Vector Analysis
Spring 2014

Exercise 6

Recital Wed 23.4.

1. The region *V* bounded by the planes $x = 0, y = 0$ and $z = 2$ and the plane $z = x^2 + y^2$ where $x \geq 0$ ja $y \geq 0.$ Calculate the volume integral of the function $f(x, y, z) = x$ over *V*. This is calculated in the Lecture Notes (page 73) by performing first the *z*-integration. Do it now so that you start with the *x*-integration.

2. Consider the object
$$S = \left\{ (x, y, z) \middle| 0 \leq x \leq 1, x^2 \leq y \leq x, x - y \leq z \leq x + y \right\}.$$
   a) Calculate the volume of *S*.
   b) Calculate the mass of *S* when the density is given by $\rho(x, y, z) = x + 2y + 4z$ (kg/m^3).

3. Consider a sphere with radius *a* and its center in the origin. The density of the sphere is proportional to the distance from the center, that is $\rho(\vec{r}) = \alpha r$, where α is a constant. Calculate the mass of the sphere
$$M = \iiint_S \rho(\vec{r}) dV.$$

4. When the density distribution $\rho(\vec{r})$ of an object is given, the radius vector of the center of the mass of the object $\vec{r}_{CM}$ is obtained from the formula (study this in Adams 14.7)
$$\vec{r}_{CM} = \frac{\iiint_{\text{object}} \rho(\vec{r}) \vec{r} dV}{\iiint_{\text{object}} \rho(\vec{r}) dV}.$$
   Calculate the radius vector of the center of mass of that part of an origin-centered sphere (radius is *a*) that is in the octant $x \geq 0, y \geq 0, z \geq 0$. The density is assumed to be constant.

5. A circular cone with the height *h* and the vertex angle α is located as shown in the figure. Its density is ρ and mass *M*. Calculate the gravitational force the cone causes on a point like mass *m* located in the origin. According to Newton's theory of gravitation the force that an object of mass *M* causes on another object of mass *m* is given by

$$\vec{F} = G\frac{mM}{d^2}\hat{e},$$

where d is the distance between the centers of mass of the objects and $\hat{e}$ is the unit vector pointing from m to M.

Hint: Calculate the force an infinitesimal mass unit $dM = \rho(\vec{r})dV$ located in the point $\vec{r}$ using the Newton's theory and integrate the over the cone. The integration can be performed e.g. using spherical or cylindrical coordinates (or both if you wish).