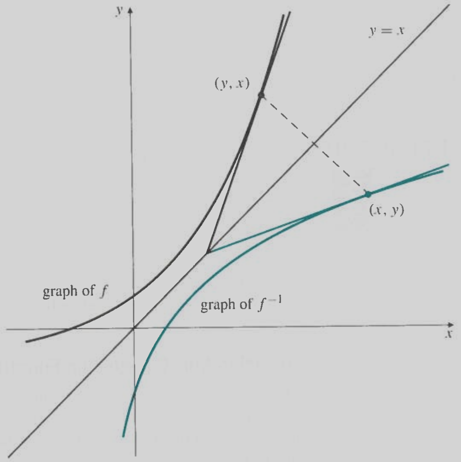


Figure 3.6 Tangents to the graphs of f and f^{-1}

Therefore, the slope of the graph of f^{-1} at (x, y) is the reciprocal of the slope of the graph of f at (y, x) (Figure 3.6) and

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'\big(f^{-1}(x)\big)}.$$

In Leibniz notation we have $\displaystyle \left.\frac{dy}{dx}\right|_{x} = \frac{1}{\left.\dfrac{dx}{dy}\right|_{y=f^{-1}(x)}}.$

EXAMPLE 4 Show that $f(x) = x^3 + x$ is one-to-one on the whole real line, and, noting that $f(2) = 10$, find $\big(f^{-1}\big)'(10)$.

Solution Since $f'(x) = 3x^2 + 1 > 0$ for all real numbers x, f is increasing and therefore one-to-one and invertible. If $y = f^{-1}(x)$, then

$$\begin{aligned} x = f(y) = y^3 + y \quad &\Longrightarrow \quad 1 = (3y^2 + 1)y' \\ &\Longrightarrow \quad y' = \frac{1}{3y^2 + 1}. \end{aligned}$$

Now $x = f(2) = 10$ implies $y = f^{-1}(10) = 2$. Thus,

$$\big(f^{-1}\big)'(10) = \left.\frac{1}{3y^2 + 1}\right|_{y=2} = \frac{1}{13}.$$

EXERCISES 3.1

Show that the functions f in Exercises 1–12 are one-to-one, and calculate the inverse functions f^{-1}. Specify the domains and ranges of f and f^{-1}.

1. $f(x) = x - 1$

2. $f(x) = 2x - 1$

3. $f(x) = \sqrt{x - 1}$

4. $f(x) = -\sqrt{x - 1}$

5. $f(x) = x^3$

6. $f(x) = 1 + \sqrt[3]{x}$

7. $f(x) = x^2, \quad x \le 0$

8. $f(x) = (1 - 2x)^3$

9. $f(x) = \dfrac{1}{x + 1}$

10. $f(x) = \dfrac{x}{1 + x}$

11. $f(x) = \dfrac{1 - 2x}{1 + x}$

12. $f(x) = \dfrac{x}{\sqrt{x^2 + 1}}$

In Exercises 13–20, f is a one-to-one function with inverse f^{-1}. Calculate the inverses of the given functions in terms of f^{-1}.

13. $g(x) = f(x) - 2$

14. $h(x) = f(2x)$

15. $k(x) = -3f(x)$

16. $m(x) = f(x - 2)$

17. $p(x) = \dfrac{1}{1 + f(x)}$

18. $q(x) = \dfrac{f(x) - 3}{2}$

19. $r(x) = 1 - 2f(3 - 4x)$

20. $s(x) = \dfrac{1 + f(x)}{1 - f(x)}$

In Exercises 21–23, show that the given function is one-to-one and find its inverse.

21. $f(x) = \begin{cases} x^2 + 1 & \text{if } x \ge 0 \\ x + 1 & \text{if } x < 0 \end{cases}$

22. $g(x) = \begin{cases} x^3 & \text{if } x \ge 0 \\ x^{1/3} & \text{if } x < 0 \end{cases}$

23. $h(x) = x|x| + 1$

24. Find $f^{-1}(2)$ if $f(x) = x^3 + x$.

25. Find $g^{-1}(1)$ if $g(x) = x^3 + x - 9$.

26. Find $h^{-1}(-3)$ if $h(x) = x|x| + 1$.

27. Assume that the function $f(x)$ satisfies $f'(x) = \dfrac{1}{x}$ and that f is one-to-one. If $y = f^{-1}(x)$, show that $dy/dx = y$.

28. Find $\big(f^{-1}\big)'(x)$ if $f(x) = 1 + 2x^3$.

29. Show that $f(x) = \dfrac{4x^3}{x^2 + 1}$ has an inverse and find $\big(f^{-1}\big)'(2)$.

30. Find $\big(f^{-1}\big)'(-2)$ if $f(x) = x\sqrt{3 + x^2}$.

31. If $f(x) = x^2/(1 + \sqrt{x})$, find $f^{-1}(2)$ correct to 5 decimal places.

32. If $g(x) = 2x + \sin x$, show that g is invertible, and find $g^{-1}(2)$ and $(g^{-1})'(2)$ correct to 5 decimal places.

33. Show that $f(x) = x \sec x$ is one-to-one on $(-\pi/2, \pi/2)$. What is the domain of $f^{-1}(x)$? Find $(f^{-1})'(0)$.

34. If f and g have respective inverses f^{-1} and g^{-1}, show that the composite function $f \circ g$ has inverse $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

35. For what values of the constants a, b, and c is the function $f(x) = (x - a)/(bx - c)$ self-inverse?

36. Can an even function be self-inverse? an odd function?

37. In this section it was claimed that an increasing (or decreasing) function defined on a single interval is necessarily one-to-one. Is the converse of this statement true? Explain.

38. Repeat Exercise 37 with the added assumption that f is continuous on the interval where it is defined.

3.2 Exponential and Logarithmic Functions

To begin we review exponential and logarithmic functions as you may have encountered them in your previous mathematical studies. In the following sections we will approach these functions from a different point of view and learn how to find their derivatives.

Exponentials

An **exponential function** is a function of the form $f(x) = a^x$, where the **base** a is a positive constant and the **exponent** x is the variable. Do not confuse such functions with **power** functions such as $f(x) = x^a$, where the base is variable and the exponent is constant. The exponential function a^x can be defined for integer and rational exponents x as follows:

DEFINITION 4

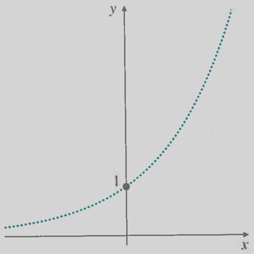


Figure 3.7 $y = 2^x$ for rational x

Eksponenttifunktio

Olkoon a on mv. positiivinen reaaliluku. Tavoitteenamme on määritellä muotoa $f(x) = a^x$ oleva eksponenttifunktio kaikille reaaliluvuille x. Aiemmin määriteltiin jo luvun potenssi luonnollisille luvuille n:

$$a^n = a \cdot a \cdot \ldots \cdot a \quad (n \text{ tekijää}); \quad n \in \mathbb{N}$$

Määritellään nyt:

