Vektorianalyysi
k. 2014

Ex Tempore 7
Ma 31.3.

1.

a) Mitkä ovat pisteen $(x, y) = (1,1)$ napakoordinaatit (r, θ)?

b) Mitkä ovat sylinterikoordinaatiston pisteen $(\rho, \theta, z) = (\sqrt{2}, \frac{\pi}{4}, 1)$ karteesiset koordinaatit (x, y, z)?

c) Mitkä ovat pisteen $(x, y, z) = (1, 1, \sqrt{2})$ pallokoordinaatit (r, ϕ, θ)?

2. Alue A määritellään seuraavasti: $A = \{x \leq 0, y \geq 0, x^2 + y^2 \leq 1\}$.

a) Laske alueen A pinta-ala tekemällä muuttujanvaihto napakoordinaatteihin.
b) Laske integraali $\iint_A dA\ (3x^2 + 3y^2)$.

3. Kuinka suuri osa maapallon pinnasta on pohjoisten leveyspiirien 60° ja 70° välisellä alueella?

4. Laske funktion $f = \arctan \frac{y}{x}$ pintaintegraali yli alueen

$$D = \{0 \leq x \leq 1, 0 \leq y \leq \sqrt{1 - x^2}\}.$$

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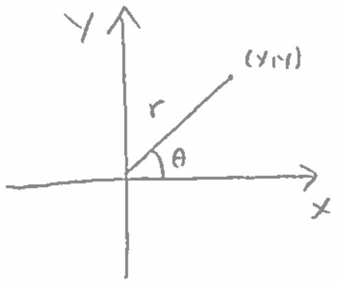
$\boxed{1.}$ a) Napakoordinaattimuunnos

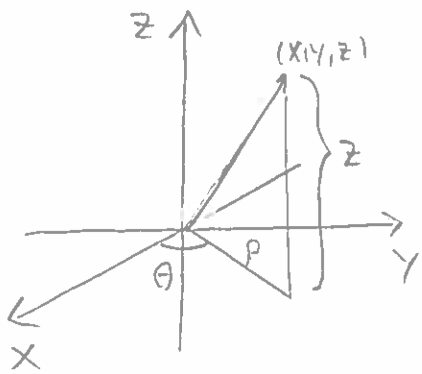


$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$(x,y) = (1,1)$$
$$\Rightarrow (r,\theta) = \left(\sqrt{2}, \tfrac{\pi}{4}\right)$$

$$\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \arctan\left(\frac{y}{x}\right) \end{cases}$$

b) Sylinterikoordinaattimuunnos

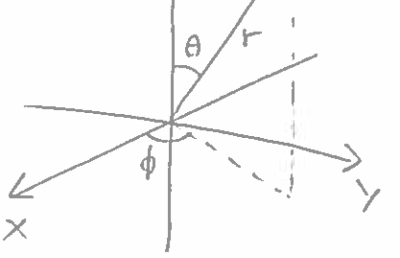


$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases}$$

$$(\rho,\theta,z) = \left(\sqrt{2}, \tfrac{\pi}{2}, 1\right)$$
$$\Rightarrow (x,y,z) = (1,1,1)$$

$$\begin{cases} \rho = \sqrt{x^2+y^2} \\ \theta = \arctan\left(\frac{y}{x}\right) \\ z = z \end{cases}$$

c) Pallokoordinaattimuunnos



$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

$$(x,y,z) = (1,1,\sqrt{2})$$
$$\Rightarrow (r,\phi,\theta) = \left(2, \tfrac{\pi}{4}, \tfrac{\pi}{4}\right)$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \phi = \arctan\left(\frac{y}{x}\right) \\ \theta = \arctan\left(\frac{z}{\sqrt{x^2+y^2}}\right) \end{cases}$$

<u>Huom.</u> Muista tarkistaa arctanin antamat numerot!

