Vektorianalyysi

k. 2014

Ex Tempore 3

Ke 12.3.

1. Kirjoita seuraavien käyrien tangentti- ja normaalivektorit $\hat{T}(t)$ ja $\hat{N}(t)$:
   a. $\vec{r}(t) = t\hat{i} - 2t^2\hat{j} + 3t^3\hat{k}$
   b. $\vec{r}(t) = a\sin(\omega t)\hat{i} + a\cos(\omega t)\hat{j}$
   c. $\vec{r}(t) = a\cos t\hat{i} + b\sin t\hat{j} + t\hat{k}$
2. Määritä ympyräheliksin

$$\vec{r}(t) = a\cos t\hat{i} + a\sin t\hat{j} + bt\hat{k}$$

   kaarevuus.
3. Missä pisteessä käyrän $y = \ln x$ kaarevuus on suurimmillaan ja mikä tämä suurin arvo on? Vihje: Käytä käyrän parametrina koordinaattia x.

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$\boxed{1.}$ $\hat{T}(t) = \dfrac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|}$, $\hat{N}(t) = \dfrac{\frac{d\hat{T}}{dt}}{\left|\frac{d\hat{T}}{dt}\right|}$

a.) $\vec{r}(t) = t\hat{i} - 2t^2\hat{j} + 3t^3$

$\dfrac{d\vec{r}}{dt} = \hat{i} - 4t\hat{j} + 9t^2\hat{k}$, $\left|\dfrac{d\vec{r}}{dt}\right| = \sqrt{1 + 16t^2 + 81t^4}$

$\hat{T}(t) = \dfrac{\hat{i} - 4t\hat{j} + 9t^2\hat{k}}{\sqrt{1 + 16t^2 + 81t^4}}$

$$\frac{\partial \hat{T}}{\partial t} = \frac{-4\hat{j} + 18t\hat{k}}{\sqrt{1+16t^2+81t^4}} - \frac{(\hat{i} - 4t\hat{j} + 9t^2\hat{k})}{2(1+16t^2+81t^4)^{3/2}}(32t + 324t^3)$$

$$= \frac{(-4\hat{j} + 18t\hat{k})(1+16t^2+81t^4) - (\hat{i} - 4t\hat{j} + 9t^2\hat{k})(16t + 162t^3)}{(1 + 16t^2 + 81t^4)^{3/2}}$$

$$= \Big[-\hat{i}\,2t(8 + 81t^2) + \hat{j}(\cancel{64t^2} + 648t^4 - 4 - \cancel{64t^2} - 324t^4)$$
$$+ \hat{k}(-144t^3 - 14\cancel{58}t^5 + 18t + 288t^3 + 14\cancel{58}t^5)\Big] / (\ldots)^{3/2}$$

$$= \left[-\hat{i}\,2t(8+81t^2) + \hat{j}(324t^4 - 4) + \hat{k}(144t^3 + 18t)\right]/(\ldots)^{3/2}$$

$$= \left[-\hat{i}\,2t(8+81t^2) + \hat{j}\,4(81t^4 - 1) + \hat{k}\,18t(8t^2+1)\right]/(\ldots)^{3/2}$$

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$\boxed{1.}$ b.)

$$\vec{r}(t) = a\sin(\omega t)\hat{\imath} + a\cos(\omega t)\hat{\jmath}$$

$$\frac{d\vec{r}}{dt} = a\omega\cos(\omega t)\hat{\imath} - a\omega\sin(\omega t)\hat{\jmath}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \left((a\omega\cos(\omega t))^2 + (a\omega\sin(\omega t))^2\right)^{1/2} = a\omega$$

$$\hat{T} = \cos(\omega t)\hat{\imath} - \sin(\omega t)\hat{\jmath}$$

$$\frac{d\hat{T}}{dt} = -\omega\sin(\omega t)\hat{\imath} - \omega\cos(\omega t)\hat{\jmath}$$

$$\left|\frac{d\hat{T}}{dt}\right| = \omega$$

$$\hat{N} = -\sin(\omega t)\hat{\imath} - \omega\cos(\omega t)\hat{\jmath} = -\frac{1}{a}\vec{r}(t)$$

c.)

$$\vec{r}(t) = a\cos t\,\hat{\imath} + b\sin t\,\hat{\jmath} + t\hat{k}$$

$$\frac{d\vec{r}}{dt} = -a\sin t\,\hat{\imath} + b\cos t\,\hat{\jmath} + \hat{k}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \left((a\sin t)^2 + (b\cos t)^2 + 1\right)^{1/2}$$

$$\hat{T} = \frac{-a\sin t\,\hat{\imath} + b\cos t\,\hat{\jmath} + \hat{k}}{\sqrt{a^2\sin^2 t + b^2\cos^2 t + 1}}$$

$$\frac{\partial\hat{T}}{\partial t} = \frac{-a\cos t\,\hat{\imath} - b\sin t\,\hat{\jmath}}{\sqrt{a^2\sin^2 t + b^2\cos^2 t + 1}} + (-a\sin t\,\hat{\imath} + b\cos t\,\hat{\jmath} + \hat{k})\cdot\left(-\frac{1}{\cancel{2}}\right)\frac{(a^2\cdot\cancel{2}\sin t\cos t - \cancel{2}b^2\cos t\sin t)}{(a^2\sin^2 t + b^2\cos^2 t + 1)^{3/2}}$$