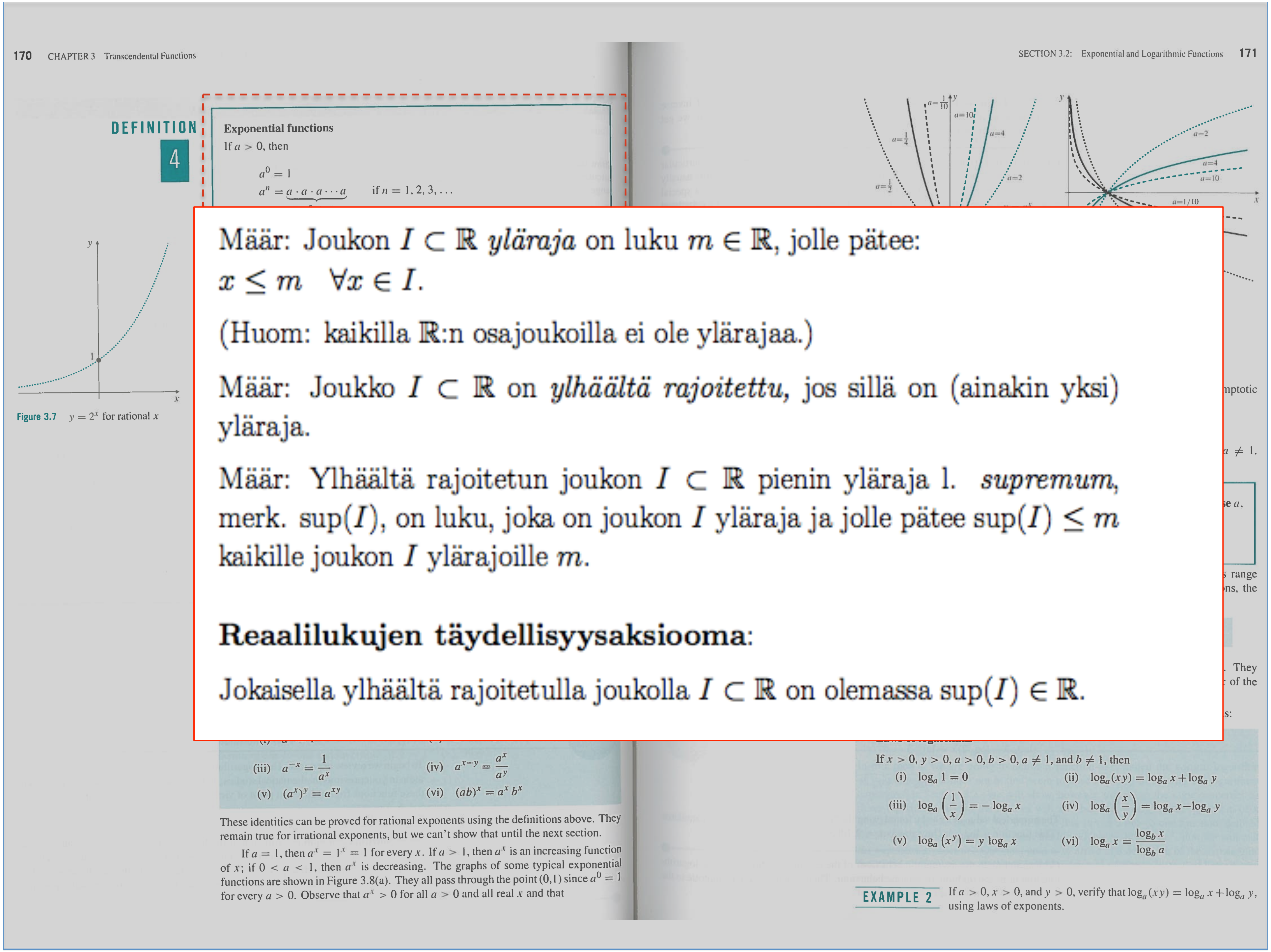
$$\begin{aligned} a^{-n} &= \frac{1}{a^n}; \quad a \neq 0 \quad \text{ja} \\ a^0 &= 1, \end{aligned}$$

jolloin luvun a potenssi (ja siis eksponenttifunktio) on tullut määritellyksi kaikille kokonaisluvuille n.

Määritellään edelleen luvun a (> 0) **m:s juuri** $\sqrt[m]{a}$ *positiivisena* lukuna jolle pätee: $(\sqrt[m]{a})^m = a$ kaikille $m \in \mathbb{N}$. Tämän avulla voidaan eksponenttifunktion määritelmä laajentaa edelleen kaikille muotoa $x = n/m$ oleville rationaaliluvuille. Määrittelemme siis (a -kantaisen) eksponenttifunktion tässä vaiheessa kuvauksena

$$\begin{aligned} &f : \mathbb{Q} \to \{y \in \mathbb{R} | y > 0\} \\ &f(x) = a^x = \sqrt[m]{a^n}; \quad n, m \in \mathbb{Z}, \quad x = n/m. \end{aligned}$$

Eksponenttifunktion määritelmä voidaan laajentaa edelleen irrationaaliluvuille ja siten koko reaalilukujen joukkoon. Tämä tehdään seuraavassa kirjan esityksestä poikkeavalla tavalla käyttämällä reaalilukujen täydellisyysominaisuutta (aksiooma).



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SECTION 3.2: Exponential and Logarithmic Functions 171

DEFINITION 4

Exponential functions

If $a > 0$, then

$$a^0 = 1$$
$$a^n = \underbrace{a \cdot a \cdot a \cdots a} \qquad \text{if } n = 1, 2, 3, \dots$$

Figure 3.7 $y = 2^x$ for rational x

(iii) $a^{-x} = \dfrac{1}{a^x}$ (iv) $a^{x-y} = \dfrac{a^x}{a^y}$

(v) $(a^x)^y = a^{xy}$ (vi) $(ab)^x = a^x b^x$

These identities can be proved for rational exponents using the definitions above. They remain true for irrational exponents, but we can't show that until the next section.

If $a = 1$, then $a^x = 1^x = 1$ for every x. If $a > 1$, then a^x is an increasing function of x; if $0 < a < 1$, then a^x is decreasing. The graphs of some typical exponential functions are shown in Figure 3.8(a). They all pass through the point $(0,1)$ since $a^0 = 1$ for every $a > 0$. Observe that $a^x > 0$ for all $a > 0$ and all real x and that

If $x > 0$, $y > 0$, $a > 0$, $b > 0$, $a \neq 1$, and $b \neq 1$, then

(i) $\log_a 1 = 0$ (ii) $\log_a(xy) = \log_a x + \log_a y$

(iii) $\log_a\left(\dfrac{1}{x}\right) = -\log_a x$ (iv) $\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$

(v) $\log_a(x^y) = y \log_a x$ (vi) $\log_a x = \dfrac{\log_b x}{\log_b a}$

EXAMPLE 2 If $a > 0$, $x > 0$, and $y > 0$, verify that $\log_a(xy) = \log_a x + \log_a y$, using laws of exponents.

Määr: Joukon $I \subset \mathbb{R}$ *yläraja* on luku $m \in \mathbb{R}$, jolle pätee:
$x \leq m \quad \forall x \in I$.

