Vektorianalyysi
k. 2014

Ex Tempore 10
Ke 9.4.

1. Laske tilavuusintegraali

$$I = \iiint_D yz^2 dxdydz,$$

jossa integroimisalue D on tasojen $x = 0, y = 0, z = 0$ ja $3x + 2y + 6z = 6$ rajoittava suljettu alue. Piirrä ensin D avuksesi.

2. Johda pallokoordinaatiston

$$\begin{aligned} x &= r\sin\theta\cos\phi, \qquad r = \sqrt{x^2 + y^2 + z^2} \\ y &= r\sin\theta\sin\phi, \\ z &= r\cos\theta, \end{aligned}$$

tilavuusintegroinnin Jacobin determinantti

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin\theta.$$

3. Origokeskisen pallokuoren $1 \le r \le 2$ tiheys on (μ on vakio)

$$\rho = \frac{\mu}{x^2 + y^2 + z^2}.$$

Laske pallokuoren massa

$$\iiint_D \rho dV.$$

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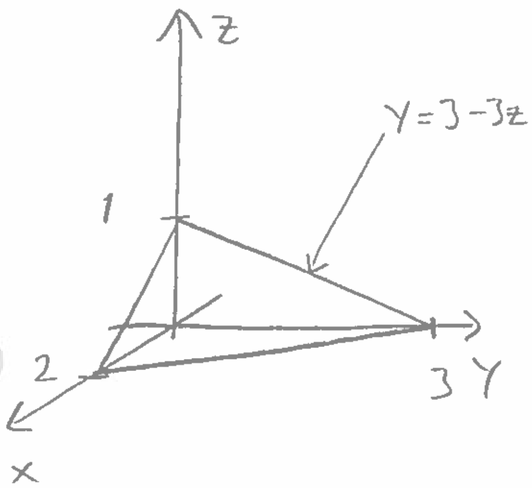
$\boxed{1.}$ D on alue jota rajaavat tasot $x=0$, $y=0$, $z=0$ ja $3x+2y+6z=6 \Rightarrow x = 2-\frac{2}{3}y-2z$.



Laske $I = \int_D dV\, yz^2$

Integroidaan järjestyksessä x, y, z.

$x \in [0, 2-\frac{2}{3}y-2z]$

$y \in [0, 3-3z]$

$z \in [0,1]$

$$\Rightarrow I = \int_0^1 dz \int_0^{3-3z} dy \int_0^{2-\frac{2}{3}y-2z} dx\, yz^2 = \int_0^1 dz \int_0^{3-3z} dy\, yz^2(2-\frac{2}{3}y-2z)$$

$$= \int_0^1 dz \int_0^{3-3z} dy\, (2yz^2 - \frac{2}{3}y^2z^2 - 2yz^3)$$

$$= \int_0^1 dz \Big/_0^{3-3z} y^2z^2 - \frac{2}{9}y^3z^2 - y^2z^3$$

$$= \int_0^1 dz\Big((3-3z)^2z^2 - \frac{2}{9}\underbrace{(3-3z)^3}_{=27-81z+81z^2-27z^3}z^2 - (3-3z)^2z^3\Big)$$

$$= \int_0^1 dz\Big(9z^2 - \cancel{18z^3} + 9z^4 - 2(3z^2 - \cancel{9z^3} + \cancel{9z^4} - 3z^5) - 9z^3 + \cancel{18z^4} - 9z^5\Big)$$

$$= \int_0^1 dz(-3z^5 + 9z^4 - 9z^3 + 3z^2).$$

$$= \Big/_0^1 -\frac{1}{2}z^6 + \frac{9}{5}z^5 - \frac{9}{4}z^4 + z^3 = -\frac{1}{2} + \frac{9}{5} - \frac{9}{4} + 1$$

$$= -\frac{10}{20} + \frac{36}{20} - \frac{45}{20} + \frac{20}{20} = \frac{56-55}{20} = \frac{1}{20}$$

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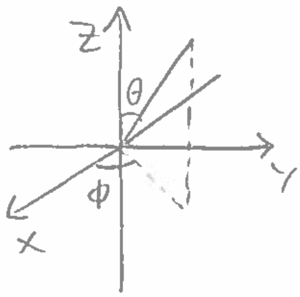

$\boxed{2.}$

$$\begin{cases} x = r\cos\phi\sin\theta \\ y = r\sin\phi\sin\theta \\ z = r\cos\theta \end{cases}$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\phi\sin\theta & \sin\phi\sin\theta & \cos\theta \\ -r\sin\phi\sin\theta & r\cos\phi\sin\theta & 0 \\ r\cos\phi\cos\theta & r\sin\phi\cos\theta & -r\sin\theta \end{vmatrix}$$

$$= \cos\phi\sin\theta\left(-r^2\sin^2\theta\cos\phi - 0\right) - \sin\phi\sin\theta\left(r^2\sin\phi\sin^2\theta - 0\right)$$
$$+\cos\theta\left(-r^2\sin^2\phi\sin\theta\cos\theta - r^2\cos^2\phi\sin\theta\cos\theta\right)$$

$$= -r^2\sin\theta\sin^2\theta\cos^2\phi - r^2\sin\theta\sin^2\phi\sin^2\theta$$
$$-r^2\cos^2\theta\sin\theta$$

$$= -r^2\sin\theta\left(\underbrace{\sin^2\theta\cos^2\phi + \sin^2\phi\sin^2\theta}_{=\sin^2\theta} + \cos^2\theta\right)$$

$$= -r^2\sin\theta$$

$$\Rightarrow \quad |J| = r^2\sin\theta$$

Muuttujanvaihdossa käytetään Jacobin determinantin itseisarvoa, koska muuttujien järjestyksellä ei saa olla merkitystä.

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$\boxed{3.}$ $1 \leq r \leq 2$, $\theta \in [0,\pi]$, $\phi \in [0,2\pi]$

$$\rho(\vec{r}) = \frac{\mu}{x^2+y^2+z^2} = \frac{\mu}{r^2}$$

$$M = \int dV \rho(\vec{r}) = \int_1^2 dr \int_0^\pi d\theta \int_0^{2\pi} d\phi \, \cancel{r^2} \sin\theta \cdot \frac{\mu}{\cancel{r^2}}$$

$$= \mu \cdot 2\pi \underbrace{\int_1^2 dr}_{=1} \underbrace{\int_0^\pi d\theta \sin\theta}_{2} = 4\pi\mu.$$