Merkitään $\alpha := (a^2\sin^2 t + b^2\cos^2 t + 1)^{3/2}$

$$\frac{\partial\hat{T}}{\partial t} = \frac{(a^2\sin^2 t + b^2\cos^2 t + 1)(-a\cos t\,\hat{\imath} - b\sin t\,\hat{\jmath}) + (a\sin t\,\hat{\imath} - b\cos t\,\hat{\jmath} - \hat{k})((a^2 - b^2)\sin t\cos t)}{\alpha}$$

$$= \frac{1}{\alpha}\Big(\hat{\imath}\big(a(\cancel{a^2} - b^2)\sin^2 t\cos t - a(a^2\cancel{\sin^2 t}\cos t + b^2\cos^3 t + \cos t)\big)$$
$$- \hat{\jmath}\big(b(a^2\sin^3 t + b^2\cancel{\cos^2 t}\sin t + \sin t) + b(a^2 - \cancel{b^2})\sin t\cos^2 t\big) - \hat{k}(a^2 - b^2)\sin t\cos t\Big)$$

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$\boxed{1.}$ ... jatkuu

$$\frac{\partial \hat{T}}{\partial t} = \frac{1}{\alpha}\Big(\hat{\imath}\big(-ab^2\overbrace{(\sin^2 t+\cos^2 t)}^{=1}\cos t - a\cos t\big)$$
$$-\hat{\jmath}\big(ba^2\sin t\overbrace{(\sin^2 t+\cos^2 t)}^{=1} + b\sin t\big)$$
$$-\hat{k}\,(a^2-b^2)\sin t\cos t\Big)$$

$$= \frac{1}{\alpha}\Big(-\hat{\imath}\,a\cos t\,(b^2+1) - \hat{\jmath}\,b\sin t\,(a^2+1) - \hat{k}(a^2-b^2)\sin t\cos t\Big)$$

$$\hat{N} = \frac{\frac{\partial \hat{T}}{\partial t}}{\left|\frac{\partial \hat{T}}{\partial t}\right|}$$

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$\boxed{2.}$ Kaarevuuden määritelmä $K(t) = \frac{|\vec{v}\times\vec{a}|}{|\vec{v}|^3}$

$$\vec{r}(t) = a\cos t\,\hat{\imath} + a\sin t\,\hat{\jmath} + bt\hat{k}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = -a\sin t\,\hat{\imath} + a\cos t\,\hat{\jmath} + b\hat{k}$$

$$|\vec{v}| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2+b^2}$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = -a\cos t\,\hat{\imath} - a\sin t\,\hat{\jmath}$$

$$\vec{v}\times\vec{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = \hat{\imath}(0 + ba\sin t) - \hat{\jmath}(0 + ba\cos t)$$

$$+ \hat{k}(a^2\sin^2 t + a^2\cos^2 t)$$

$$= ba\sin t\,\hat{\imath} - ba\cos t\,\hat{\jmath} + a^2\hat{k}$$

$$|\vec{v}\times\vec{a}| = \sqrt{a^2b^2 + a^4} = a\sqrt{a^2+b^2}$$

$$K(t) = \frac{|\vec{v}\times\vec{a}|}{|\vec{v}|^3} = \frac{a\sqrt{a^2+b^2}}{(a^2+b^2)^{3/2}} = \frac{a}{a^2+b^2}$$

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$\boxed{3.}$ $y = \ln x$

Parametrisoidaan käyrä: $x = t$, $y = \ln t$, $t > 0$

$\Rightarrow \vec{r}(t) = t\hat{\imath} + \ln t\,\hat{\jmath}$

Kaarevuuden määritelmä: $K(t) = \dfrac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$,

missä $\vec{v} = \dfrac{d\vec{r}}{dt}$ ja $\vec{a} = \dfrac{d^2\vec{r}}{dt^2}$.

$$\frac{d\vec{r}}{dt} = \hat{\imath} + \frac{1}{t}\hat{\jmath}, \quad \frac{d^2\vec{r}}{dt^2} = -\frac{1}{t^2}\hat{\jmath}$$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & \frac{1}{t} & 0 \\ 0 & -\frac{1}{t^2} & 0 \end{vmatrix} = \hat{\imath}(0-0) - \hat{\jmath}(0-0) + \hat{k}\left(-\frac{1}{t^2} - 0\right)$$

$$= -\frac{1}{t^2}$$

$$|\vec{v}| = \sqrt{1 + \left(\frac{1}{t}\right)^2} = \sqrt{\frac{t^2+1}{t^2}} = \frac{\sqrt{t^2+1}}{t}$$

$$\Rightarrow K(t) = \frac{\frac{1}{t^2}}{\left(\frac{\sqrt{t^2+1}}{t}\right)^3} = \frac{1}{t^2} \cdot \frac{t^3}{(t^2+1)^{3/2}} = \frac{t}{(t^2+1)^{3/2}}$$

Maksimoidaan $K(t)$ etsimällä derivaatan nollakohdat.

$$\frac{dK(t)}{dt} = \frac{1}{(t^2+1)^{3/2}} + t \cdot \left(-\frac{3}{2}\right) \cdot \frac{2t}{(t^2+1)^{5/2}} = \frac{t^2+1-3t^2}{(t^2+1)^{5/2}} = 0$$

$$\Rightarrow -2t^2+1 = 0 \Rightarrow t = \underset{(-)}{+}\frac{1}{\sqrt{2}} \Rightarrow K\left(\frac{1}{\sqrt{2}}\right) = \frac{\frac{1}{\sqrt{2}}}{\left(\frac{1}{2}+1\right)^{3/2}} = \frac{1}{\sqrt{2}} \cdot \left(\frac{2}{3}\right)^{3/2}$$

$$= \frac{2}{3^{3/2}}$$