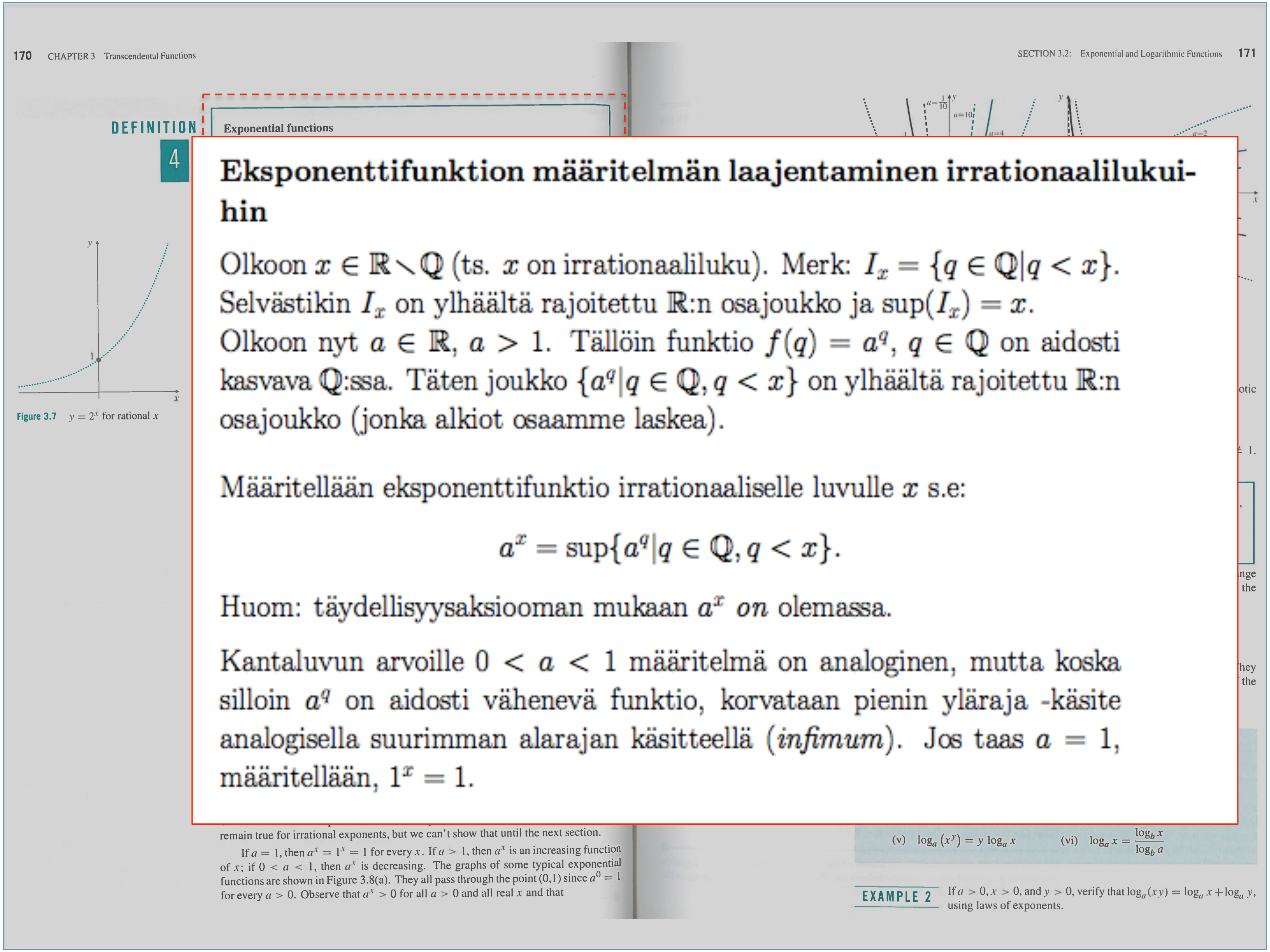
(Huom: kaikilla $\mathbb{R}$:n osajoukoilla ei ole ylärajaa.)

Määr: Joukko $I \subset \mathbb{R}$ on *ylhäältä rajoitettu*, jos sillä on (ainakin yksi) yläraja.

Määr: Ylhäältä rajoitetun joukon $I \subset \mathbb{R}$ pienin yläraja l. *supremum*, merk. $\sup(I)$, on luku, joka on joukon I yläraja ja jolle pätee $\sup(I) \leq m$ kaikille joukon I ylärajoille m.

Reaalilukujen täydellisyysaksiooma:

Jokaisella ylhäältä rajoitetulla joukolla $I \subset \mathbb{R}$ on olemassa $\sup(I) \in \mathbb{R}$.



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DEFINITION 4

Exponential functions

Figure 3.7 $y = 2^x$ for rational x

Eksponenttifunktion määritelmän laajentaminen irrationaalilukuihin

Olkoon $x \in \mathbb{R} \setminus \mathbb{Q}$ (ts. x on irrationaaliluku). Merk: $I_x = \{q \in \mathbb{Q} | q < x\}$. Selvästikin I_x on ylhäältä rajoitettu $\mathbb{R}$:n osajoukko ja $\sup(I_x) = x$. Olkoon nyt $a \in \mathbb{R}$, $a > 1$. Tällöin funktio $f(q) = a^q$, $q \in \mathbb{Q}$ on aidosti kasvava $\mathbb{Q}$:ssa. Täten joukko $\{a^q | q \in \mathbb{Q}, q < x\}$ on ylhäältä rajoitettu $\mathbb{R}$:n osajoukko (jonka alkiot osaamme laskea).

Määritellään eksponenttifunktio irrationaaliselle luvulle x s.e:

$$a^x = \sup\{a^q | q \in \mathbb{Q}, q < x\}.$$

Huom: täydellisyysaksiooman mukaan a^x *on* olemassa.

Kantaluvun arvoille $0 < a < 1$ määritelmä on analoginen, mutta koska silloin a^q on aidosti vähenevä funktio, korvataan pienin yläraja -käsite analogisella suurimman alarajan käsitteellä (*infimum*). Jos taas $a = 1$, määritellään, $1^x = 1$.

remain true for irrational exponents, but we can't show that until the next section.

If $a = 1$, then $a^x = 1^x = 1$ for every x. If $a > 1$, then a^x is an increasing function of x; if $0 < a < 1$, then a^x is decreasing. The graphs of some typical exponential functions are shown in Figure 3.8(a). They all pass through the point $(0,1)$ since $a^0 = 1$ for every $a > 0$. Observe that $a^x > 0$ for all $a > 0$ and all real x and that

(v) $\log_a(x^y) = y \log_a x$ (vi) $\log_a x = \dfrac{\log_b x}{\log_b a}$

EXAMPLE 2 If $a > 0$, $x > 0$, and $y > 0$, verify that $\log_a(xy) = \log_a x + \log_a y$, using laws of exponents.

DEFINITION 4

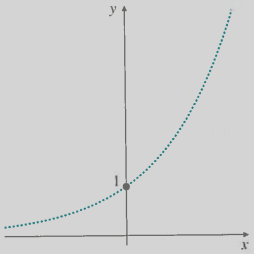
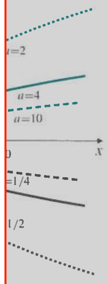


Figure 3.7 $y = 2^x$ for rational x



Eksponenttifunktio kantaluvulle $a > 0$ on näin tullut määriteltyä kaikille reaaliluvuille kuvauksena $f : \mathbb{R} \to \{y | y \in \mathbb{R}, y > 0\}$, $f(x) = a^x$. Voidaan todistaa, että näin määritelty eksponenttifunktio on jatkuva ja derivoituva koko $\mathbb{R}$:ssä ja että sille pätevät samat laskusäännöt kuin alkuperäiselle kokonaislukujen joukossa määritellylle eksponenttifunktiolle. Vaikka määritelmä on hyvin formaalinen, se antaa kuitenkin käytännön menetelmän laskea funktion $f(x) = a^x$; $a \in \mathbb{R}$, $a > 0$ likiarvoja mielivaltaisen tarkasti myös irrationaaliselle luvulle x. Esimerkiksi laskimet ja tietokoneet laskevat eksponenttifunktion numeeriset (liki)arvot (ja kaiken muunkin) käyttäen vain rationaalilukuja. Tämä on aina mahdollista, koska rationaalilukujen joukko on reaalilukujen joukon *tiheä* osajoukko.

Huom: Vaihtamalla x:n ja a:n roolit, voimme e.o. tarkastelun perusteella määritellä myös potenssifunktion positiivisille reaaliluvuille kuvauksena

$$f : \{x \in \mathbb{R} | x > 0\} \to \{y \in \mathbb{R} | y > 0\}$$
$$f(x) = x^a; a \in \mathbb{R}.$$

Jos $a > 0$ on potenssifunktio määritelty myös arvolle $x = 0$ (ts. $f(0) = 0$). Jos $a < 0$, on potenssifunktio määrittelemätön pisteessä $x = 0$.

for every $a > 0$. Observe that $a^x > 0$ for all $a > 0$ and all real x and that

EXAMPLE 2 If $a > 0$, $x > 0$, and $y > 0$, verify that $\log_a(xy) = \log_a x + \log_a y$, using laws of exponents.