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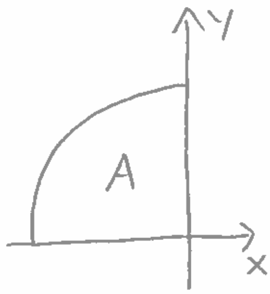
$\boxed{2.}$ $A = \{x \le 0,\ y \ge 0,\ x^2+y^2 \le 1\}$

Piirrä aina ensin kuva!



Parametrisoidaan alue A.

$$\left.\begin{array}{ll} x = r\cos\theta, & r \in [0,1] \\ y = r\sin\theta, & \theta \in [\frac{\pi}{2}, \pi] \end{array}\right\} \text{ Napakoordinaattimuunnos.}$$

$$\vec{r}(r,\theta) = r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath}$$

$$\frac{\partial \vec{r}}{\partial r} = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$$

$$\frac{\partial \vec{r}}{\partial \theta} = -r\sin\theta\,\hat{\imath} + r\cos\theta\,\hat{\jmath}$$

$$\Rightarrow \frac{\partial \vec{r}}{\partial r} \times \frac{\partial \vec{r}}{\partial \theta} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= \hat{k}(r\cos^2\theta + r\sin^2\theta) = r\hat{k}$$

a.) $$\int_A ds = \int_0^1 dr \int_{\frac{\pi}{2}}^{\pi} d\theta\, \underbrace{r}_{=|J| = \left|\frac{\partial \vec{r}}{\partial r} \times \frac{\partial \vec{r}}{\partial \theta}\right|} = \frac{\pi}{2}\int_0^1 dr\, r = \underline{\underline{\frac{\pi}{4}}}$$

b.) $$\int_A ds\,\underbrace{(3x^2+3y^2)}_{3(x^2+y^2) = 3r^2} = \int_0^1 dr \int_{\frac{\pi}{2}}^{\pi} d\theta\ 3r^2 \cdot r = \frac{3\pi}{2}\int_0^1 dr\, r^3 = \underline{\frac{3\pi}{8}}$$

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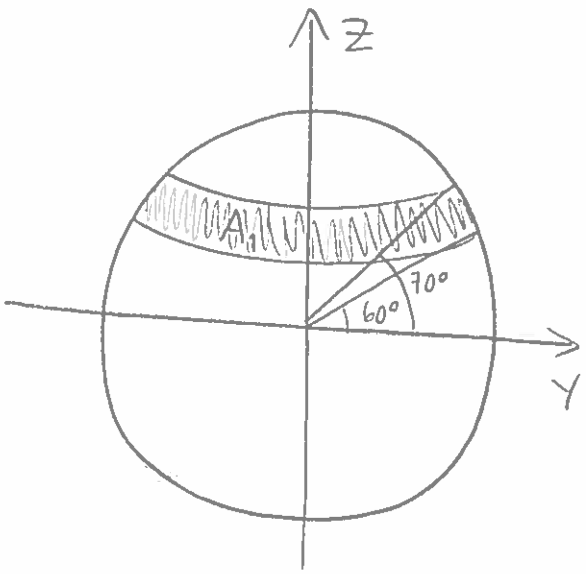
3.



Lasketaan pohjoisten leveyspiirien 60° ja 70° väliin jäävä pinta-ala.

Pinta on osa pallopintaa, joten käytetään pallokoordinaattimuunnosta.

Huom. Muunnoksessa kulma θ mitataan z-akselista.

Parametrisoidaan pinta

$$\begin{cases} x = R\sin\theta\cos\phi \\ y = R\sin\theta\sin\phi \\ z = R\cos\theta \end{cases}$$