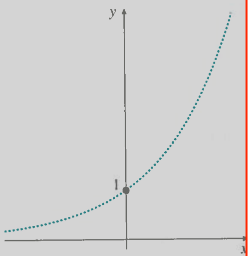
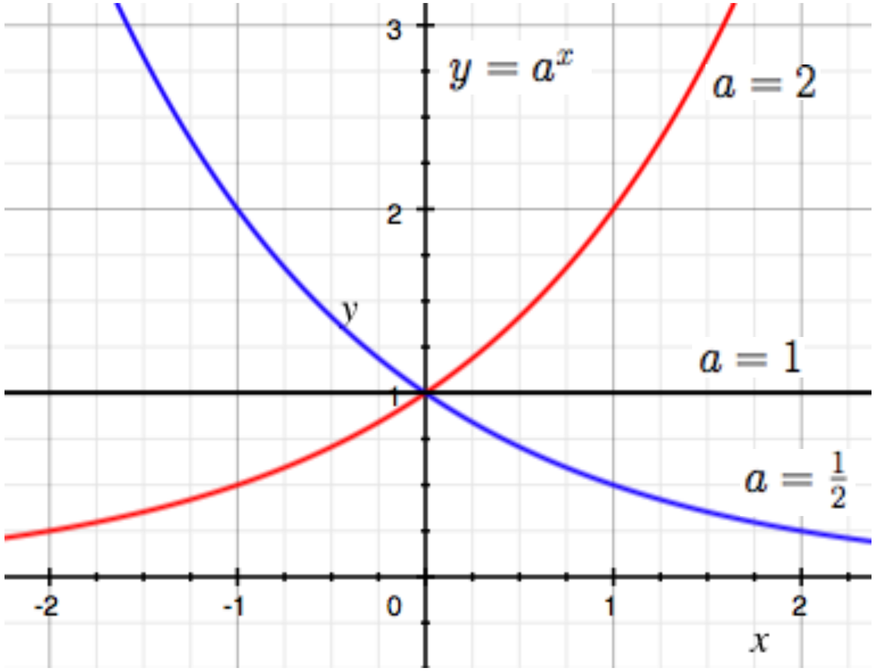


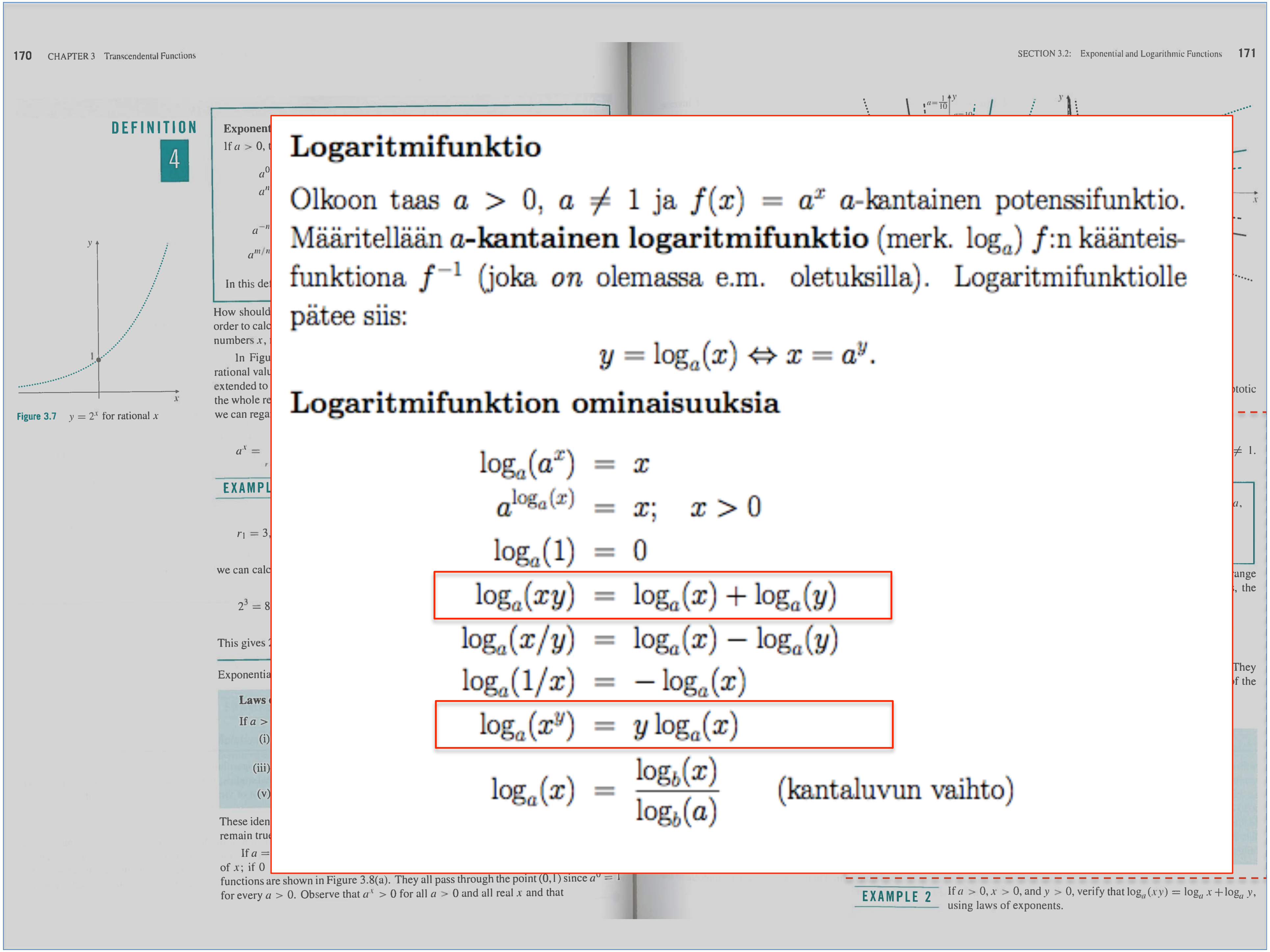
Figure 3.7 $y = 2^x$ for rational x

Eksponenttifunktion perusominaisuuksia

$$\begin{aligned} a^0 &= 1 \\ a^{-x} &= 1/a^x \\ a^{x+y} &= a^x a^y \\ a^{x-y} &= a^x/a^y \\ (a^x)^y &= a^{xy} \\ (ab)^x &= a^x b^x \end{aligned}$$

Huom: eksponenttifunktioita kantaluvulle $a < 0$ ei voida määritellä reaalilukuille. Sen sijaan kompleksilukujen joukossa tämäkin tapaus voidaan käsitellä.





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DEFINITION 4

Figure 3.7 $y = 2^x$ for rational x

functions are shown in Figure 3.8(a). They all pass through the point (0,1) since $a^0 = 1$ for every $a > 0$. Observe that $a^x > 0$ for all $a > 0$ and all real x and that

EXAMPLE 2 If $a > 0$, $x > 0$, and $y > 0$, verify that $\log_a(xy) = \log_a x + \log_a y$, using laws of exponents.

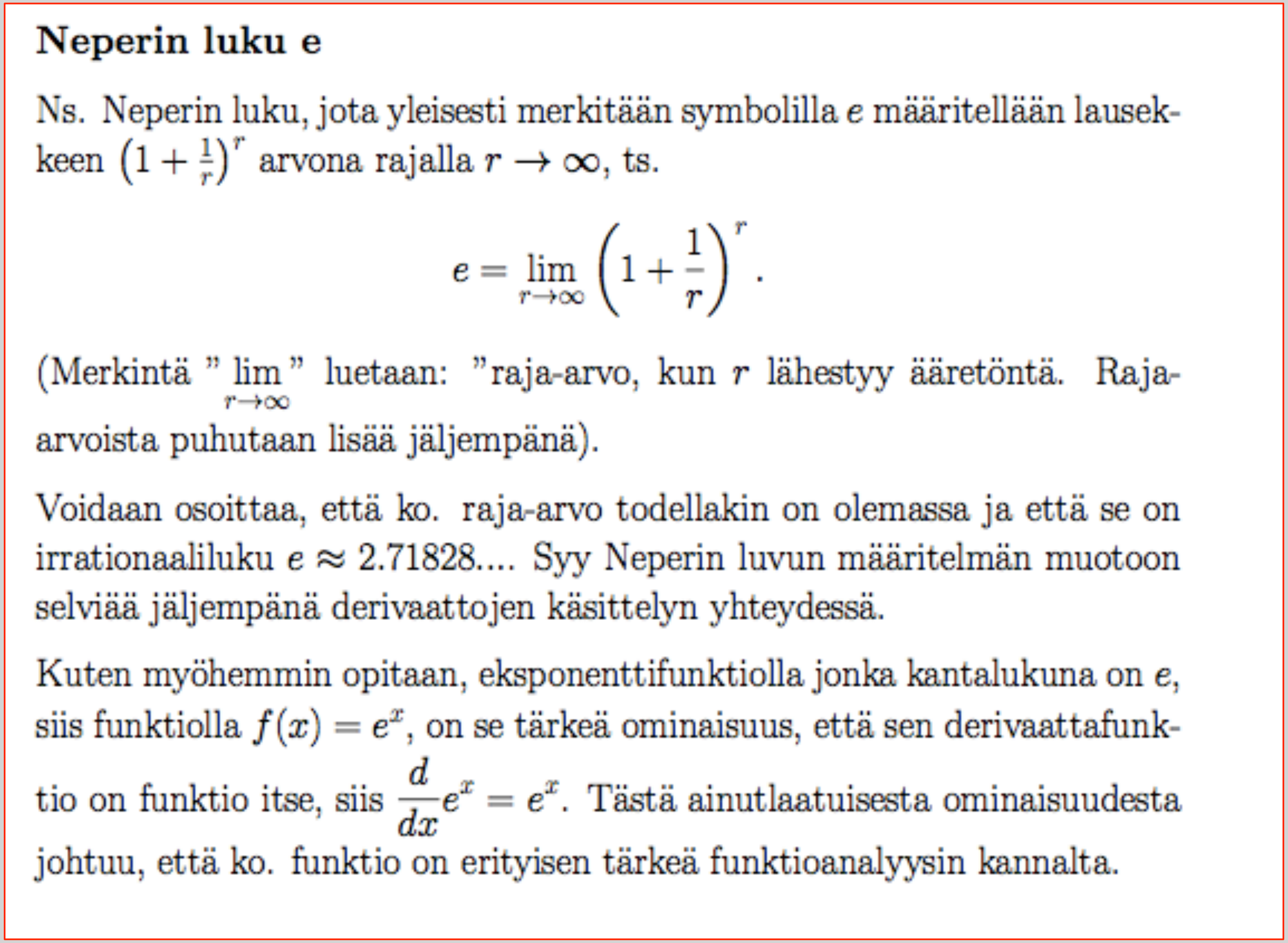
Logaritmifunktio

Olkoon taas $a > 0$, $a \neq 1$ ja $f(x) = a^x$ a-kantainen potenssifunktio. Määritellään **a-kantainen logaritmifunktio** (merk. $\log_a$) f:n käänteisfunktiona f^{-1} (joka *on* olemassa e.m. oletuksilla). Logaritmifunktiolle pätee siis:

$$y = \log_a(x) \Leftrightarrow x = a^y.$$

Logaritmifunktion ominaisuuksia

$$\begin{aligned}
\log_a(a^x) &= x \\
a^{\log_a(x)} &= x; \quad x > 0 \\
\log_a(1) &= 0 \\
\boxed{\log_a(xy) = \log_a(x) + \log_a(y)} \\
\log_a(x/y) &= \log_a(x) - \log_a(y) \\
\log_a(1/x) &= -\log_a(x) \\
\boxed{\log_a(x^y) = y\log_a(x)} \\
\log_a(x) &= \frac{\log_b(x)}{\log_b(a)} \qquad \text{(kantaluvun vaihto)}
\end{aligned}$$



Neperin luku e

Ns. Neperin luku, jota yleisesti merkitään symbolilla e määritellään lausekkeen $\left(1+\frac{1}{r}\right)^r$ arvona rajalla $r \to \infty$, ts.

$$e = \lim_{r\to\infty}\left(1+\frac{1}{r}\right)^r .$$

(Merkintä "$\lim\limits_{r\to\infty}$" luetaan: "raja-arvo, kun r lähestyy ääretöntä. Raja-arvoista puhutaan lisää jäljempänä).

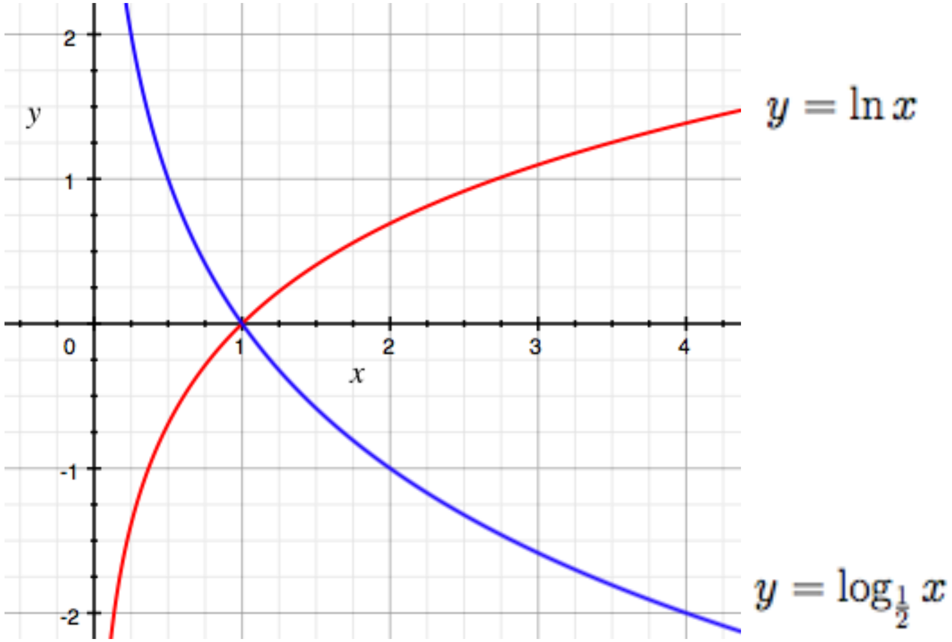
Voidaan osoittaa, että ko. raja-arvo todellakin on olemassa ja että se on irrationaaliluku $e \approx 2.71828\ldots$. Syy Neperin luvun määritelmän muotoon selviää jäljempänä derivaattojen käsittelyn yhteydessä.