$R =$ vakio, koska ollaan pallopinnalla

$\phi \in [0, 2\pi[$

$\theta \in [90° - 70°, 90° - 60°] = [\frac{\pi}{9}, \frac{\pi}{6}]$

$$\vec{r}(\theta,\phi) = R\sin\theta\cos\phi\,\hat{\imath} + R\sin\theta\sin\phi\,\hat{\jmath} + R\cos\theta\,\hat{k}$$

$$\frac{\partial\vec{r}}{\partial\theta} = R\cos\theta\cos\phi\,\hat{\imath} + R\cos\theta\sin\phi\,\hat{\jmath} - R\sin\theta\,\hat{k}$$

$$\frac{\partial\vec{r}}{\partial\phi} = -R\sin\theta\sin\phi\,\hat{\imath} + R\sin\theta\cos\phi\,\hat{\jmath}$$

$$\frac{\partial\vec{r}}{\partial\theta} \times \frac{\partial\vec{r}}{\partial\phi} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ R\cos\theta\cos\phi & R\cos\theta\sin\phi & -R\sin\theta \\ -R\sin\theta\sin\phi & R\sin\theta\cos\phi & 0 \end{vmatrix}$$

$$= -R^2\sin^2\theta\cos\phi\,\hat{\imath} + R^2\sin^2\theta\sin\phi\,\hat{\jmath} + \hat{k}\overbrace{(R^2\sin\theta\cos^2\phi\cos\theta + R^2\sin^2\phi\cos\theta\sin\theta)}^{R^2\sin\theta\cos\theta}$$

$$\left|\frac{\partial\vec{r}}{\partial\theta} \times \frac{\partial\vec{r}}{\partial\phi}\right| = \left(R^4\sin^4\theta\cos^2\phi + R^4\sin^4\theta\sin^2\phi + R^4\sin^2\theta\cos^2\theta\right)^{1/2}$$

$$= R^2\left(\sin^4\theta + \sin^2\theta\cos^2\theta\right)^{1/2} = \underline{R^2\sin\theta} \geq 0 \quad \text{kun } \theta \in [0, \pi]$$

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$\boxed{3.}$ $$A_1 = \int_{A_1} dS = \int_0^{2\pi} d\phi \int_{\frac{\pi}{9}}^{\frac{\pi}{6}} d\theta \underbrace{R^2 \sin\theta}_{|J|} = 2\pi R^2 \Big/_{\frac{\pi}{9}}^{\pi/6} -\cos\theta$$

$$= 2\pi R^2 \left(\cos\left(\tfrac{\pi}{9}\right) - \cos\left(\tfrac{\pi}{6}\right)\right)$$

Väliin jäävän pinta-alan osuus koko pinta-alasta on

$$\frac{A_1}{4\pi R^2} = \frac{1}{2}\left(\cos\left(\tfrac{\pi}{9}\right) - \cos\left(\tfrac{\pi}{6}\right)\right) \simeq 3{,}7\%$$

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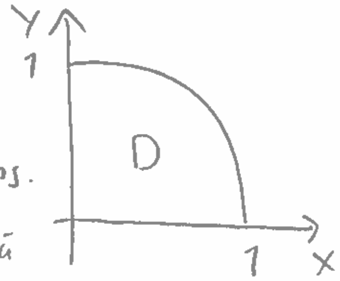
$\boxed{4.}$ $\int\limits_D ds \, \arctan\left(\frac{y}{x}\right)$, $\quad D = \left\{0 \le x \le 1, \quad 0 \le y \le \sqrt{1-y^2}\right\}$



Tehdään napakoordinaattimuunnos. Parametrisaatio kuten tehtävässä 2, mutta $r \in [0,1]$ ja $\theta \in [0, \frac{\pi}{2}]$.

Nyt $x = r\cos\theta$, $y = r\sin\theta$ ja $|J| = r$.

$$\Rightarrow \int\limits_D ds \, \arctan\left(\frac{y}{x}\right) = \int_0^1 dr \int_0^{\frac{\pi}{2}} d\theta \, r \cdot \underbrace{\arctan\left(\frac{r\sin\theta}{r\cos\theta}\right)}_{=\theta}$$

$$= \int_0^1 dr \, r \int_0^{\frac{\pi}{2}} d\theta \, \theta = \int_0^1 dr \, r \underbrace{\Big/_0^{\frac{\pi}{2}} \frac{1}{2}\theta^2}_{\frac{\pi^2}{8}} = \frac{\pi^2}{8} \Big/_0^1 \frac{1}{2} r^2 = \underline{\frac{\pi^2}{16}}.$$