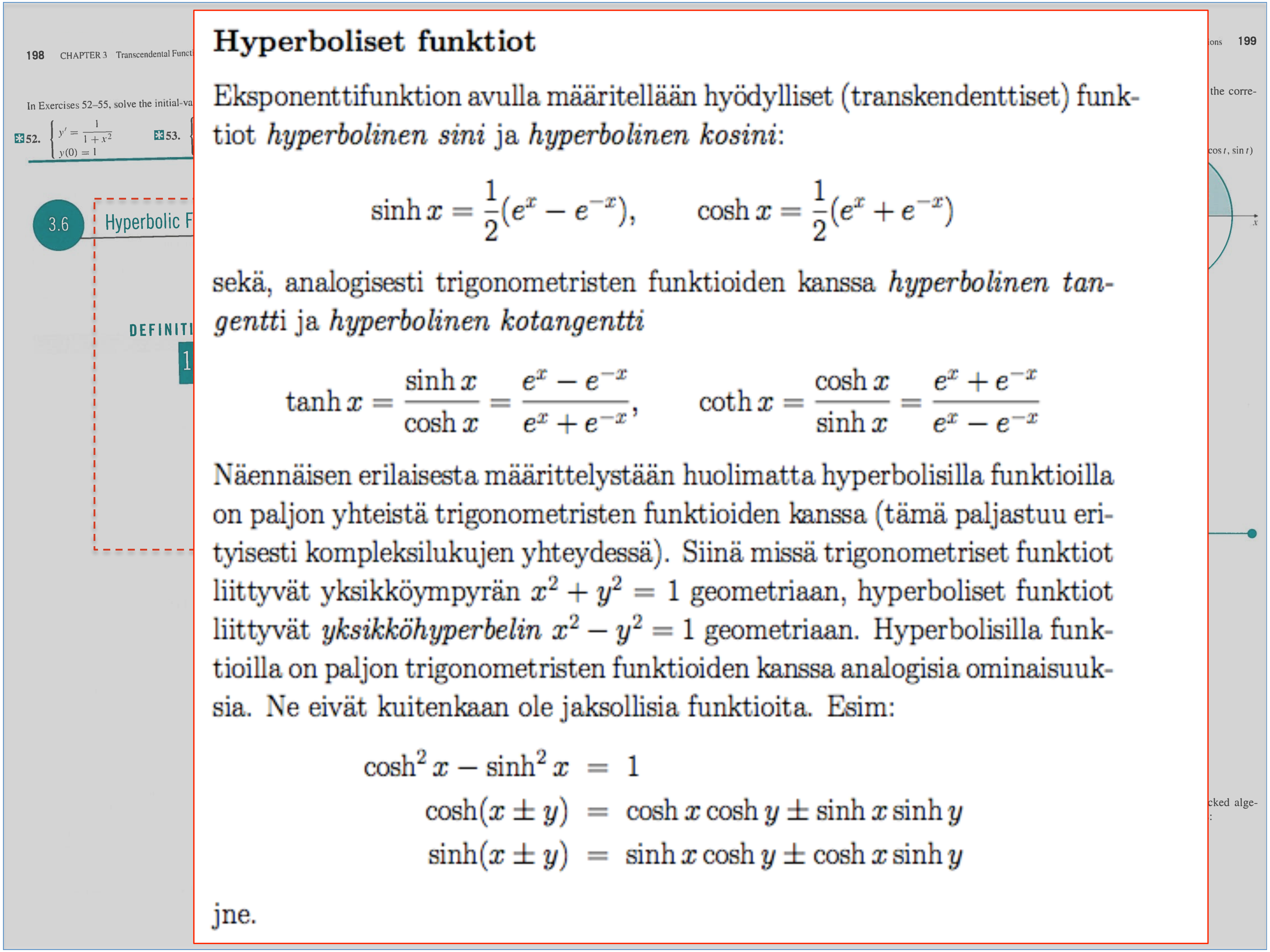
Kuten myöhemmin opitaan, eksponenttifunktiolla jonka kantalukuna on e, siis funktiolla $f(x) = e^x$, on se tärkeä ominaisuus, että sen derivaattafunktio on funktio itse, siis $\dfrac{d}{dx}e^x = e^x$. Tästä ainutlaatuisesta ominaisuudesta johtuu, että ko. funktio on erityisen tärkeä funktioanalyysin kannalta.

Luonnollinen logaritmi

Eksponenttifunktion e^x käänteisfunktio on e -kantainen logaritmi $\log_e(x)$. Sitä kutsutaan *luonnolliseksi logaritmiksi* ja merkitään $\ln(x)$.

Huom: Yleisen logaritmin kantaluvun vaihtokaavan mukaan voidaan mv. a -kantainen logaritmi kirjoittaa luonnollisen logaritmin avulla: $\log_a x = \ln x / \ln a$. Samoin voidaan a -kantainen eksponenttifunktio kirjoittaa e -kantaisena eksponenttifunktiona: $a^x = e^{x \ln a}$. Näin ollen kaikki logaritmi- eksponenttifunktiot voidaan aina lausua funktioiden $\ln x$ ja e^x avulla. Yleisen käytännön mukaan, jos puhutaan vain 'logaritmista' tai 'eksponettifunktiosta' määrittelemättä kantalukua, tarkoitetaan useimmiten nimenomaan funktioita $\ln x$ ja e^x (logaritmifunktion kohdalla tosin joskus 10 -kantaista tai harvemmin 2 -kantaista logaritmia).





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In Exercises 52–55, solve the initial-va

52. $\begin{cases} y' = \dfrac{1}{1+x^2} \\ y(0) = 1 \end{cases}$ 53.

3.6 Hyperbolic F

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cos t, sin t)

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Hyperboliset funktiot

Eksponenttifunktion avulla määritellään hyödylliset (transkendenttiset) funktiot *hyperbolinen sini* ja *hyperbolinen kosini*:

$$\sinh x = \frac{1}{2}(e^x - e^{-x}), \qquad \cosh x = \frac{1}{2}(e^x + e^{-x})$$

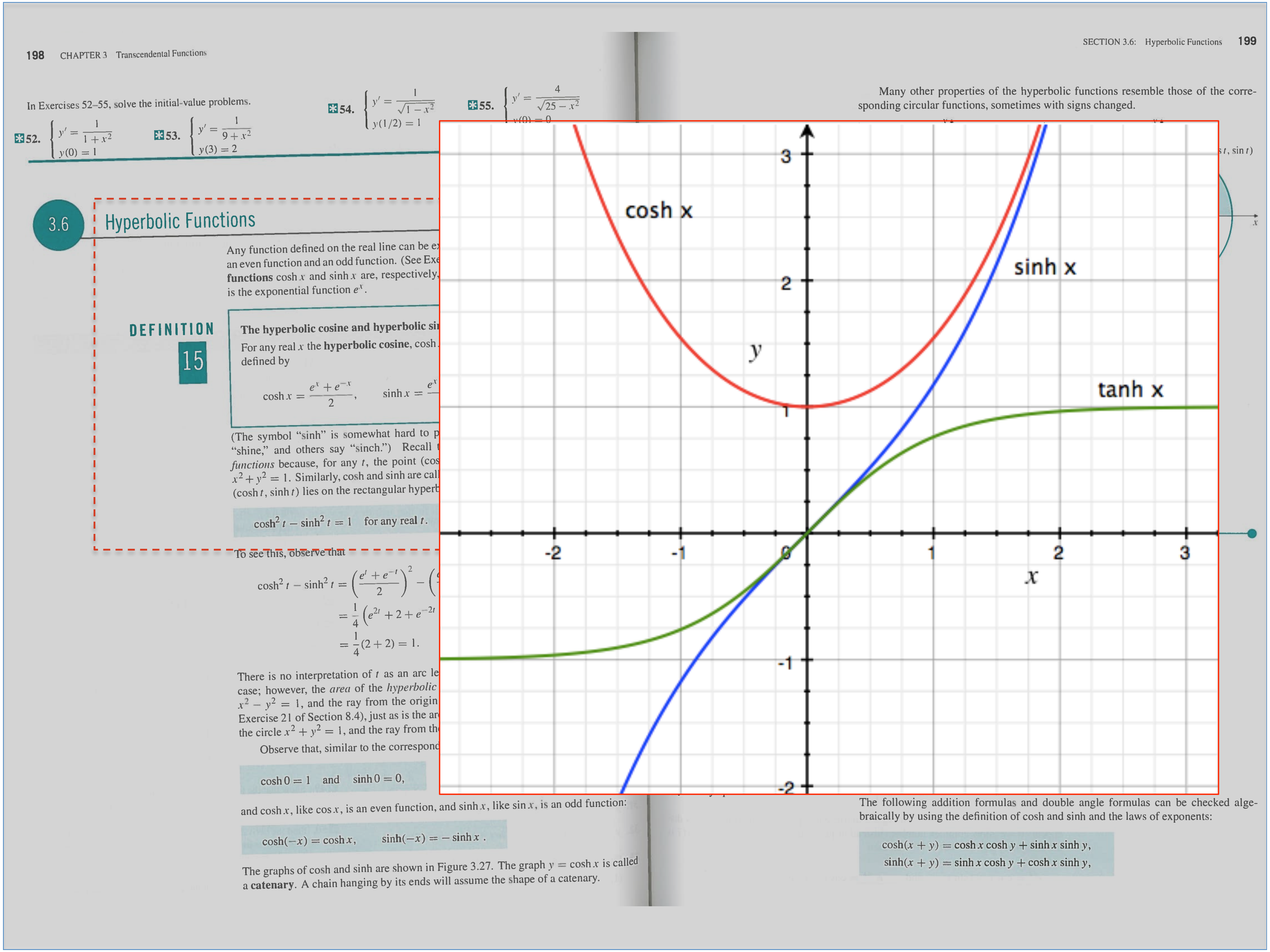
sekä, analogisesti trigonometristen funktioiden kanssa *hyperbolinen tangentti* ja *hyperbolinen kotangentti*

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Näennäisen erilaisesta määrittelystään huolimatta hyperbolisilla funktioilla on paljon yhteistä trigonometristen funktioiden kanssa (tämä paljastuu erityisesti kompleksilukujen yhteydessä). Siinä missä trigonometriset funktiot liittyvät yksikköympyrän $x^2 + y^2 = 1$ geometriaan, hyperboliset funktiot liittyvät *yksikköhyperbelin* $x^2 - y^2 = 1$ geometriaan. Hyperbolisilla funktioilla on paljon trigonometristen funktioiden kanssa analogisia ominaisuuksia. Ne eivät kuitenkaan ole jaksollisia funktioita. Esim:

$$\begin{aligned} \cosh^2 x - \sinh^2 x &= 1 \\ \cosh(x \pm y) &= \cosh x \cosh y \pm \sinh x \sinh y \\ \sinh(x \pm y) &= \sinh x \cosh y \pm \cosh x \sinh y \end{aligned}$$

jne.



198 CHAPTER 3 Transcendental Functions

In Exercises 52–55, solve the initial-value problems.

52. $\begin{cases} y' = \dfrac{1}{1+x^2} \\ y(0) = 1 \end{cases}$

53. $\begin{cases} y' = \dfrac{1}{9+x^2} \\ y(3) = 2 \end{cases}$

54. $\begin{cases} y' = \dfrac{1}{\sqrt{1-x^2}} \\ y(1/2) = 1 \end{cases}$

55. $\begin{cases} y' = \dfrac{4}{\sqrt{25-x^2}} \\ y(0) = 0 \end{cases}$

3.6 Hyperbolic Functions

Any function defined on the real line can be ex
an even function and an odd function. (See Exe
functions cosh x and sinh x are, respectively,
is the exponential function e^x.

DEFINITION 15

The hyperbolic cosine and hyperbolic si

For any real x the **hyperbolic cosine**, cosh
defined by

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x}{}$$

(The symbol "sinh" is somewhat hard to p
"shine," and others say "sinch.") Recall t
functions because, for any t, the point (cos
$x^2 + y^2 = 1$. Similarly, cosh and sinh are call
(cosh t, sinh t) lies on the rectangular hyperb

$$\cosh^2 t - \sinh^2 t = 1 \quad \text{for any real } t.$$

To see this, observe that

$$\cosh^2 t - \sinh^2 t = \left(\frac{e^t + e^{-t}}{2}\right)^2 - ($$
$$= \frac{1}{4}\left(e^{2t} + 2 + e^{-2t}\right.$$
$$= \frac{1}{4}(2+2) = 1.$$

There is no interpretation of t as an arc le
case; however, the *area* of the *hyperbolic*
$x^2 - y^2 = 1$, and the ray from the origin
Exercise 21 of Section 8.4), just as is the ar
the circle $x^2 + y^2 = 1$, and the ray from th

Observe that, similar to the correspond

$$\cosh 0 = 1 \quad \text{and} \quad \sinh 0 = 0,$$

and cosh x, like cos x, is an even function, and sinh x, like sin x, is an odd function:

$$\cosh(-x) = \cosh x, \qquad \sinh(-x) = -\sinh x.$$

The graphs of cosh and sinh are shown in Figure 3.27. The graph $y = \cosh x$ is called a **catenary**. A chain hanging by its ends will assume the shape of a catenary.

SECTION 3.6: Hyperbolic Functions 199

Many other properties of the hyperbolic functions resemble those of the corresponding circular functions, sometimes with signs changed.

t, sin t)

The following addition formulas and double angle formulas can be checked algebraically by using the definition of cosh and sinh and the laws of exponents:

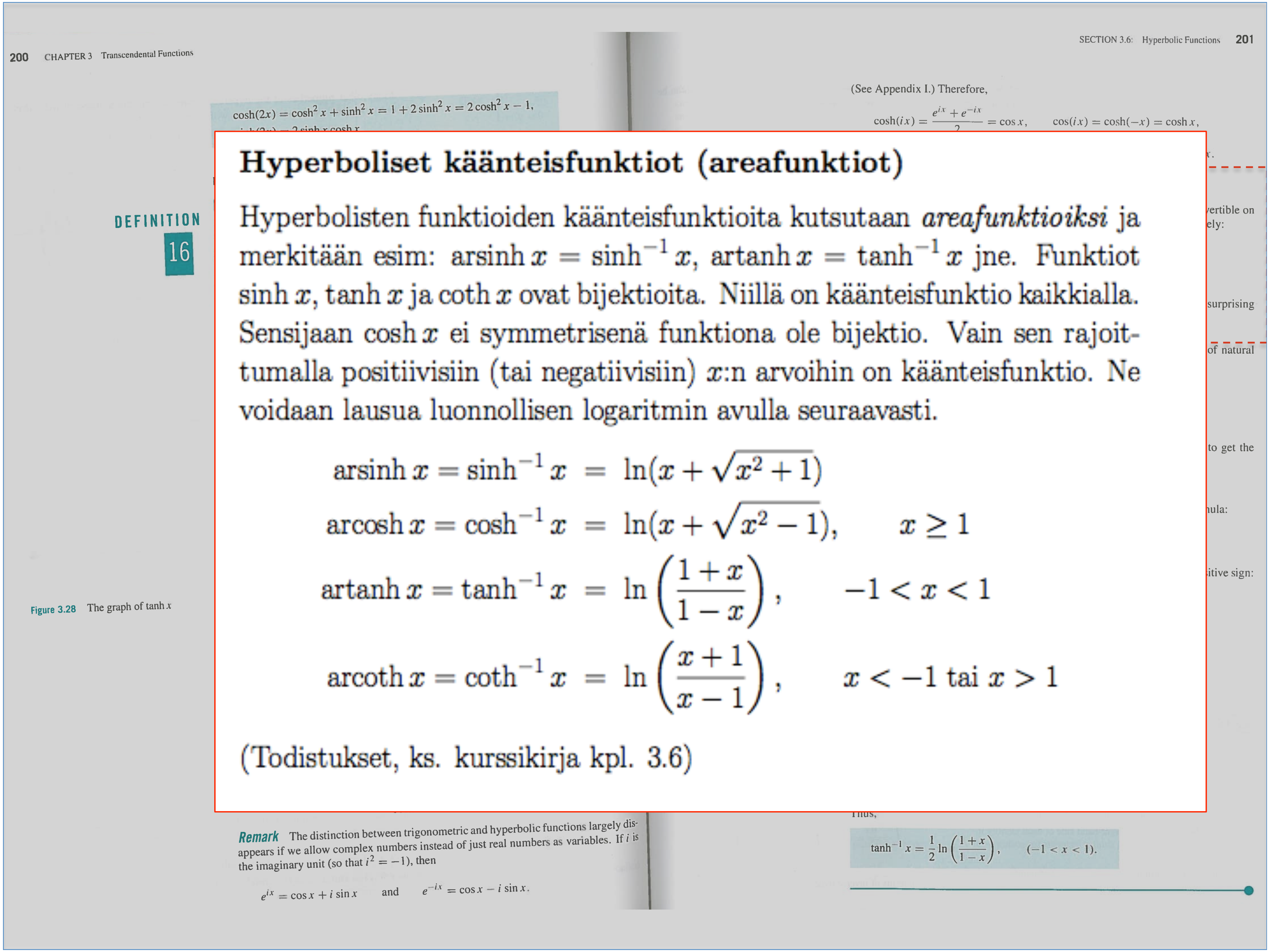
$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y,$$
$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y,$$

Hyperboliset käänteisfunktiot (areafunktiot)

Hyperbolisten funktioiden käänteisfunktioita kutsutaan *areafunktioiksi* ja merkitään esim: arsinh $x = \sinh^{-1} x$, artanh $x = \tanh^{-1} x$ jne. Funktiot $\sinh x$, $\tanh x$ ja $\coth x$ ovat bijektioita. Niillä on käänteisfunktio kaikkialla. Sensijaan $\cosh x$ ei symmetrisenä funktiona ole bijektio. Vain sen rajoittumalla positiivisiin (tai negatiivisiin) x:n arvoihin on käänteisfunktio. Ne voidaan lausua luonnollisen logaritmin avulla seuraavasti.

$$\begin{aligned}
\operatorname{arsinh} x = \sinh^{-1} x &= \ln(x+\sqrt{x^2+1}) \\
\operatorname{arcosh} x = \cosh^{-1} x &= \ln(x+\sqrt{x^2-1}), \qquad x \geq 1 \\
\operatorname{artanh} x = \tanh^{-1} x &= \ln\left(\frac{1+x}{1-x}\right), \qquad -1 < x < 1 \\
\operatorname{arcoth} x = \coth^{-1} x &= \ln\left(\frac{x+1}{x-1}\right), \qquad x < -1 \text{ tai } x > 1
\end{aligned}$$

(Todistukset, ks. kurssikirja kpl. 3.6)



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SECTION 3.6: Hyperbolic Functions 201

$\cosh(2x) = \cosh^2 x + \sinh^2 x = 1 + 2\sinh^2 x = 2\cosh^2 x - 1,$

DEFINITION 16

Figure 3.28 The graph of tanh x

Remark The distinction between trigonometric and hyperbolic functions largely disappears if we allow complex numbers instead of just real numbers as variables. If i is the imaginary unit (so that $i^2 = -1$), then

$$e^{ix} = \cos x + i \sin x \quad \text{and} \quad e^{-ix} = \cos x - i \sin x.$$

(See Appendix I.) Therefore,

$$\cosh(ix) = \frac{e^{ix}+e^{-ix}}{2} = \cos x, \qquad \cos(ix) = \cosh(-x) = \cosh x,$$

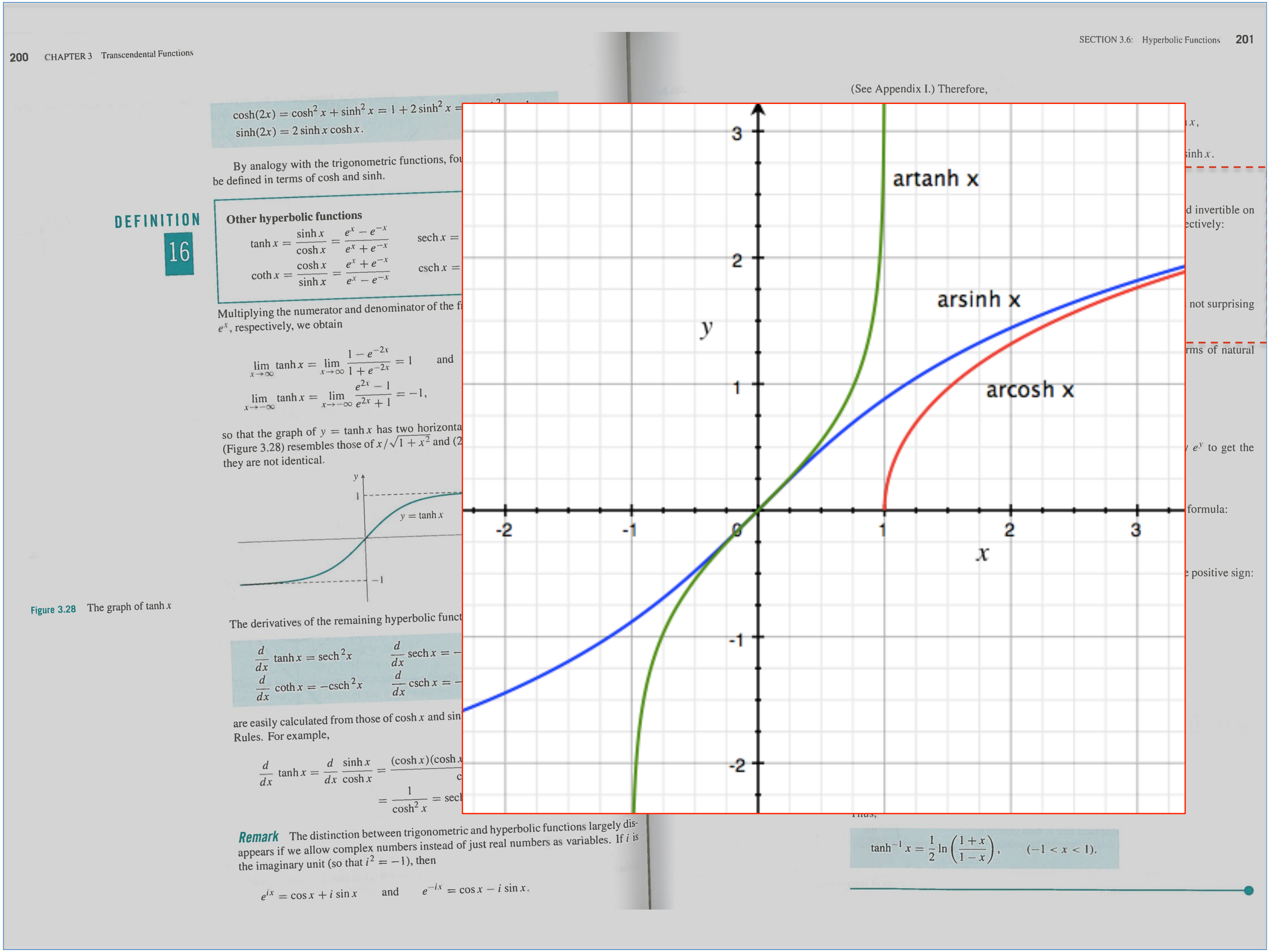
surprising

of natural

to get the

Thus,

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \qquad (-1 < x < 1).$$



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$$\cosh(2x) = \cosh^2 x + \sinh^2 x = 1 + 2\sinh^2 x =$$
$$\sinh(2x) = 2\sinh x \cosh x.$$

By analogy with the trigonometric functions, fou
be defined in terms of cosh and sinh.

DEFINITION 16

Other hyperbolic functions

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \operatorname{sech} x =$$
$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} \qquad \operatorname{csch} x =$$

Multiplying the numerator and denominator of the fi
e^x, respectively, we obtain

$$\lim_{x\to\infty} \tanh x = \lim_{x\to\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = 1 \qquad \text{and}$$
$$\lim_{x\to-\infty} \tanh x = \lim_{x\to-\infty} \frac{e^{2x} - 1}{e^{2x} + 1} = -1,$$

so that the graph of $y = \tanh x$ has two horizonta
(Figure 3.28) resembles those of $x/\sqrt{1+x^2}$ and (2
they are not identical.

Figure 3.28 The graph of tanh x

The derivatives of the remaining hyperbolic funct

$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x \qquad \frac{d}{dx}\operatorname{sech} x = -$$
$$\frac{d}{dx}\coth x = -\operatorname{csch}^2 x \qquad \frac{d}{dx}\operatorname{csch} x = -$$

are easily calculated from those of $\cosh x$ and sin
Rules. For example,

$$\frac{d}{dx}\tanh x = \frac{d}{dx}\frac{\sinh x}{\cosh x} = \frac{(\cosh x)(\cosh x}{c}$$
$$= \frac{1}{\cosh^2 x} = \text{sec}$$

Remark The distinction between trigonometric and hyperbolic functions largely disappears if we allow complex numbers instead of just real numbers as variables. If i is the imaginary unit (so that $i^2 = -1$), then

$$e^{ix} = \cos x + i\sin x \qquad \text{and} \qquad e^{-ix} = \cos x - i\sin x.$$

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(See Appendix I.) Therefore,

x,

$\sinh x$.

d invertible on
ectively:

not surprising

rms of natural

e^y to get the

formula:

positive sign:

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \qquad (-1 < x < 1).$$